CSE 135: Introduction to Theory of Computation
Pumping Lemma

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Finite Languages

Definition

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$\{0, 1, 00, 10\}$ is a finite language, however, $(00 \cup 11)^*$ is not.

Finiteness and Regularity

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Proof.

Let $L = \{w_1, w_2, \ldots w_n\}$. Then $R = w_1 \cup w_2 \cup \cdots \cup w_n$ is a regular expression defining L. $\square$

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Above is a weak argument because $E = \{w \in \{0,1\}^* \mid w$ has an equal number of 01 and 10 substrings$\}$ is regular!

Proving Non-Regularity

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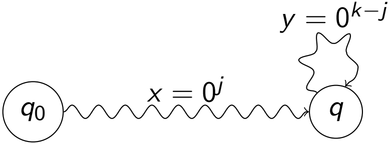
- Let $x = 0^j$, $y = 0^{k-j}$, and $z = 0^{n-k}1^n$; so $xyz = 0^n1^n$. $\cdots\!\!\longrightarrow$

Proving Non-Regularity

Proof (contd).



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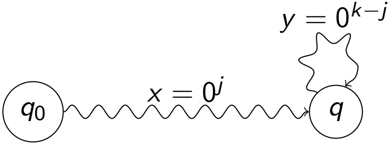

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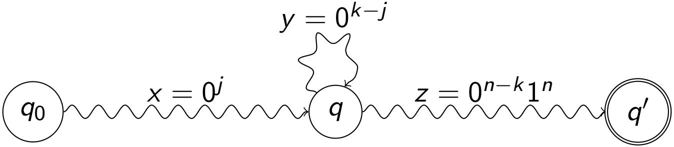

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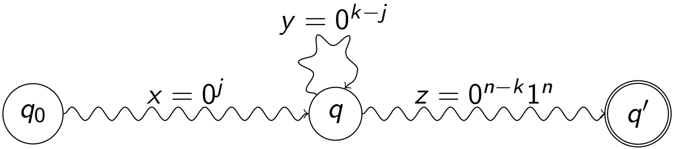

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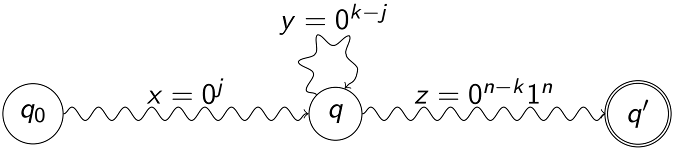

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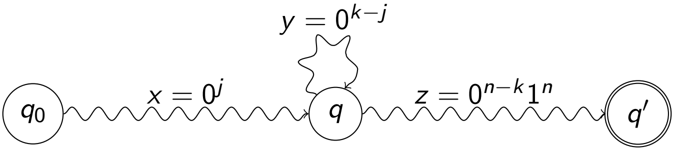

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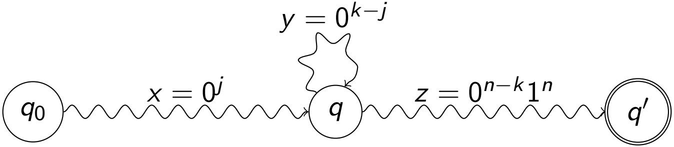

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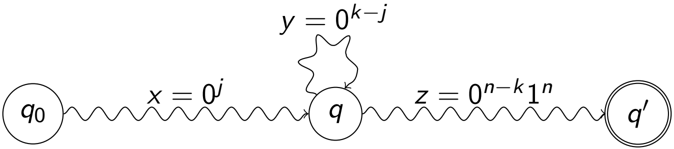

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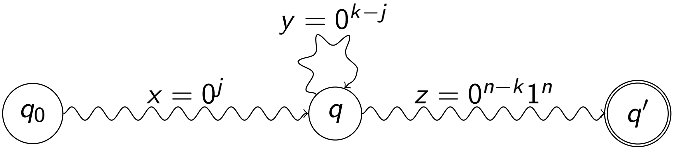

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- So M accepts $0^{n-k+j}1^n$ as well.

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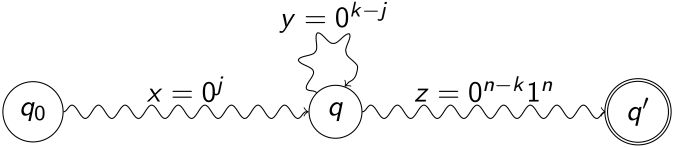

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- So M accepts $0^{n-k+j}1^n$ as well. But, $0^{n-k+j}1^n \notin L_{\mathrm{eq}}$! □

Pumping Lemma: Overview

Pumping Lemma

The lemma generalizes this argument. Gives the template of an argument that can be used to easily prove that many languages are non-regular.

Pumping Lemma

The Statement

Lemma

If L is regular then there is a number p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

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Proof.

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- Since $s_0, s_1, \ldots, s_i, \ldots s_p$ are $p + 1$ states, there must be j, k, $0 \leq j < k \leq p$ such that $s_j = s_k$ $(= q$ say$)$.

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- Take $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$, and $z = w_{k+1} \cdots w_n$
- Observe that since $j < k \leq p$, we have $|xy| \leq p$ and $|y| > 0$.

$\cdots\longrightarrow$

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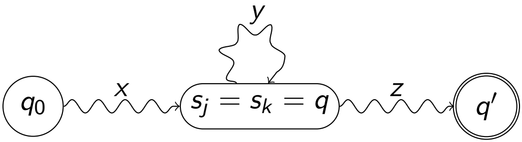
□

Completing the Proof

Proof (contd).



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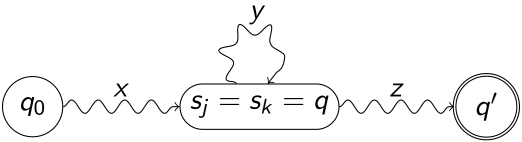

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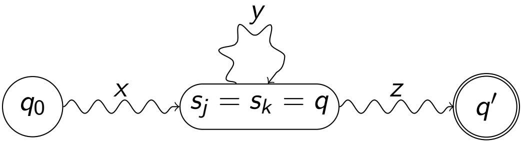

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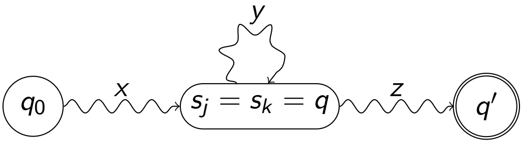

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- Observe,
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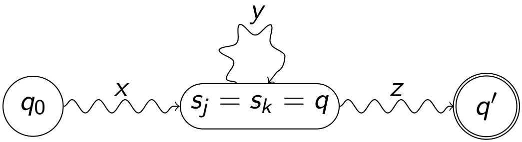

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- Similarly, $\hat{\delta}(q_0, xy^iz) = \hat{\delta}(q_0, xyz) \in F$ and so $xy^iz \in L$ $\square$

Finite Languages and Pumping Lemma

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Answer

Yes, they do. Let p be larger than the longest string in the language. Then the condition "$\forall w \in L$ with $|w| \geq p, \ldots$" is vaccuously satisfied as there are no strings in the language longer than p!

Using the Pumping Lemma

L regular implies that L satisfies the condition in the pumping lemma.

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Using the Pumping Lemma

L regular implies that L satisfies the condition in the pumping lemma. If L is not regular pumping lemma says nothing about L!

Using the Pumping Lemma

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Pumping Lemma, in contrapositive

If L does not satisfy the pumping condition, then L not regular.

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Pumping Lemma, in contrapositive

If L does not satisfy the pumping condition, then L not regular.

Pumping Condition

$$\exists p. \quad \forall w \in L. \text{ with } |w| \geq p \quad \exists x, y, z \in \Sigma^*. \ w = xyz$$

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Equivalent to showing that if (1), (2) then (3) does not. In other words, we can find i such that $xy^i z \notin L$

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Think of using the Pumping Lemma as a game between you and an opponent.

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Pumping Lemma: If L is regular, opponent has a winning strategy (no matter what you do).
Contrapositive: If you can beat the opponent, L not regular.
Your strategy should work for any p and any subdivision that the opponent may come up with.

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Since $r + t < p$, $xy^0 z \notin L_{0n1n}$. Contradiction! □

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A Tale of two Proofs

Non Pumping Lemma

Suppose L_{eq} is recognized by DFA M with p states. Consider the input $0^p 1^p$.

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Suppose L_{eq} is regular. Let p be pumping length for L_{eq}. Consider $w = 0^p 1^p$.

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Suppose L_{eq} is recognized by DFA M with p states. Consider the input 0^p1^p. There exist j, k and state q such that

- $j < k$ and $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$

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Suppose L_{eq} is regular. Let p be pumping length for L_{eq}. Consider $w = 0^p1^p$. There exist x, y, z such that

- $w = xyz$, $|xy| \leq p$, $|y| > 0$: so for some r, s, t, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$, with $s > 0$.

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- But $0^j 0^{(p-k)} 1^p \notin L_{\mathrm{eq}}$.

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- $xy^i z \in L_{\mathrm{eq}}$ for all i: so $xy^0 z \in L_{\mathrm{eq}}$.
- But $xy^0 z = 0^{p-s} 1^p \notin L_{\mathrm{eq}}$

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Proposition

$L_p = \{0^i \mid i \text{ prime}\}$ is not regular

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Is $L_{xx} = \{xx \mid x \in \{0,1\}^*\}$ is regular?

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- Another bad choice $(01)^p(01)^p$.

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Since $r + t < p$, $xy^0z \notin L_{xx}$. Contradiction! □

Lessons on Expressivity

Limits of Finite Memory

Finite automata cannot

- "keep track of counts": e.g., L_{0n1n} not regular.
- "compare far apart pieces" of the input: e.g. L_{xx} not regular.
- do "computations that require it to look at global properties" of the input. e.g. L_{prime} not regular.

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. . . and pumping lemma provides one way to find out some of these limitations.