Finite Languages

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A language is finite if it has finitely many strings.
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**Example**
\{0, 1, 00, 10\} is a finite language
Finite Languages

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Example
\{0, 1, 00, 10\} is a finite language, however, \((00 \cup 11)^*\) is not.
Finiteness and Regularity

Proposition

*If* \( L \) *is finite then* \( L \) *is regular.*
Proposition

If $L$ is finite then $L$ is regular.

Proof.

Let $L = \{w_1, w_2, \ldots w_n\}$. Then $R = w_1 \cup w_2 \cup \cdots \cup w_n$ is a regular expression defining $L$. □
Are all languages regular?

Proposition

The language \( L_{eq} = \{ w \in \{0, 1\}^* | w \text{ has an equal number of 0s and 1s} \} \) is not regular.

Proof

No DFA has enough states to keep track of the number of 0s and 1s it might see. □

Above is a weak argument because \( E = \{ w \in \{0, 1\}^* | w \text{ has an equal number of 01 and 10 substrings} \} \) is regular!
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Suppose (for contradiction) $L_{eq}$ is recognized by DFA $M = (Q, \{0,1\}, \delta, q_0, F)$, where $|Q| = n$. 

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The language $L_{eq} = \{w \in \{0, 1\}^* | w \text{ has an equal number of 0s and 1s}\}$ is not regular.

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- There must be $j < k \leq n$ such that $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k)$ ($= q$ say).
Proving Non-Regularity

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Suppose (for contradiction) $L_{eq}$ is recognized by DFA $M = (Q, \{0, 1\}, \delta, q_0, F)$, where $|Q| = n$.

- There must be $j < k \leq n$ such that $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k)$ (say).
- Let $x = 0^j$, $y = 0^{k-j}$, and $z = 0^{n-k}1^n$; so $xyz = 0^n1^n$. \[\rightarrow\]
Proving Non-Regularity

Proof (contd).

We have $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$
Proving Non-Regularity

Proof (contd).

- We have $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$
- Since $0^n1^n \in L_{eq}$, $\hat{\delta}(q_0, 0^n1^n) \in F$. 

\[ y = 0^{k-j} \]
Proving Non-Regularity

Proof (contd).

We have $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$

Since $0^n1^n \in L_{eq}$, $\hat{\delta}(q_0, 0^n1^n) \in F.$

$\hat{\delta}(q_0, 0^n1^n) = \delta(\hat{\delta}(q_0, 0^k), 0^{n-k}1^n)$
Proving Non-Regularity

Proof (contd).

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Since \( 0^n1^n \in L_{eq} \), \( \hat{\delta}(q_0, 0^n1^n) \in F \).

\[
\hat{\delta}(q_0, 0^n1^n) = \hat{\delta}(\hat{\delta}(q_0, 0^k), 0^{n-k}1^n) \quad \text{(since} \ \hat{\delta}(q, uv) = \hat{\delta}(\hat{\delta}(q, u), v))
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$$\hat{\delta}(q_0, 0^n1^n) = \hat{\delta}(\hat{\delta}(q_0, 0^k), 0^{n-k}1^n)$$

(since $\hat{\delta}(q, uv) = \hat{\delta}(\hat{\delta}(q, u), v)$)

$$= \hat{\delta}(\hat{\delta}(q_0, 0^j), 0^{n-k}1^n)$$
Proving Non-Regularity

Proof (contd).

We have $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$

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\[
\begin{align*}
\hat{\delta}(q_0, 0^n1^n) &= \hat{\delta}(\hat{\delta}(q_0, 0^k), 0^{n-k}1^n) \\
&= \hat{\delta}(\hat{\delta}(q_0, 0^j), 0^{n-k}1^n) \\
&= \hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k)
\end{align*}
\]
Proving Non-Regularity

Proof (contd).

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= \hat{\delta}(q_0, 0^{n-k+j}1^n)
\]

(since \( \delta(q, uv) = \delta(\delta(q, u), v) \))

\( (\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) ) \)
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(since $\delta(q, uv) = \delta(\hat{\delta}(q, u), v)$)

$\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k)$

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- So $M$ accepts $0^{n-k+j}1^n$ as well.
Proving Non-Regularity

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(since \( \hat{\delta}(q, uv) = \hat{\delta}(\hat{\delta}(q, u), v) \))

So \( M \) accepts \( 0^{n-k+j}1^n \) as well. But, \( 0^{n-k+j}1^n \notin L_{eq}! \)
Pumping Lemma: Overview

Pumping Lemma

The lemma generalizes this argument. Gives the template of an argument that can be used to easily prove that many languages are non-regular.
Pumping Lemma

The Statement

Lemma

If $L$ is regular then there is a number $p$ (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and
Pumping Lemma

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1. \(|y| > 0\)
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1. $|y| > 0$
2. $|xy| \leq p$
3. $\forall i \geq 0. \ xy^i z \in L$
Proving the Pumping Lemma

Proof.
Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = L$ and let $p = |Q|$. 

Let $w = w_1w_2\cdots w_n \in L$ be such that $n \geq p$.

For $1 \leq i \leq n$, let $s_i = \hat{\delta}(q_0, w_1\cdots w_i)$; define $s_0 = q_0$.

Since $s_0, s_1, \ldots, s_i, \ldots, s_p$ are $p + 1$ states, there must be $j, k$, $0 \leq j < k \leq p$ such that $s_j = s_k (= q$ say).

Take $x = w_1\cdots w_j$, $y = w_{j+1}\cdots w_k$, and $z = w_{k+1}\cdots w_n$.

Observe that since $j < k \leq p$, we have $|xy| \leq p$ and $|y| > 0$.··→
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Proving the Pumping Lemma

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Since $s_0, s_1, \ldots, s_i, \ldots, s_p$ are $p + 1$ states, there must be $j, k$, $0 \leq j < k \leq p$ such that $s_j = s_k$ (say).

Take $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$, and $z = w_{k+1} \cdots w_n$.

Observe that since $j < k \leq p$, we have $|xy| \leq p$ and $|y| > 0$. 
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Since $s_0, s_1, \ldots, s_i, \ldots s_p$ are $p + 1$ states, there must be $j, k, 0 \leq j < k \leq p$ such that $s_j = s_k (= q$ say).
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- Since $s_0, s_1, \ldots, s_i, \ldots s_p$ are $p + 1$ states, there must be $j, k$, $0 \leq j < k \leq p$ such that $s_j = s_k$ ($= q$ say).
- Take $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$, and $z = w_{k+1} \cdots w_n$
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Let \( M = (Q, \Sigma, \delta, q_0, F) \) be a DFA such that \( L(M) = L \) and let \( p = |Q| \). Let \( w = w_1 w_2 \cdots w_n \in L \) be such that \( n \geq p \). For \( 1 \leq i \leq n \), let \( s_i = \hat{\delta}(q_0, w_1 \cdots w_i) \); define \( s_0 = q_0 \).

- Since \( s_0, s_1, \ldots, s_i, \ldots s_p \) are \( p + 1 \) states, there must be \( j, k \), \( 0 \leq j < k \leq p \) such that \( s_j = s_k \) (\( = q \) say).
- Take \( x = w_1 \cdots w_j \), \( y = w_{j+1} \cdots w_k \), and \( z = w_{k+1} \cdots w_n \)
- Observe that since \( j < k \leq p \), we have \( |xy| \leq p \) and \( |y| > 0 \).
Proof . . .

Technical Claim

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For all $i \geq 1$, $\hat{\delta}(xy^i) = \hat{\delta}(q_0, x)$. 
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We will prove it by induction on $i$.

- **Base Case:** By our assumption that $s_j = s_k$ and the definition of $x$ and $y$, we have $\hat{\delta}(q_0, xy) = s_k = s_j = \hat{\delta}(q_0, x)$. 

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- **Induction Step:** We have

\[
\hat{\delta}(q_0, xy^{\ell+1}) = \hat{\delta}(\hat{\delta}(q_0, xy^\ell), y)
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□
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For all $i \geq 1$, $\hat{\delta}(x y^i) = \hat{\delta}(q_0, x)$.

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  $\hat{\delta}(q_0, x y^{\ell+1}) = \hat{\delta}(\hat{\delta}(q_0, x y^\ell), y) = \hat{\delta}(\hat{\delta}(q_0, x), y)$

\[ \square \]
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For all \( i \geq 1, \hat{\delta}(xy^i) = \hat{\delta}(q_0, x) \).

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We will prove it by induction on \( i \).

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\[
\hat{\delta}(q_0, xy^{\ell+1}) = \hat{\delta}(\hat{\delta}(q_0, xy^\ell), y) \\
= \hat{\delta}(\hat{\delta}(q_0, x), y) \\
= \hat{\delta}(q_0, xy) = \hat{\delta}(q_0, x) \\
\]

\[\Box\]
Completing the Proof

Proof (contd).

\[ q_0 \xrightarrow{x} s_j = s_k = q \xrightarrow{z} q' \]

\[ \hat{\delta}(q_0, \alpha) = \hat{\delta}(q_0, \beta) \] for all \( \alpha = \beta \)

\[ \text{Since } w \in L, \hat{\delta}(q_0, w) = \hat{\delta}(q_0, \alpha) \in F \]

\[ \text{Observe, } \hat{\delta}(q_0, \alpha) = \hat{\delta}(\hat{\delta}(q_0, \beta), \gamma) = \hat{\delta}(\hat{\delta}(q_0, \alpha), \gamma) = \hat{\delta}(q_0, w). \text{ So } \alpha = \beta \]

\[ \text{Similarly, } \hat{\delta}(q_0, \alpha \beta) = \hat{\delta}(q_0, w) \]
Completing the Proof

Proof (contd).

We have \( \hat{\delta}(q_0, xy^i) = \hat{\delta}(q_0, x) \) for all \( i \geq 1 \)
Completing the Proof

Proof (contd).

We have $\hat{\delta}(q_0, x y^i) = \hat{\delta}(q_0, x)$ for all $i \geq 1$

Since $w \in L$, we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, xyz) \in F$
We have $\hat{\delta}(q_0, xy^i) = \hat{\delta}(q_0, x)$ for all $i \geq 1$

Since $w \in L$, we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, xyz) \in F$

Observe,

$\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(\hat{\delta}(q_0, xy), z) = \hat{\delta}(q_0, w)$. So $xz \in L$
Completing the Proof

Proof (contd).

We have \( \hat{\delta}(q_0, xy^i) = \hat{\delta}(q_0, x) \) for all \( i \geq 1 \)

Since \( w \in L \), we have \( \hat{\delta}(q_0, w) = \hat{\delta}(q_0, xyz) \in F \)

Observe,
\[
\hat{\delta}(q_0, xz) = \hat{\delta}(\hat{\delta}(q_0, x), z) = \hat{\delta}(\hat{\delta}(q_0, xy), z) = \hat{\delta}(q_0, w).
\]
So \( xz \in L \)

Similarly, \( \hat{\delta}(q_0, xy^i z) = \hat{\delta}(q_0, xyz) \in F \) and so \( xy^i z \in L \)
Finite Languages and Pumping Lemma

Question
Do finite languages really satisfy the condition in the pumping lemma?
Finite Languages and Pumping Lemma

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Do finite languages really satisfy the condition in the pumping lemma?

Recall Pumping Lemma: If $L$ is regular then there is a number $p$ (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $|y| > 0$
2. $|xy| \leq p$
3. $\forall i \geq 0. \ xy^i z \in L$

Answer
Yes, they do. Let $p$ be larger than the longest string in the language. Then the condition "$\forall w \in L$ with $|w| \geq p$, ...

is vacuously satisfied as there are no strings in the language longer than $p$!
Finite Languages and Pumping Lemma

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Do finite languages really satisfy the condition in the pumping lemma?

Recall Pumping Lemma: If \( L \) is regular then there is a number \( p \) (the pumping length) such that \( \forall w \in L \) with \( |w| \geq p \), \( \exists x, y, z \in \Sigma^* \) such that \( w = xyz \) and
1. \( |y| > 0 \)
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3. \( \forall i \geq 0. \) \( xy^i z \in L \)

Answer
Yes, they do. Let \( p \) be larger than the longest string in the language. Then the condition “\( \forall w \in L \) with \( |w| \geq p \), …” is vacuously satisfied as there are no strings in the language longer than \( p \)!
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma. If $L$ is not regular
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma. If $L$ is not regular pumping lemma says nothing about $L$!
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

**Pumping Lemma, in contrapositive**

If $L$ does not satisfy the pumping condition, then $L$ not regular.
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

Pumping Lemma, in contrapositive
If $L$ does not satisfy the pumping condition, then $L$ not regular.

Pumping Condition

\[
\exists p. \quad \forall w \in L. \text{ with } |w| \geq p \quad \exists x, y, z \in \Sigma^*. \ w = xyz
\]

(1) $|y| > 0$

(2) $|xy| \leq p$

(3) $\forall i \geq 0. \ xy^i z \in L$
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

Pumping Lemma, in contrapositive
If $L$ does not satisfy the pumping condition, then $L$ not regular.

Negation of the Pumping Condition

\[ \overline{p}. \quad \forall w \in L. \text{ with } |w| \geq p \quad \exists x, y, z \in \Sigma^*. w = xyz \]
\[ (1) \quad |y| > 0 \]
\[ (2) \quad |xy| \leq p \]
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Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

Pumping Lemma, in contrapositive
If $L$ does not satisfy the pumping condition, then $L$ not regular.

Negation of the Pumping Condition

\[ \forall p. \quad \exists w \in L. \text{ with } |w| \geq p \quad \exists x, y, z \in \Sigma^*. \quad w = xyz \]

\begin{align*}
(1) \quad & |y| > 0 \\
(2) \quad & |xy| \leq p \\
(3) \quad & \forall i \geq 0. \; xy^iz \in L
\end{align*}
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

Pumping Lemma, in contrapositive
If $L$ does not satisfy the pumping condition, then $L$ not regular.

Negation of the Pumping Condition

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\forall p. \exists w \in L. \text{ with } |w| \geq p \quad \forall x, y, z \in \Sigma^*. \ w = xyz \\
(1) \ |y| > 0 \\
(2) \ |xy| \leq p \\
(3) \ \forall i \geq 0. \ xy^i z \in L
\]
Using the Pumping Lemma

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If $L$ does not satisfy the pumping condition, then $L$ not regular.

Negation of the Pumping Condition

$$\forall p. \ \exists w \in L. \ \text{with} \ |w| \geq p \ \forall x, y, z \in \Sigma^*. \ w = xyz$$

(1) $|y| > 0$
(2) $|xy| \leq p$
(3) $\forall i \geq 0. \ xy^i z \in L$

(not all of them hold)
Using the Pumping Lemma

$L$ regular implies that $L$ satisfies the condition in the pumping lemma.

Pumping Lemma, in contrapositive
If $L$ does not satisfy the pumping condition, then $L$ not regular.

Negation of the Pumping Condition

\[ \forall p. \ \exists w \in L. \text{ with } |w| \geq p \quad \forall x, y, z \in \Sigma^*. \ w = xyz \]

\[
(1) \quad |y| > 0 \\
(2) \quad |xy| \leq p \\
(3) \quad \forall i \geq 0. \ xy^i z \in L
\]

not all of them hold

Equivalent to showing that if (1), (2) then (3) does not. In other words, we can find $i$ such that $xy^i z \notin L$
Think of using the Pumping Lemma as a game between you and an opponent.
Think of using the Pumping Lemma as a game between you and an opponent.

$L$ Task: To show that $L$ is not regular
Think of using the Pumping Lemma as a game between you and an opponent.

- \( L \) Task: To show that \( L \) is not regular
- \( \forall p. \) Opponent picks \( p \)

Pumping Lemma: If \( L \) is regular, opponent has a winning strategy (no matter what you do).

Contrapositive: If you can beat the opponent, \( L \) not regular.

Your strategy should work for any \( p \) and any subdivision that the opponent may come up with.
Think of using the Pumping Lemma as a game between you and an opponent.

- $L$ Task: To show that $L$ is not regular
- $\forall p.$ Opponent picks $p$
- $\exists w.$ Pick $w$ that is of length at least $p$
Think of using the Pumping Lemma as a game between you and an opponent.

Let $L$ be the language.

- Task: To show that $L$ is not regular
- $\forall p$. Opponent picks $p$
- $\exists w$. Pick $w$ that is of length at least $p$
- $\forall x, y, z$. Opponent divides $w$ into $x, y, z$ such that $|y| > 0$, and $|xy| \leq p$
Think of using the Pumping Lemma as a game between you and an opponent.

$L$

Task: To show that $L$ is not regular

$\forall p$. Opponent picks $p$

$\exists w$. Pick $w$ that is of length at least $p$

$\forall x, y, z$. Opponent divides $w$ into $x, y, z$ such that $|y| > 0$, and $|xy| \leq p$

$\exists k$. You pick $k$ and win if $xy^kz \notin L$
Game View

Think of using the Pumping Lemma as a game between you and an opponent.

\[ L \]

Task: To show that \( L \) is not regular
\[ \forall p. \]
Opponent picks \( p \)
\[ \exists w. \]
Pick \( w \) that is of length at least \( p \)
\[ \forall x, y, z \]
Opponent divides \( w \) into \( x, y, \) and \( z \) such that
\[ |y| > 0, \text{ and } |xy| \leq p \]
\[ \exists k. \]
You pick \( k \) and win if \( xy^kz \notin L \)

**Pumping Lemma:** If \( L \) is regular, opponent has a winning strategy (no matter what you do).
Think of using the Pumping Lemma as a game between you and an opponent.

L

Task: To show that $L$ is not regular

$\forall p$. Opponent picks $p$

$\exists w$. Pick $w$ that is of length at least $p$

$\forall x, y, z$. Opponent divides $w$ into $x, y, z$ such that $|y| > 0$, and $|xy| \leq p$

$\exists k$. You pick $k$ and win if $xy^kz \notin L$

Pumping Lemma: If $L$ is regular, opponent has a winning strategy (no matter what you do).

Contrapositive: If you can beat the opponent, $L$ not regular.
Think of using the Pumping Lemma as a game between you and an opponent.

- **L** Task: To show that $L$ is not regular
- $\forall p$. Opponent picks $p$
- $\exists w$. Pick $w$ that is of length at least $p$
- $\forall x, y, z$. Opponent divides $w$ into $x, y, z$ such that $|y| > 0$, and $|xy| \leq p$
- $\exists k$. You pick $k$ and win if $xy^kz \notin L$

**Pumping Lemma**: If $L$ is regular, opponent has a winning strategy (no matter what you do).

**Contrapositive**: If you can beat the opponent, $L$ not regular.
Your strategy should work for any $p$ and any subdivision that the opponent may come up with.
Example 1

Proposition

\[ L_{01n} = \{0^n1^n \mid n \geq 0\} \text{ is not regular.} \]
Example 1

Proposition
$L_{0^n1^n} = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof.
Suppose $L_{0^n1^n}$ is regular. Let $p$ be the pumping length for $L_{0^n1^n}$.
Example 1

Proposition
\[ L_{0^n1^n} = \{0^n1^n \mid n \geq 0\} \text{ is not regular.} \]

Proof.
Suppose \( L_{0^n1^n} \) is regular. Let \( p \) be the pumping length for \( L_{0^n1^n} \).

- Consider \( w = 0^p1^p \)
Example 1

Proposition

\[ L_{0^n1^n} = \{0^n1^n \mid n \geq 0\} \text{ is not regular.} \]

Proof.

Suppose \( L_{0^n1^n} \) is regular. Let \( p \) be the pumping length for \( L_{0^n1^n} \).

▶ Consider \( w = 0^p1^p \)

▶ Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_{0^n1^n} \), for all \( i \).
Example I

Proposition
\[ L_{0n1n} = \{0^n1^n \mid n \geq 0\} \text{ is not regular.} \]

Proof.
Suppose \( L_{0n1n} \) is regular. Let \( p \) be the pumping length for \( L_{0n1n} \).

- Consider \( w = 0^p1^p \)
- Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_{0n1n} \), for all \( i \).
- Since \( |xy| \leq p \), \( x = 0^r \), \( y = 0^s \) and \( z = 0^t1^p \). Further, as \( |y| > 0 \), we have \( s > 0 \).
Example I

Proposition
$L_{0n1n} = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof.
Suppose $L_{0n1n}$ is regular. Let $p$ be the pumping length for $L_{0n1n}$.

- Consider $w = 0^p1^p$
- Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_{0n1n}$, for all $i$.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$. Further, as $|y| > 0$, we have $s > 0$.

$$xy^0z = 0^r0^t1^p = 0^{r+t}1^p$$

Contradiction! $\square$
Example I

Proposition
$L_{01n} = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof.
Suppose $L_{01n}$ is regular. Let $p$ be the pumping length for $L_{01n}$.

- Consider $w = 0^p1^p$
- Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_{01n}$, for all $i$.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$. Further, as $|y| > 0$, we have $s > 0$.

$$xy^0z = 0^r0^t1^p = 0^{r+t}1^p$$

Since $r + t < p$, $xy^0z \not\in L_{01n}$. Contradiction! □
Example II

Proposition

\[ L_{eq} = \{ w \in \{0, 1\}^* \mid w \text{ has an equal number of 0s and 1s} \} \text{ is not regular.} \]
Example II

Proposition
$L_{eq} = \{w \in \{0, 1\}^* \mid w \text{ has an equal number of } 0\text{s and } 1\text{s}\}$ is not regular.

Proof.
Suppose $L_{eq}$ is regular. Let $p$ be the pumping length for $L_{eq}$.
Example II

Proposition
$L_{eq} = \{ w \in \{0,1\}^* \mid w \text{ has an equal number of } 0\text{s and } 1\text{s}\}$ is not regular.

Proof.
Suppose $L_{eq}$ is regular. Let $p$ be the pumping length for $L_{eq}$.

- Consider $w = 0^p1^p$
Example II

Proposition

\[ L_{eq} = \{ w \in \{0, 1\}^* \mid w \text{ has an equal number of 0s and 1s} \} \text{ is not regular.} \]

Proof.

Suppose \( L_{eq} \) is regular. Let \( p \) be the pumping length for \( L_{eq} \).

- Consider \( w = 0^p1^p \)
- Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_{eq} \), for all \( i \).
Example II

Proposition
\[ L_{eq} = \{ w \in \{0, 1\}^* \mid w \ has \ an \ equal \ number \ of \ 0s \ and \ 1s \} \ is \ not \ regular. \]

Proof.
Suppose \( L_{eq} \) is regular. Let \( p \) be the pumping length for \( L_{eq} \).

▶ Consider \( w = 0^p1^p \)

▶ Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_{eq} \), for all \( i \).

▶ Since \( |xy| \leq p \), \( x = 0^r \), \( y = 0^s \) and \( z = 0^t1^p \). Further, as \( |y| > 0 \), we have \( s > 0 \).

\[
xy^0z = 0^r\epsilon 0^t1^p = 0^{r+t}1^p
\]

Since \( r + t < p \), \( xy^0z \not\in L_{eq} \). Contradiction! \( \Box \)
A Tale of two Proofs

Non Pumping Lemma
Suppose $L_{eq}$ is recognized by DFA $M$ with $p$ states. Consider the input $0^p1^p$.

Pumping Lemma
Suppose $L_{eq}$ is regular. Let $p$ be pumping length for $L_{eq}$. Consider $w = 0^p1^p$. 
A Tale of two Proofs

Non Pumping Lemma
Suppose $L_{\text{eq}}$ is recognized by DFA $M$ with $p$ states. Consider the input $0^p1^p$. There exist $j, k$ and state $q$ such that

Pumping Lemma
Suppose $L_{\text{eq}}$ is regular. Let $p$ be pumping length for $L_{\text{eq}}$. Consider $w = 0^p1^p$. There exist $x, y, z$ such that
A Tale of two Proofs

**Non Pumping Lemma**
Suppose $L_{eq}$ is recognized by DFA $M$ with $p$ states. Consider the input $0^p1^p$. There exist $j$, $k$ and state $q$ such that

- $j < k$ and
  
  $$\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$$

**Pumping Lemma**
Suppose $L_{eq}$ is regular. Let $p$ be pumping length for $L_{eq}$. Consider $w = 0^p1^p$. There exist $x$, $y$, $z$ such that

- $w = xyz$, $|xy| \leq p$, $|y| > 0$: so for some $r$, $s$, $t$, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$, with $s > 0$. 
A Tale of two Proofs

Non Pumping Lemma
Suppose $L_{eq}$ is recognized by DFA $M$ with $p$ states. Consider the input $0^p1^p$. There exist $j$, $k$ and state $q$ such that

- $j < k$ and 
  $\hat{\delta}(q_0, 0^j) = \hat{\delta}(q_0, 0^k) = q$
- Since $0^p1^p \in L_{eq}$, $0^k0^{(p-k)}1^p$ is accepted by $M$ and so is $0^j0^{(p-k)}1^p$.

Pumping Lemma
Suppose $L_{eq}$ is regular. Let $p$ be pumping length for $L_{eq}$. Consider $w = 0^p1^p$. There exist $x, y, z$ such that

- $w = xyz$, $|xy| \leq p$, $|y| > 0$: so for some $r, s, t$, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$, with $s > 0$.
- $xy^iz \in L_{eq}$ for all $i$: so $xy^0z \in L_{eq}$. 
A Tale of two Proofs

Non Pumping Lemma
Suppose $L_{eq}$ is recognized by DFA $M$ with $p$ states. Consider the input $0^p1^p$. There exist $j, k$ and state $q$ such that

- $j < k$ and
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- Since $0^p1^p \in L_{eq}$,
  $0^k0^{(p-k)}1^p$ is accepted by $M$ and so is $0^j0^{(p-k)}1^p$.

- But $0^j0^{(p-k)}1^p \notin L_{eq}$.

Pumping Lemma
Suppose $L_{eq}$ is regular. Let $p$ be pumping length for $L_{eq}$. Consider $w = 0^p1^p$. There exist $x, y, z$ such that

- $w = xyz$, $|xy| \leq p$, $|y| > 0$: so for some $r, s, t$, $x = 0^r$, $y = 0^s$ and $z = 0^t1^p$, with $s > 0$.

- $xy^iz \in L_{eq}$ for all $i$: so $xy^0z \in L_{eq}$.

- But $xy^0z = 0^{p-s}1^p \notin L_{eq}$.
Example III

Proposition

\[ L_p = \{0^i \mid i \text{ prime}\} \text{ is not regular} \]
Example III

Proposition
$L_p = \{0^i \mid i \text{ prime}\} \text{ is not regular}$

Proof.
Suppose $L_p$ is regular. Let $p$ be the pumping length for $L_p$. Consider $w = 0^m$, where $m \geq p + 2$ and $m$ is prime. Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all $i$. Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$. $xy^{r+t}z \neq 0^{r(0^s)(r+t)}0^t = 0^{r+s(r+t)}+t$. Now $r+s(r+t) + t = (r+t)(s+1)$. Further $m = r+s+t \geq p+2$ and $s > 0$ mean that $t \geq 2$ and $s+1 \geq 2$. Thus, $xy^{r+t}z \not\in L_p$. Contradiction! □
Example III

Proposition
$L_p = \{0^i \mid i \text{ prime}\}$ is not regular

Proof.
Suppose $L_p$ is regular. Let $p$ be the pumping length for $L_p$.

- Consider $w = 0^m$, where $m \geq p + 2$ and $m$ is prime.
Example III

Proposition

$L_p = \{0^i \mid i \text{ prime}\}$ is not regular

Proof.

Suppose $L_p$ is regular. Let $p$ be the pumping length for $L_p$.

- Consider $w = 0^m$, where $m \geq p + 2$ and $m$ is prime.
- Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all $i$. 

Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$.

$\boxed{\text{Contradiction!} \; \square}$
Example III

Proposition
$L_p = \{0^i \mid i \text{ prime}\}$ is not regular

Proof.
Suppose $L_p$ is regular. Let $p$ be the pumping length for $L_p$.

- Consider $w = 0^m$, where $m \geq p + 2$ and $m$ is prime.
- Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all $i$.
- Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$. 
Example III

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\[ L_p = \{0^i \mid i \text{ prime}\} \text{ is not regular} \]

Proof.
Suppose \( L_p \) is regular. Let \( p \) be the pumping length for \( L_p \).

- Consider \( w = 0^m \), where \( m \geq p + 2 \) and \( m \) is prime.
- Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_p \), for all \( i \).
- Thus, \( x = 0^r \), \( y = 0^s \) and \( z = 0^t \). Further, as \( |y| > 0 \), we have \( s > 0 \). \( xy^{r+t}z = 0^r(0^s)^{(r+t)}0^t = 0^{r+s(r+t)+t} \).
Example III

Proposition

\[ L_p = \{0^i \mid i \text{ prime}\} \text{ is not regular} \]

Proof.

Suppose \( L_p \) is regular. Let \( p \) be the pumping length for \( L_p \).

- Consider \( w = 0^m \), where \( m \geq p + 2 \) and \( m \) is prime.
- Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^i z \in L_p \), for all \( i \).
- Thus, \( x = 0^r \), \( y = 0^s \) and \( z = 0^t \). Further, as \( |y| > 0 \), we have \( s > 0 \). \( xy^{r+t}z = 0^r (0^s)^{r+t}0^t = 0^{r+s(r+t)+t} \). Now \( r + s(r + t) + t = (r + t)(s + 1) \). Further \( m = r + s + t \geq p + 2 \) and \( s > 0 \) mean that \( t \geq 2 \) and \( s + 1 \geq 2 \). Thus, \( xy^{r+t}z \not\in L_p \). Contradiction! \( \square \)
Example IV

Question
Is $L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?
Example IV

Question
Is \( L_{xx} = \{ xx \mid x \in \{0, 1\}^* \} \) is regular?

Suppose \( L_{xx} \) is regular, and let \( p \) be the pumping length of \( L_{xx} \).
Question

Is $L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?

Suppose $L_{xx}$ is regular, and let $p$ be the pumping length of $L_{xx}$.

- Consider $w = 0^p0^p \in L$.
Question

Is \( L_{xx} = \{xx \mid x \in \{0, 1\}^*\} \) is regular?

Suppose \( L_{xx} \) is regular, and let \( p \) be the pumping length of \( L_{xx} \).

- Consider \( w = 0^p0^p \in L \).
- Can we find substrings \( x, y, z \) satisfying the conditions in the pumping lemma?
  
  Yes! Consider \( x = \epsilon, y = 00, z = 0^{2p-2} \).
  
  Does this mean \( L_{xx} \) satisfies the pumping lemma? Does it mean it is regular?

  No! We have chosen a bad \( w \). To prove that the pumping lemma is violated, we only need to exhibit some \( w \) that cannot be pumped.

  Another bad choice (01).
Example IV

Question
Is $L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?

Suppose $L_{xx}$ is regular, and let $p$ be the pumping length of $L_{xx}$.

- Consider $w = 0^p0^p \in L$.
- Can we find substrings $x, y, z$ satisfying the conditions in the pumping lemma? Yes! Consider $x = \epsilon, y = 00, z = 0^{2p-2}$.
Example IV

Question
Is $L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?

Suppose $L_{xx}$ is regular, and let $p$ be the pumping length of $L_{xx}$.

▶ Consider $w = 0^p0^p \in L$.

▶ Can we find substrings $x, y, z$ satisfying the conditions in the pumping lemma? Yes! Consider $x = \epsilon, y = 00, z = 0^{2p-2}$.

▶ Does this mean $L_{xx}$ satisfies the pumping lemma? Does it mean it is regular?
Example IV

**Question**

Is $L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?

Suppose $L_{xx}$ is regular, and let $p$ be the pumping length of $L_{xx}$.

- Consider $w = 0^p0^p \in L$.
- Can we find substrings $x, y, z$ satisfying the conditions in the pumping lemma? Yes! Consider $x = \epsilon, y = 00, z = 0^{2p-2}$.
- Does this mean $L_{xx}$ satisfies the pumping lemma? Does it mean it is regular?
  - No! We have chosen a bad $w$. To prove that the pumping lemma is violated, we only need to exhibit some $w$ that cannot be pumped.
Question

Is \( L_{xx} = \{xx \mid x \in \{0, 1\}^*\} \) is regular?

Suppose \( L_{xx} \) is regular, and let \( p \) be the pumping length of \( L_{xx} \).

- Consider \( w = 0^p0^p \in L \).
- Can we find substrings \( x, y, z \) satisfying the conditions in the pumping lemma? Yes! Consider \( x = \epsilon, y = 00, z = 0^{2p-2} \).
- Does this mean \( L_{xx} \) satisfies the pumping lemma? Does it mean it is regular?
  - No! We have chosen a bad \( w \). To prove that the pumping lemma is violated, we only need to exhibit some \( w \) that cannot be pumped.
- Another bad choice \( (01)^p(01)^p \).
Example IV
Reloaded

Proposition

\[ L_{xx} = \{ xx \mid x \in \{0, 1\}^* \} \text{ is not regular.} \]
Example IV
Reloaded

**Proposition**

\[ L_{xx} = \{ xx \mid x \in \{0, 1\}^* \} \text{ is not regular.} \]

**Proof.**

Suppose \( L_{xx} \) is regular. Let \( p \) be the pumping length for \( L_{xx} \).
Example IV
Reloaded

**Proposition**

$L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is not regular.

**Proof.**

Suppose $L_{xx}$ is regular. Let $p$ be the pumping length for $L_{xx}$.

- Consider $w = 0^p10^p1$. 

Since $r + t < p$, $xy0z \not\in L_{xx}$. Contradiction!

□
Example IV
Reloaded

Proposition

\[ L_{xx} = \{ xx \mid x \in \{0, 1\}^* \} \text{ is not regular.} \]

Proof.

Suppose \( L_{xx} \) is regular. Let \( p \) be the pumping length for \( L_{xx} \).

\begin{itemize}
  \item Consider \( w = 0^p10^p1 \).
  \item Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_p \), for all \( i \).
\end{itemize}
Example IV
Reloaded

Proposition
\( L_{xx} = \{xx \mid x \in \{0, 1\}^*\} \) is not regular.

Proof.
Suppose \( L_{xx} \) is regular. Let \( p \) be the pumping length for \( L_{xx} \).

- Consider \( w = 0^p10^p1 \).
- Since \( |w| > p \), there are \( x, y, z \) such that \( w = xyz \), \( |xy| \leq p \), \( |y| > 0 \), and \( xy^iz \in L_p \), for all \( i \).
- Since \( |xy| \leq p \), \( x = 0^r \), \( y = 0^s \) and \( z = 0^t10^p1 \). Further, as \( |y| > 0 \), we have \( s > 0 \).
Example IV
Reloaded

Proposition
\[ L_{xx} = \{xx \mid x \in \{0, 1\}^*\} \text{ is not regular.} \]

Proof.
Suppose \( L_{xx} \) is regular. Let \( p \) be the pumping length for \( L_{xx} \).

- Consider \( w = 0^p10^p1 \).
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- Since \( |xy| \leq p \), \( x = 0^r \), \( y = 0^s \) and \( z = 0^t10^p1 \). Further, as \( |y| > 0 \), we have \( s > 0 \).

\[ xy^0z = 0^r0^t10^p1 = 0^r+t10^p1 \]
Example IV

Reloaded

Proposition

$L_{xx} = \{xx \mid x \in \{0, 1\}^*\}$ is not regular.

Proof.

Suppose $L_{xx}$ is regular. Let $p$ be the pumping length for $L_{xx}$.

- Consider $w = 0^p10^p1$.
- Since $|w| > p$, there are $x, y, z$ such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all $i$.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t10^p1$. Further, as $|y| > 0$, we have $s > 0$.

$$xy^0z = 0^r0^t10^p1 = 0^{r+t}10^p1$$

Since $r + t < p$, $xy^0z \not\in L_{xx}$. Contradiction! □
Lessons on Expressivity

Limits of Finite Memory

Finite automata cannot

- “keep track of counts”: e.g., $L_{0n1n}$ not regular.
- “compare far apart pieces” of the input: e.g. $L_{xx}$ not regular.
- do “computations that require it to look at global properties” of the input. e.g. $L_{prime}$ not regular.
Lessons on Expressivity

Limits of Finite Memory

Finite automata cannot

▶ “keep track of counts”: e.g., $L_{0n1n}$ not regular.
▶ “compare far apart pieces” of the input: e.g. $L_{xx}$ not regular.
▶ do “computations that require it to look at global properties” of the input. e.g. $L_{prime}$ not regular.

...and pumping lemma provides one way to find out some of these limitations.