CSE 135: Introduction to Theory of Computation
Optimal DFA

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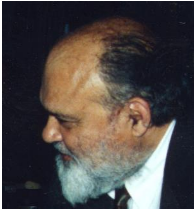
Optimal Algorithms for Regular Languages



Anil Nerode

Myhill-Nerode Theorem

There is a "unique" "optimal" "algorithm" for every problem that can be solved using finite memory.

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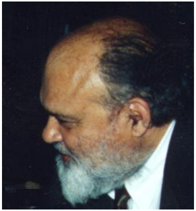

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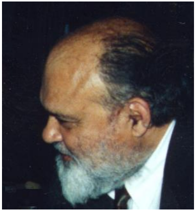

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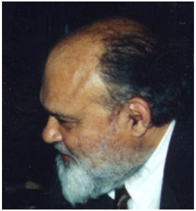

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There is a "unique" "optimal" "algorithm" for every problem that can be solved using finite memory.

- "algorithm" here means a deterministic machine
- "optimal" means requires least memory, i.e., has fewest states
- "unique" means that any two DFAs with fewest states for a language are "isomorphic"

Roadmap

- DFA minimization: Minimize a given DFA M by merging "indistinguishable" states.

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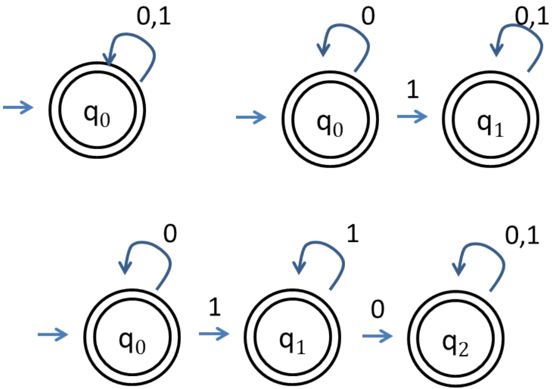
- In DFA minimization, a minimal (locally optimal) DFA is a minimum (globally optimal) DFA.
- Test if two given DFAs are equivalent.
- Revisit Myhill-Nerode Theorem.

Many DFAs for the same language



Minimization

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Applications

Algorithms using DFAs run in time directly related to the number of states of DFA. Implementation of the DFA itself takes memory proportional to log number of states. So constructing small DFAs is very critical.

Algorithm

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Seems not strong enough.

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We will say that p and q are distinguishable when this happens.

Indistinguishability

We say that two states p and q of M are indistinguishable/equivalent if

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Indistinguishability defines an equivalence class: $A \equiv A$ (reflexivity), $A \equiv B \Leftrightarrow B \equiv A$ (symmetricity), $A \equiv B$ and $B \equiv C \Rightarrow$ (transivity). So let's use $p \equiv q$ to say that two states p and q are indistinguishable.

Gradually Refine Indistinguishability

Recall

$$p \equiv q : \qquad \forall w.\ \hat{\delta}(p, w) \in F \text{ iff } \hat{\delta}(q, w) \in F$$

For each $k \geq 0$, define

$$p \equiv_k q : \qquad \forall w \text{ with } |w| \leq k.\ \hat{\delta}(p, w) \in F \text{ iff } \hat{\delta}(q, w) \in F$$

An equivalence class partitions states into disjoint groups. $\equiv_0$ has two groups: $Q \setminus F$ and F.

Gradually Refine Indistinguishability

Suppose that we know for each pair of two states p, q if $p \equiv_k q$ or not. How can we examine if $p \equiv_{k+1} q$ or not?

If $p \not\equiv_k q$, then we have $p \not\equiv_{k+1} q$. Each group of states can be only refined!
If $p \equiv_k q$, then we need to do more work:

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Do you see why?

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Do you see why?

$$\begin{array}{ll} & p \equiv_{k+1} q \\ \Leftrightarrow & \forall u \text{ with } |u| \leq k \forall a \in \Sigma . \hat{\delta}(p, au) \in F \text{ iff } \hat{\delta}(q, au) \in F \\ \Leftrightarrow & \forall u \text{ with } |u| \leq k \forall a \in \Sigma . \hat{\delta}(\delta(p, a), u) \in F \text{ iff } \hat{\delta}(\delta(q, a), u) \in F \\ \Leftrightarrow & \forall a \in \Sigma \forall u \text{ with } |u| \leq k . \hat{\delta}(\delta(p, a), u) \in F \text{ iff } \hat{\delta}(\delta(q, a), u) \in F \\ \Leftrightarrow & \forall a \in \Sigma . \delta(p, a) \equiv_k \delta(q, a) \end{array}$$

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- If $p \in F$ and $q \notin F$ then p and q are distinuishable
- If for some a, $\delta(p, a) = p'$ and $\delta(q, a) = q'$, and p' and q' are distinguishable, then p and q are distinguishable

Distinguishability

An Algorithm

Let `distinct` be a table with an entry for each pair of states. Initially all entries are 0.

```
if p ∈ F and q ∉ F (or vice versa)
then distinct(p, q) := 1
repeat
for each pair (p,q) and symbol a
    if distinct(δ(p,a), δ(q,a)) = 1,
    then distinct(p, q) := 1
until no changes in table
```

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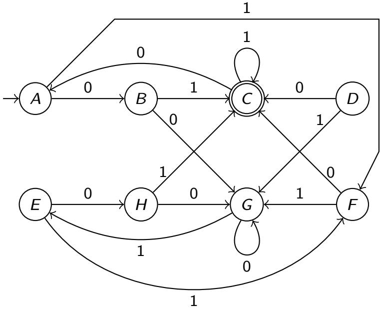
3. Collapse pairs that are not distinguishable

Example



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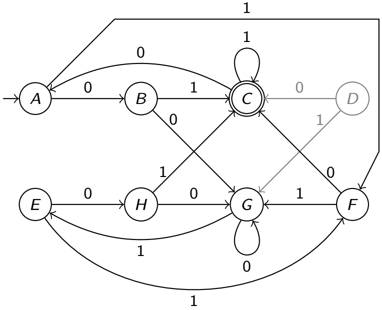
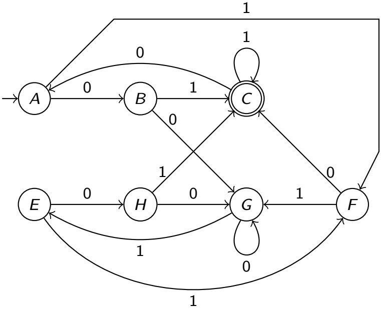


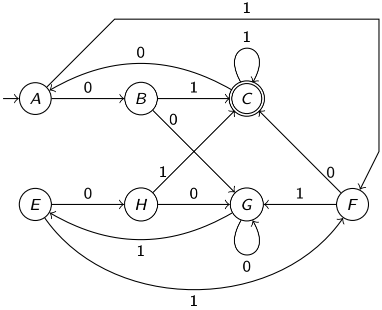
Example

	A	B	C	E	F	G
B						
C	★	★				
E			★			
F			★			
G			★			
H			★			



Example

	A	B	C	E	F	G
B	⋆					
C	⋆	⋆				
E		⋆	⋆			
F	⋆	⋆	⋆	⋆		
G		⋆	⋆		⋆	
H	⋆		⋆	⋆	⋆	⋆



Example

B	$\star$					
C	$\star$	$\star$				
E		$\star$	$\star$			
F	$\star$	$\star$	$\star$	$\star$		
G	$\star$	$\star$	$\star$	$\star$	$\star$	
H	$\star$		$\star$	$\star$	$\star$	$\star$
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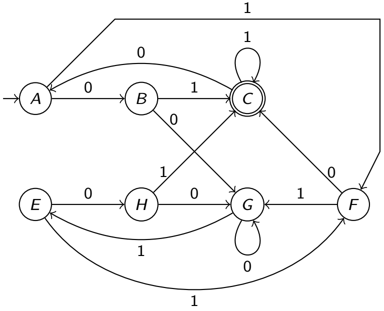

Example

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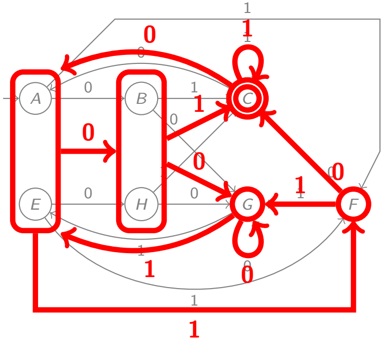
C	$\star$	$\star$				
E		$\star$	$\star$			
F	$\star$	$\star$	$\star$	$\star$		
G	$\star$	$\star$	$\star$	$\star$	$\star$	
H	$\star$		$\star$	$\star$	$\star$	$\star$



Example

$$
\begin{aligned}
\equiv_0: &\quad \{A, B, E, F, G, H\}, \{C\} \\
\equiv_1: &\quad \{A, E, G\}, \{B, H\}, \{F\}, \{C\} \\
\equiv_2: &\quad \{A, E\}, \{G\}, \{B, H\}, \{F\}, \{C\} \\
\equiv_3: &\quad \{A, E\}, \{G\}, \{B, H\}, \{F\}, \{C\}
\end{aligned}
$$

No change from $\equiv_2$ to $\equiv_3$, so stop.

Roadmap

- DFA minimization: Minimize a given DFA M by merging "indistinguishable" states (possibly locally optimal) – done
- In DFA minimization, a minimal (locally optimal) DFA is a minimum (globally optimal) DFA.
- Test if the two given DFAs are equivalent.
- Revisit Myhill-Nerode Theorem.

Decide if two given DFAs accpet the same language

We would like to test if two DFAs $M = (Q^M, \Sigma^M, \delta^M, q_0^M, F^M)$ and $N = (Q^N, \Sigma^N, \delta^N, q_0^N, F^N)$ accept the same language or not.

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1. Run the table-filling algorithm on both DFAs simultaneously.
2. M and N accept the same language iff $q_0^M \equiv q_0^N$.

Table-filling Algorithm Gives a Globally Optimal DFA

Very short proof sketch: $M = (Q^M, \Sigma^M, \delta^M, q_0^M, F^M)$: DFA output by the algorithm
$N = (Q^N, \Sigma^N, \delta^N, q_0^N, F^N)$: a globally optimal DFA
For the sake of contradiction suppose $|Q^N| < |Q^M|$. Then one can find $q_i^M \not\equiv q_j^M$ such that $q_i^M \equiv q_t^N \equiv q_j^M$ for some $q_t^N \in Q^N$.

Isomorphism

Definition

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be two DFAs. A function $f : Q_1 \to Q_2$ is said to be isomorphism iff

- f is bijective, i.e., one-to-one and onto
- $f(q_1) = q_2$
- For every $p \in Q_1$ and $a \in \Sigma$, $f(\delta_1(p, a)) = \delta_2(f(p), a)$
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M_1 and M_2 are said to be isomorphic if there is an isomorphism f from M_1 to M_2.
Thus, if M_1 and M_2 are isomorphic then they are the "same" machine except for possibly renaming states.

Myhill-Nerode Theorem

implies...

Theorem

For any regular language L, threre is a unique (upto isomorphism) DFA with fewest states that recognizes L.