CSE 135: Introduction to Theory of Computation
Closure Properties

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- Very useful in studying the properties of one language by relating it to other (better understood) languages
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- Today: A variety of operations which preserve regularity
  - i.e., the universe of regular languages is closed under these operations

Closure Properties

Definition

Regular Languages are closed under an operation op on languages if

$$L_1, L_2, \ldots L_n \text{ regular} \implies L = \mathrm{op}(L_1, L_2, \ldots L_n) \text{ is regular}$$

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Regular languages are closed under

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Regular languages are closed under

- "halving", i.e., L regular $\implies \frac{1}{2}L$ regular.
- "reversing", i.e., L regular $\implies L^{\mathsf{rev}}$ regular.

Operations from Regular Expressions

Proposition

Regular Languages are closed under $\cup$, $\circ$ *and* *.

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Proof.

(Summarizing previous arguments.)

- L_1, L_2 regular $\implies \exists$ regexes R_1, R_2 s.t. $L_1 = L(R_1)$ and $L_2 = L(R_2)$.
  - $\implies L_1 \cup L_2 = L(R_1 \cup R_2) \implies L_1 \cup L_2$ regular.

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  - $\implies L_1 \cup L_2 = L(R_1 \cup R_2) \implies L_1 \cup L_2$ regular.
  - $\implies L_1 \circ L_2 = L(R_1 \circ R_2) \implies L_1 \circ L_2$ regular.
  - $\implies L_1^* = L(R_1^*) \implies L_1^*$ regular. $\square$

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What happens if M (above) was an NFA?

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Observe that $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$. Since regular languages are closed under union and complementation, we have

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- $\overline{L_1}$ and $\overline{L_2}$ are regular
- $\overline{L_1} \cup \overline{L_2}$ is regular
- Hence, $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ is regular. $\square$

Is there a direct proof for intersection (yielding a smaller DFA)?

Cross-Product Construction

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFAs recognizing L_1 and L_2, respectively.

Idea: Run M_1 and M_2 in parallel on the same input and accept if both M_1 and M_2 accept.

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Consider $M = (Q, \Sigma, \delta, q_0, F)$ defined as follows

- $Q = Q_1 \times Q_2$
- $q_0 = \langle q_1, q_2 \rangle$
- $\delta(\langle p_1, p_2 \rangle, a) = \langle \delta_1(p_1, a), \delta_2(p_2, a) \rangle$
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M accepts $L_1 \cap L_2$ (exercise)

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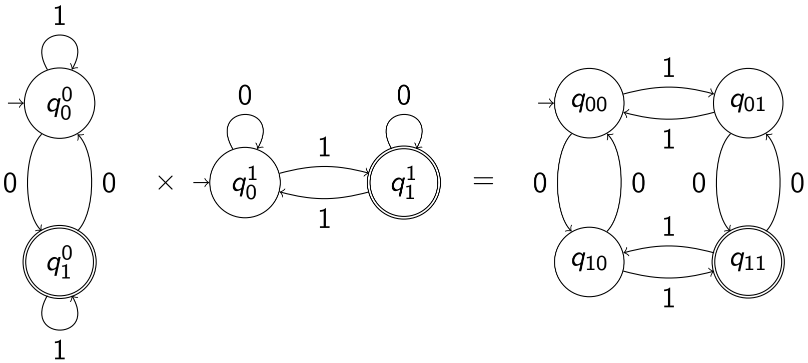
M accepts $L_1 \cap L_2$ (exercise)
What happens if M_1 and M_2 where NFAs? Still works! Set $\delta(\langle p_1, p_2 \rangle, a) = \delta_1(p_1, a) \times \delta_2(p_2, a)$.

An Example



Homomorphism

Definition

A homomorphism is function $h : \Sigma^* \to \Delta^*$ defined as follows:

- $h(\epsilon) = \epsilon$ and for $a \in \Sigma$, $h(a)$ is any string in Δ^*
- For $a = a_1 a_2 \ldots a_n \in \Sigma^*$ $(n \geq 2)$, $h(a) = h(a_1)h(a_2)\ldots h(a_n)$.

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Exercise: $h(L_1 \cup L_2) = h(L_1) \cup h(L_2)$.

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Exercise: $h(L_1 \cup L_2) = h(L_1) \cup h(L_2)$. $h(L_1 \circ L_2) = h(L_1) \circ h(L_2)$, and $h(L^*) = h(L)^*$.

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- Show that $L(h(R)) = h(L(R))$

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- Define homomorphism as an operation on regular expressions
- Show that $L(h(R)) = h(L(R))$
- Let R be such that $L = L(R)$. Let $R' = h(R)$. Then $h(L) = L(R')$.

$\cdot\cdot\longrightarrow$

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Formally $h(R)$ is defined inductively as follows.

$$\begin{array}{ll} h(\emptyset) = \emptyset & h(R_1 R_2) = h(R_1) h(R_2) \\ h(\epsilon) = \epsilon & h(R_1 \cup R_2) = h(R_2) \cup h(R_2) \\ h(a) = h(a) & h(R^*) = (h(R))^* \end{array}$$

Proof of Claim

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For any regular expression R, $L(h(R)) = h(L(R))$.

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By induction on the number of operations in R

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- Base Cases: For $R = \epsilon$ or $\emptyset$, $h(R) = R$ and $h(L(R)) = L(R)$.

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  Other cases ($R = R_1R_2$ and $R = R_1^*$) similar. $\square$

Nonregularity and Homomorphism

If L is not regular, is $h(L)$ also not regular?

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- No! Consider $L = \{0^n 1^n \mid n \geq 0\}$ and $h(0) = a$ and $h(1) = \epsilon$. Then $h(L) = a^*$.

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Applying a homomorphism can "simplify" a non-regular language into a regular language.

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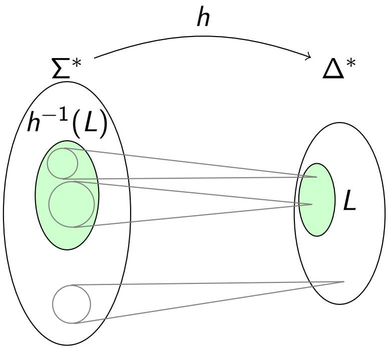

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Example

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- $h^{-1}(1001) = \{ba\}$, $h^{-1}(010110) = \{aab\}$

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Let $\Sigma = \{a, b\}$, and $\Delta = \{0, 1\}$. Let $L = (00 \cup 1)^*$ and $h(a) = 01$ and $h(b) = 10$.

- $h^{-1}(1001) = \{ba\}$, $h^{-1}(010110) = \{aab\}$
- $h^{-1}(L) = \ (ba)^*$
- What is $h(h^{-1}(L))$?

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- What is $h(h^{-1}(L))$? $(1001)^* \subsetneq L$

Note: In general $h(h^{-1}(L)) \subseteq L \subseteq h^{-1}(h(L))$, but neither containment is necessarily an equality.

Closure under Inverse Homomorphism

Proposition

Regular languages are closed under inverse homomorphism, i.e., if L is regular and h is a homomorphism then $h^{-1}(L)$ is regular.

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We will use the representation of regular languages in terms of DFA to argue this.

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We will use the representation of regular languages in terms of DFA to argue this.
Given a DFA M recognizing L, construct an DFA M' that accepts $h^{-1}(L)$

- Intuition: On input w M' will run M on $h(w)$ and accept if M does.

$\cdot\cdot\longrightarrow$

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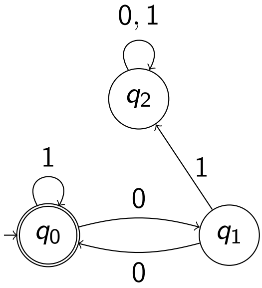

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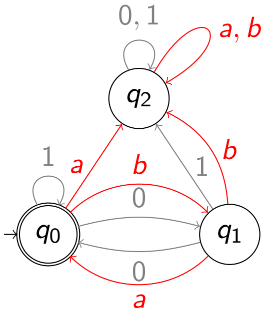

Closure under Inverse Homomorphism

Formal Construction

- Let $M = (Q, \Delta, \delta, q_0, F)$ accept $L \subseteq \Delta^*$ and let $h : \Sigma^* \to \Delta^*$ be a homomorphism
- Define $M' = (Q', \Sigma, \delta', q_0', F')$, where
  - $Q' = Q$
  - $q_0' = q_0$
  - $F' = F$, and
  - $\delta'(q, a) = \hat{\delta}_M(q, h(a))$; M' on input a simulates M on $h(a)$
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- M' accepts $h^{-1}(L)$
- Because $\forall w.\ \hat{\delta}_{M'}(q_0, w) = \hat{\delta}_M(q_0, h(w))$

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Problem

Show that $L = \{a^n b a^n \mid n \geq 0\}$ *is not regular*

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More formally, we will show that by applying a sequence of "regularity preserving" operations to L we can get K. Then, since K is not regular, L cannot be regular. $\cdot\cdot\!\longrightarrow$

Proving Non-Regularity

Using Closure Properties

Proof (contd).

To show that by applying a sequence of "regularity preserving" operations to $L = \{a^n b a^n \mid n \geq 0\}$ we can get $K = \{0^n 1^n \mid n \geq 0\}$.

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Proof (contd).

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  - $L_3 = h_2(L_2) = \{0^n 1^n \mid n \geq 0\} = K$
- Now if L is regular then so are L_1, L_2, L_3, and K. But K is not regular, and so L is not regular. $\square$

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- Define a homomorphism g where $g(0) = 0, g(1) = 1, g(a) = \epsilon, g(b) = \epsilon$. Then $g\Big((0 \cup 1)^*(a \cup b)^* \cap h^{-1}(L)\Big)$ is regular.

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- Do you see $head(L) = g\Big((0 \cup 1)^*(a \cup b)^* \cap h^{-1}(L)\Big)$?

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- Note: **Do not use pumping lemma to prove regularity!!**

A list of Regularity-Preserving Operations

Regular languages are closed under the following operations.

- Regular Expression operations
- Boolean operations: union, intersection, complement
- Homomorphism
- Inverse Homomorphism

(And several other operations...)