CSE 135: Introduction to Theory of Computation
Regular Expressions

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- A simple but powerful collection of operations:
  - Union, Concatenation and Kleene Closure

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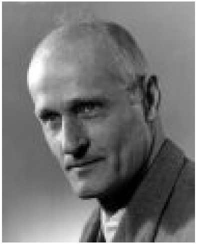
Regular Expressions

A Simple Programming Language



Stephen Cole Kleene

A regular expression is a formula for representing a (complex) language in terms of "elementary" languages combined using the three operations union, concatenation and Kleene closure.

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Also will sometimes omit $\circ$: e.g. will write RS instead of $R \circ S$

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For any regex R, there is an NFA N_R s.t. $L(N_R) = L(R)$.

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We will build the NFA N_R for R, inductively, based on the number of operators in R, $\#(R)$.

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- Induction Step: Consider R with $\#(R) = n + 1$. Based on the form of R, the NFA N_R will be built using the induction hypothesis.

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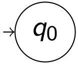

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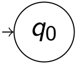

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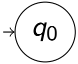

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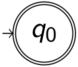

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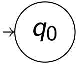

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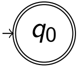

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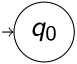

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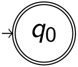

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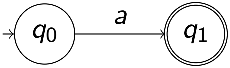

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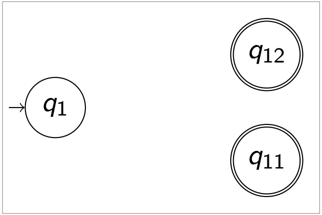

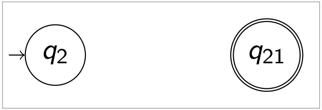

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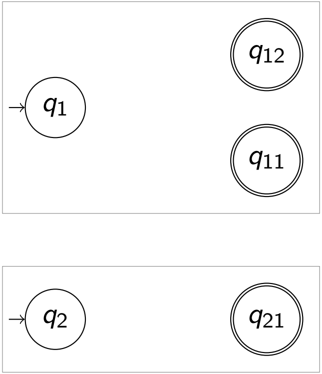

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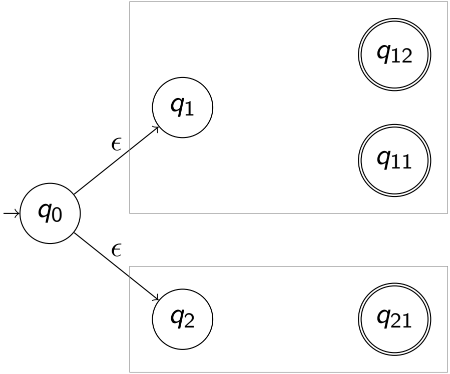

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Formal Definition

Case $R = R_1 \cup R_2$

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ (with $Q_1 \cap Q_2 = \emptyset$) such that $L(N_1) = L(R_1)$ and $L(N_2) = L(R_2)$. The NFA $N = (Q, \Sigma, \delta, q_0, F)$ is given by

- $Q = Q_1 \cup Q_2 \cup \{q_0\}$, where $q_0 \notin Q_1 \cup Q_2$
- $F = F_1 \cup F_2$
- δ is defined as follows

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \\ \delta_2(q, a) & \text{if } q \in Q_2 \\ \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

Induction Step: Union

Correctness Proof

Need to show that $w \in L(N)$ iff $w \in L(N_1) \cup L(N_2)$.

$\Rightarrow$ $w \in L(N)$ implies $q_0 \xrightarrow{w}_N q$ for some $q \in F$.

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- $\Leftarrow$ $w \in L(N_1) \cup L(N_2)$. Consider $w \in L(N_1)$; case of $w \in L(N_2)$ is similar.

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Induction Step: Union

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Induction Step: Concatenation

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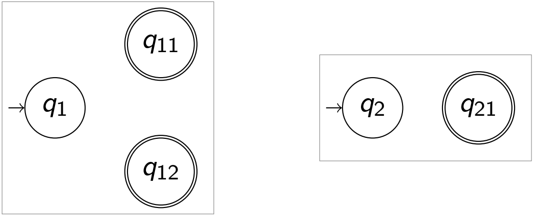

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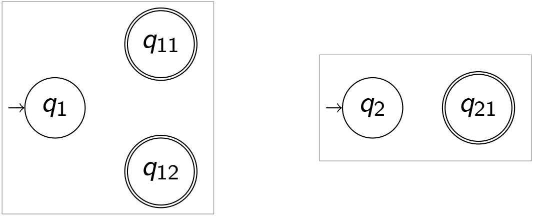

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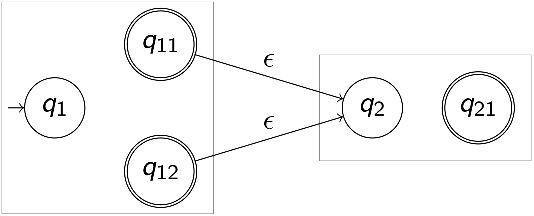

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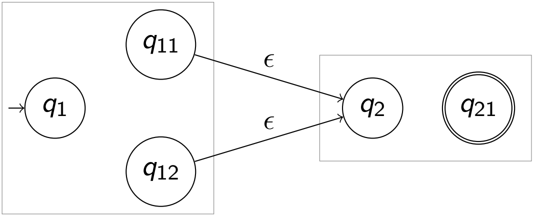

Formal definition and proof of correctness left as exercise.

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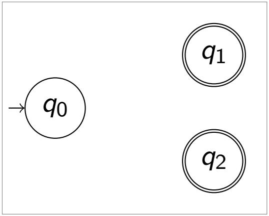

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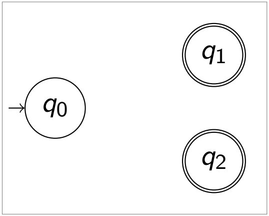

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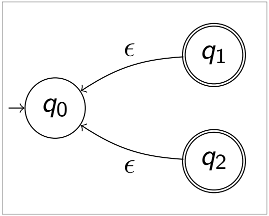

Induction Step: Kleene Closure

First Attempt

Case $R = R_1^*$

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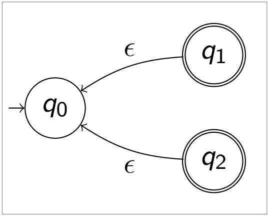

Problem: May not accept ϵ! One can show that $L(N) = (L(N_1))^+$.

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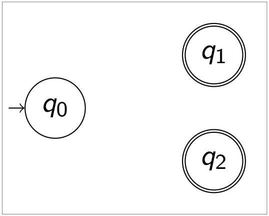

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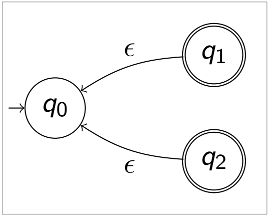

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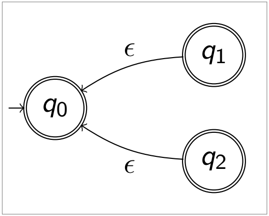

Induction Step: Kleene Closure

Second Attempt

Case $R = R_1^*$

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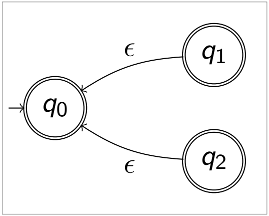
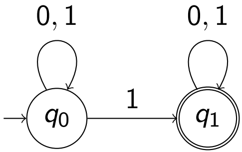


Problem: May accept strings that are not in $(L(N_1))^*$!

Example demonstrating the problem



Example NFA N

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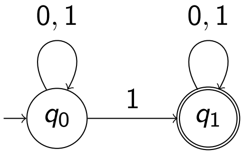

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$$L(N) = (0 \cup 1)^* 1 (0 \cup 1)^*.$$

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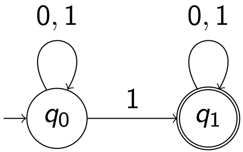

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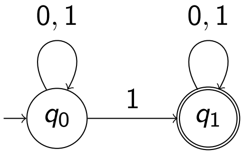

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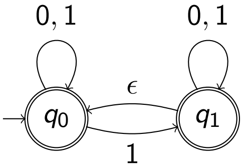

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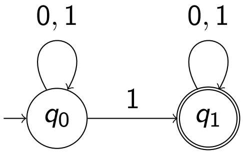

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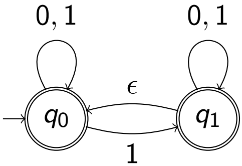

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$L(N) = (0 \cup 1)^*1(0 \cup 1)^*$. Thus, $(L(N))^* = \epsilon \cup (0 \cup 1)^*1(0 \cup 1)^*$.
The previous construction, gives an NFA that accepts $0 \notin (L(N))^*$!

Induction Step: Kleene Closure

Correct Construction

Case $R = R_1^*$

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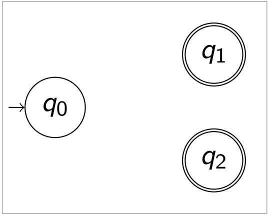

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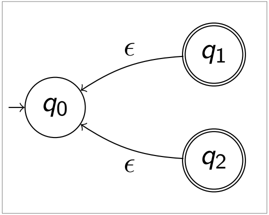

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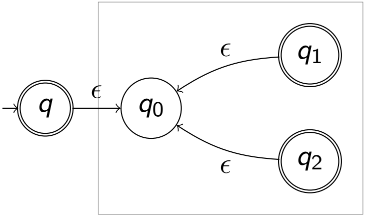

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Regular Expressions to NFA

To Summarize

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- When R was an elementary regular expression, we gave an explicit construction of an NFA recognizing $L(R)$
- When $R = R_1 \operatorname{op} R_2$ (or $R = \operatorname{op}(R_1)$), we constructed an NFA N for R, using the NFAs for R_1 and R_2.

Regular Expressions to NFA

An Example

Build NFA for $(1 \cup 01)^*$

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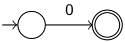

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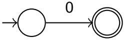
N_0 

N_1

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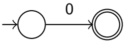

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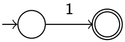

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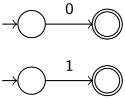

N_{01}

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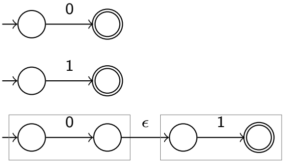

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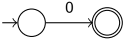

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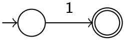

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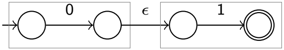

$N_{1\cup 01}$

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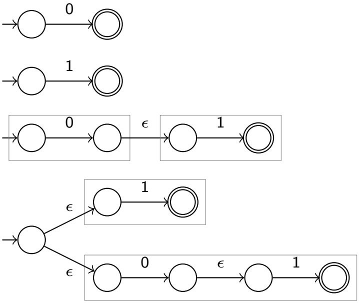
N_{01}

$N_{1 \cup 01}$



Example Continued

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$N_{(1\cup 01)^*}$

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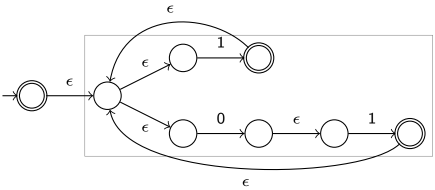

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  - Coming up: Regular languages can be represented by regular expressions (by building regex for any given DFA).