Regular Expressions and Regular Languages

Why do they have such similar names?

Theorem

$L$ is a regular language if and only if there is a regular expression $R$ such that $L(R) = L$.

That is, regular expressions have the same "expressive power" as finite automata.

Proof.

Given regular expression $R$, can construct NFA $N$ such that $L(N) = L(R)$.

Given DFA $M$, will construct regular expression $R$ such that $L(M) = L(R)$.

$\square$
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DFA to Regular Expression

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DFA to Regular Expression

- Given DFA $M$, will construct regular expression $R$ such that $L(M) = L(R)$. In two steps:
  - Construct a “Generalized NFA” (GNFA) $G$ from the DFA $M$
  - And then convert $G$ to a regex $R$
Generalized NFA

- A GNFA is similar to an NFA, but:

  - There is a single accept state.
  - The start state has no incoming transitions, and the accept state has no outgoing transitions.
  - These are "cosmetic changes": Any NFA can be converted to an equivalent NFA of this kind.
  - The transitions are labeled not by characters in the alphabet, but by regular expressions.
  - For every pair of states \((q_1, q_2)\), the transition from \(q_1\) to \(q_2\) is labeled by a regular expression \(\rho(q_1, q_2)\).

- "Generalized NFA" because a normal NFA has transitions labeled by \(\epsilon\), elements in \(\Sigma\) (a union of elements, if multiple edges between a pair of states) and \(\emptyset\) (missing edges).
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- Transition: GNFA non-deterministically reads a block of characters from the input, chooses an edge from the current state $q_1$ to another state $q_2$, and if the block of symbols matches the regex $\rho(q_1, q_2)$, then moves to $q_2$. 

- Acceptance: $G$ accepts $w$ if there exists some sequence of valid transitions such that on starting from the start state, and after finishing the entire input, $G$ is in the accept state.
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Generalized NFA: Example

Example GNFA $G$

Accepting run of $G$ on 11110100 is
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Accepting run of $G$ on 11110100 is

$q_0 \xrightarrow{1} G q_1$
Generalized NFA: Example

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Accepting run of $G$ on 11110100 is
$q_0 \xrightarrow{1} G q_1 \xrightarrow{11} G q_1$
Generalized NFA: Example

Example GNFA $G$

Accepting run of $G$ on $11110100$ is

$q_0 \xrightarrow{1} G q_1 \xrightarrow{11} G q_1 \xrightarrow{101} G q_1$
Generalized NFA: Example

Example GNFA $G$

Accepting run of $G$ on 11110100 is

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Definition
A generalized nondeterministic finite automaton (GNFA) is $G = (Q, \Sigma, q_0, q_F, \rho)$, where
- $Q$ is the finite set of states
- $\Sigma$ is the finite alphabet
- $q_0 \in Q$ initial state
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- $Q$ is the finite set of states
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- $q_0 \in Q$ initial state
- $q_F \in Q$, a single accepting state
- $\rho : (Q \setminus \{q_F\}) \times (Q \setminus \{q_0\}) \rightarrow \mathcal{R}_\Sigma$, where $\mathcal{R}_\Sigma$ is the set of all regular expressions over the alphabet $\Sigma$
Generalized NFA: Definition

Definition
For a GNFA $M = (Q, \Sigma, q_0, q_F, \rho)$ and string $w \in \Sigma^*$, we say $M$ accepts $w$ iff there exist $x_1, \ldots, x_t \in \Sigma^*$ and states $r_0, \ldots, r_t$ such that
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2. $r_0 = q_0$ and $r_t = q_F$
3. for each $i \in [1, t]$, $x_i \in L(\rho(r_{i-1}, r_i))$. 
Converting DFA to GNFA

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ can be easily converted to an equivalent GNFA $G = (Q', \Sigma, q'_0, q'_F, \rho)$:

- $Q' = Q \cup \{q'_0, q'_F\}$ where $Q \cap \{q'_0, q'_F\} = \emptyset$
- $\rho(q_1, q_2) = \begin{cases} \epsilon, & \text{if } q_1 = q'_0 \text{ and } q_2 = q_0 \\ \epsilon, & \text{if } q_1 \in F \text{ and } q_2 = q'_F \cup \{a | \delta(q_1, a) = q_2\} \\ \text{otherwise} \end{cases}$

Prove: $L(G) = L(M)$. 
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![Diagram showing the conversion process from DFA to GNFA]
Converting DFA to GNFA

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ can be easily converted to an equivalent GNFA $G = (Q', \Sigma, q_0', q_F', \rho)$:

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- $\rho(q_1, q_2) = \begin{cases} 
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\end{cases}$

Prove: $L(G) = L(M)$. 
Suppose $G$ is a GNFA with only two states, $q_0$ and $q_F$. Then $L(R) = L(G)$ where $R = \rho(q_0, q_F)$.

How about $G$ with three states?

Plan: Reduce any GNFA $G$ with $k > 2$ states to an equivalent GFA with $k - 1$ states.
GNFA to Regex

- Suppose $G$ is a GNFA with only two states, $q_0$ and $q_F$. 

- How about $G$ with three states:

  $q_0 \xrightarrow{R_4} q_F \cup q_0 \xrightarrow{R_1} q_F \cup q_0 \xrightarrow{R_3} q_F$

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![Diagram of a GNFA with three states]
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Plan: Reduce any GNFA $G$ with $k > 2$ states to an equivalent GFA with $k - 1$ states.
Definition (Deleting a GNFA State)

Given GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, and any state $q^* \in Q \setminus \{q_0, q_F\}$, define GNFA $\text{rip}(G, q^*) = (Q', \Sigma, q_0, q_F, \rho')$ as follows:

1. $Q' = Q \setminus \{q^*\}$.
2. For any $(q_1, q_2) \in Q' \setminus \{q_F\} \times Q' \setminus \{q_0\}$ (possibly $q_1 = q_2$), let $\rho'(q_1, q_2) = (R_1 R_2 R_3) \cup R_4$, where $R_1 = \rho(q_1, q^*)$, $R_2 = \rho(q^*, q^*)$, $R_3 = \rho(q^*, q_2)$ and $R_4 = \rho(q_1, q_2)$. 

Claim. For any $q^* \in Q \setminus \{q_0, q_F\}$, $G$ and $\text{rip}(G, q^*)$ are equivalent.
GNFA to Regex: From $k$ states to $k - 1$ states

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   $$\rho'(q_1, q_2) = (R_1 R_2^* R_3) \cup R_4,$$

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GNFA to Regex: From $k$ states to $k - 1$ states

\( w \in L(G) \implies w \in L(G') \)

Proof.
GNFA to Regex: From $k$ states to $k - 1$ states

$w \in L(G) \implies w \in L(G')$

Proof.

- $w \in L(G) \implies w = x_1x_2x_3 \cdots x_t$, and a sequence of states $q_0 = r_0, r_1, \ldots, r_t = q_F$ s.t. $x_i \in L(\rho(r_{i-1}, r_i))$.
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Let $(q_0 = s_0, \ldots, s_d = q_F)$ be the subsequence of states obtained by deleting all occurrences of $q^*$. 


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- Let \( (q_0 = s_0, \ldots, s_d = q_F) \) be the subsequence of states obtained by deleting all occurrences of \( q^* \).
- For any run of \( q^* \) — i.e., an interval \([a, b]\) s.t. \( r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b \) — let \( x_{[a,b]} = x_a \cdots x_b \).
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- If $s_{j-1} = r_{a-1}$ and $s_j = r_b$, then $x_{[a,b]} \in L(\rho'(s_{j-1}, s_j))$.
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- If \( s_{j-1} = r_{a-1} \) and \( s_j = r_b \), then \( x_{[a,b]} \in L(\rho'(s_{j-1}, s_j)) \).
  - Let \( R_1 = \rho(s_{j-1}, q^*) \), \( R_2 = \rho(q^*, q^*) \), \( R_3 = \rho(q^*, s_j) \) and \( R_4 = \rho(s_{j-1}, s_j) \). Then \( \rho'(s_{j-1}, s_j) = R_4 \cup (R_1 R_2^* R_3) \).
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- Let $(q_0 = s_0, \ldots, s_d = q_F)$ be the subsequence of states obtained by deleting all occurrences of $q^*$.
- For any run of $q^* \text{ — i.e., an interval } [a, b] \text{ s.t. } r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b \text{ — let } x_{[a, b]} = x_a \cdots x_b$.
- If $s_{j-1} = r_{a-1}$ and $s_j = r_b$, then $x_{[a, b]} \in L(\rho'(s_{j-1}, s_j))$
  - Let $R_1 = \rho(s_{j-1}, q^*), R_2 = \rho(q^*, q^*), R_3 = \rho(q^*, s_j)$ and $R_4 = \rho(s_{j-1}, s_j)$. Then $\rho'(s_{j-1}, s_j) = R_4 \cup (R_1 R_2^* R_3)$.
  - Case $a = b$. $(s_{j-1}, s_j) = (r_{b-1}, r_b)$ and $x_{[a, b]} = x_b \in L(R_4)$. 

··→
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Let \( (q_0 = s_0, \ldots, s_d = q_F) \) be the subsequence of states obtained by deleting all occurrences of \( q^* \).

For any run of \( q^* \) — i.e., an interval \([a, b]\) s.t. \( r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b \) — let \( x_{[a,b]} = x_a \cdots x_b \).

If \( s_{j-1} = r_{a-1} \) and \( s_j = r_b \), then \( x_{[a,b]} \in L(\rho'(s_{j-1}, s_j)) \)

Let \( R_1 = \rho(s_{j-1}, q^*), R_2 = \rho(q^*, q^*), R_3 = \rho(q^*, s_j) \) and \( R_4 = \rho(s_{j-1}, s_j) \). Then \( \rho'(s_{j-1}, s_j) = R_4 \cup (R_1 R_2^* R_3) \).

Case \( a = b \). \((s_{j-1}, s_j) = (r_{b-1}, r_b)\) and \( x_{[a,b]} = x_b \in L(R_4) \).

Case \( a = b + 1 + u \). \( x_a \in L(R_1), x_{a+1}, \ldots, x_{b-1} \in L(R_2) \) and \( x_b \in L(R_3) \). So \( x_{[a,b]} \in L(R_1 R_2^u R_3) \).
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- Let $(q_0 = s_0, \ldots, s_d = q_F)$ be the subsequence of states obtained by deleting all occurrences of $q^*$.
- For any run of $q^*$ — i.e., an interval $[a, b]$ s.t. $r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b$ — let $x_{[a,b]} = x_a \cdots x_b$.
- If $s_{j-1} = r_{a-1}$ and $s_j = r_b$, then $x_{[a,b]} \in L(\rho'(s_{j-1}, s_j))$
- Let $y_1, \ldots, y_d$ be the sequence of blocks of the form $x_{[a,b]}$. 
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Proof.

- $w \in L(G) \implies w = x_1x_2x_3 \cdots x_t$, and a sequence of states $q_0 = r_0, r_1, \ldots, r_t = q_F$ s.t. $x_i \in L(\rho(r_{i-1}, r_i))$.
- Let $(q_0 = s_0, \ldots, s_d = q_F)$ be the subsequence of states obtained by deleting all occurrences of $q^*$.
- For any run of $q^*$ — i.e., an interval $[a, b]$ s.t. $r_{a-1} \neq q^* = r_a = \ldots = r_{b-1} \neq r_b$ — let $x_{[a,b]} = x_a \cdots x_b$.
- If $s_{j-1} = r_{a-1}$ and $s_j = r_b$, then $x_{[a,b]} \in L(\rho'(s_{j-1}, s_j))$.
- Let $y_1, \ldots, y_d$ be the sequence of blocks of the form $x_{[a,b]}$.
- Then $w = y_1 \cdots y_d$ and $y_j \in L(\rho'(s_{j-1}, s_j))$. 
GNFA to Regex: From $k$ states to $k - 1$ states

$w \in L(G) \implies w \in L(G')$

Proof.

$\triangleright$ $w \in L(G) \implies w = x_1x_2x_3 \cdots x_t$, and a sequence of states $q_0 = r_0, r_1, \ldots, r_t = q_F$ s.t. $x_i \in L(\rho(r_{i-1}, r_i))$.

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$\triangleright$ Let $y_1, \ldots, y_d$ be the sequence of blocks of the form $x_{[a,b]}$.

$\triangleright$ Then $w = y_1 \cdots y_d$ and $y_j \in L(\rho'(s_{j-1}, s_j))$.

i.e., $w \in L(G) \implies w \in L(G')$. 

\(\triangleright\)
GNFA to Regex: From $k$ states to $k-1$ states

$w \in L(G') \implies w \in L(G)$

Proof (contd).
GNFA to Regex: From $k$ states to $k-1$ states

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GNFA to Regex: From \( k \) states to \( k - 1 \) states

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Proof (contd).

\[
\begin{align*}
\text{Case } y_j \in L(R_4). & \text{ Retain the block } y_j \text{ and retain } s_{j-1} \text{ and } s_j \text{ as adjacent states.} \\
\text{Case } y_j \in L(R_1 R^*_2 R_3). & \text{ } y_j = z_0 \cdots z_u + 1 \text{ where } z_0 \in L(R_1), z_1, ..., z_u \in L(R_2) \text{ and } z_{u+1} = L(R_3) \text{ (for some finite } u). \text{ Insert } u + 1 \text{ copies of } q^* \text{ between } s_{j-1} \text{ and } s_j. \text{ Divide } y_j \text{ into } u + 2 \text{ blocks } (z_0, ..., z_u+1). 
\end{align*}
\]
GNFA to Regex: From $k$ states to $k - 1$ states

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Proof (contd).

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(See notes for a formal argument.)
Proof (contd).

- \( w \in L(G') \implies w = y_1 \cdots y_d \) and a sequence of states 
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- To build a sequence of blocks \( x_1, \ldots, x_t \) and a sequence of states 
  \( q_0 = r_0, \ldots, r_t = q_F \) to show \( w \in L(G): \)
GNFA to Regex: From \(k\) states to \(k - 1\) states

\(w \in L(G') \implies w \in L(G)\)

Proof (contd).

\(\begin{align*}
&w \in L(G') \implies w = y_1 \cdots y_d \text{ and a sequence of states} \\
&q_0 = s_0, \ldots, s_d = q_F \text{ s.t. } y_j \in L(\rho'(s_{j-1}, s_j)) = \\
&L(\rho(s_{j-1}, q^*)\rho(q^*, q^*)^* \rho(q^*, r_i)) \cup \rho(s_{j-1}, s_j)) = \\
&L(R_1R_2^*R_3) \cup L(R_4). \\
\end{align*}\)

To build a sequence of blocks \(x_1, \ldots, x_t\) and a sequence of states \(q_0 = r_0, \ldots, r_t = q_F\) to show \(w \in L(G)\):

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&q_0 = s_0, \ldots, s_d = q_F \text{ s.t. } y_j \in L(\rho'(s_{j-1}, s_j)) = \\
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Proof (contd).

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      $z_1, \ldots, z_u \in L(R_2)$ and $z_{u+1} = L(R_3)$ (for some finite $u$). Insert
      $u + 1$ copies of $q^*$ between $s_{j-1}$ and $s_j$. Divide $y_j$ into $u + 2$
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GNFA to Regex: From \( k \) states to \( k - 1 \) states

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(See notes for a formal argument.)
DFA to Regex: Summary

**Lemma**

*For every DFA $M$, there is a regular expression $R$ such that $L(M) = L(R)$.***
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- So given $G$, by applying rip repeatedly (choosing $q^*$ arbitrarily each time), we can get a GNFA $G'$ with two states s.t. $L(G) = L(G')$. 
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For every DFA $M$, there is a regular expression $R$ such that $L(M) = L(R)$.

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- For any GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, for any $q^* \in Q \setminus \{q_0, q_F\}$, $G$ and $\text{rip}(G, q^*)$ are equivalent. $\text{rip}(G, q^*)$ has one fewer state than $G$.
- So given $G$, by applying rip repeatedly (choosing $q^*$ arbitrarily each time), we can get a GNFA $G'$ with two states s.t. $L(G) = L(G')$. Formally, by induction on the number of states in $G$.
- For a 2-state GNFA $G'$, $L(G') = L(R)$, where $R = \rho(q_0, q_F)$. 
DFA to Regex: Example

```
q1 -> 0 -> q1
     |        |
     1      1
     |        |
q2 -> 0 -> q2
```
DFA to Regex: Example
DFA to Regex: Example
DFA to Regex: Example
DFA to Regex: Example

\[ q_0 \xrightarrow{\epsilon 0^* 1} q_2 \]

\[ q_2 \rightarrow 0 \cup (10^* 1) \]

\[ q_2 \xrightarrow{\epsilon} q_F \]
DFA to Regex: Example

\[ q_0 \rightarrow q_2 \rightarrow q_F \]

\[ 0 \cup (10^*1) \]

\[ \epsilon 0^*1 \rightarrow \epsilon \]
DFA to Regex: Example

$q_0 \xrightarrow{0^*1(0 \cup (10^*)^*)} q_F$