Example II

The NFA "guesses" at the beginning whether it will see a multiple of 2 or 3, and then confirms that the guess was correct.
Example II

NFA accepting strings of 0s where the length is either a multiple 2 or 3

The NFA “guesses” at the beginning whether it will see a multiple of 2 or 3, and then confirms that the guess was correct.
Example III

NFA accepting strings with 001 as substring

At some point the NFA "guesses" that the pattern 001 is starting and then checks to confirm the guess.
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NFA accepting strings with 001 as substring

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Nondeterministic Finite Automata (NFA)

Definition

A nondeterministic finite automaton (NFA) is $M = (Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is the finite set of states
- $\Sigma$ is the finite alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow P(Q)$, where $P(Q)$ is the powerset of $Q$
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states
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Formal Definition

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- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the powerset of $Q$
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Example of NFA

\[ M_0 = (\{q_\epsilon, q_0, q_{00}, q_p\}, \{0, 1\}, \delta, q_\epsilon, \{q_p\}) \]

where

\[ \delta(q_\epsilon, 0) = \{q_\epsilon, q_0\} \]
\[ \delta(q_\epsilon, 1) = \{q_\epsilon\} \]
\[ \delta(q_0, 0) = \{q_{00}\} \]
\[ \delta(q_0, 1) = \{q_p\} \]
\[ \delta(q_{00}, 0) = \{q_p\} \]
\[ \delta(q_{00}, 1) = \{q_p\} \]
\[ \delta(q_p, 0) = \{q_p\} \]
\[ \delta(q_p, 1) = \{q_p\} \]

is \emptyset in all other cases.
Example of NFA

Formally, the NFA is $M_{001} = (\{q_\epsilon, q_0, q_{00}, q_p\}, \{0, 1\}, \delta, q_\epsilon, \{q_p\})$ where $\delta$ is given by

\[
\begin{align*}
\delta(q_\epsilon, 0) &= \{q_\epsilon, q_0\} & \delta(q_\epsilon, 1) &= \{q_\epsilon\} & \delta(q_0, 0) &= \{q_{00}\} \\
\delta(q_{00}, 1) &= \{q_p\} & \delta(q_p, 0) &= \{q_p\} & \delta(q_p, 1) &= \{q_p\}
\end{align*}
\]

$\delta$ is $\emptyset$ in all other cases.
Computation

Definition
For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string $w$, and states $q_1, q_2 \in Q$, we say $q_1 \xrightarrow{w} M q_2$ if there is one thread of computation on input $w$ from state $q_1$ that ends in $q_2$. 
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- $r_0 = q_1$,
- for each $i$, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = q_2$, and
- $w = x_1x_2x_3\cdots x_k$
Example Computation

$q_\epsilon \xrightarrow{0100} q_p$ because taking $r_0 = q_\epsilon$, $r_1 = q_0$, $r_2 = q_{00}$, $r_3 = q_p$, $r_4 = q_p$, $r_5 = q_p$, and $x_1 = 0$, $x_2 = \epsilon$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, we have

- $x_1 x_2 \cdots x_5 = 0\epsilon100 = 0100$
- $r_{i+1} \in \delta(r_i, x_{i+1})$
Defining $\hat{\Delta}$

**Definition**
For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string $w$, and state $q_1 \in Q$, we say $\hat{\Delta}(q_1, w)$ to denote states of all the active threads of computation on input $w$ from $q_1$. 

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$$\hat{\Delta}(q_1, w) = \{ q \in Q \mid q_1 \xrightarrow{w} M q \}$$
Example

Example NFA

\( q, \epsilon \)

\( q_0 \)

\( q_{00} \)

\( q_p \)

0, 1

0, 0

0, 1

0, 1

Computation on 0100
Example

Example NFA

Computation on 0100
Example NFA

\[ \hat{\Delta}(q_\epsilon, 0100) = \{ q_0, q_p, q_{00}, q_\epsilon \} \]

Computation on 0100
Definition
For an NFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say $M$ accepts $w$ iff $\hat{\Delta}(q_0, w) \cap F \neq \emptyset$. 
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For an NFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say $M$ accepts $w$ iff $\hat{\Delta}(q_0, w) \cap F \neq \emptyset$.

Definition
The language accepted or recognized by NFA $M$ over alphabet $\Sigma$ is $L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$.
Acceptance/Recognition

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For an NFA \( M = (Q, \Sigma, \delta, q_0, F) \) and string \( w \in \Sigma^* \), we say \( M \) accepts \( w \) iff \( \hat{\Delta}(q_0, w) \cap F \neq \emptyset \).

Definition

The language accepted or recognized by NFA \( M \) over alphabet \( \Sigma \) is \( L(M) = \{ w \in \Sigma^* \mid M \) accepts \( w \} \). A language \( L \) is said to be accepted/recognized by \( M \) if \( L = L(M) \).
Observations about NFAs

Observation 1
For NFA $M$, string $w$ and state $q_1$ it could be that

\[ \hat{\delta}(q_1, w) = \emptyset \]
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Observation 2
However, the following proposition about DFAs continues to hold for NFAs

- For NFA $M$, strings $u$ and $v$, and state $q$,

$$
\hat{\Delta}(q, uv) = \bigcup_{q' \in \hat{\Delta}(q, u)} \hat{\Delta}(q', v)
$$
Using Nondeterminism

When designing an NFA for a language
Using Nondeterminism

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- You follow the same methodology as for DFAs, like identifying what needs to be remembered
Using Nondeterminism

When designing an NFA for a language
- You follow the same methodology as for DFAs, like identifying what needs to be remembered
- But now, the machine can “guess” at certain steps
Problem
For $\Sigma = \{0, 1, 2, \#\}$, let

$$L = \{w\#c \mid w \in \{0, 1, 2\}^*, c \in \{0, 1, 2\}, \text{ and } c \text{ occurs in } w\}$$

So $1011\#0 \in L$ but $1011\#2 \not\in L$. Design an NFA recognizing $L$. 
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Solution

- Read symbols of $w$, i.e., portion of input before $\#$ is seen
- Guess at some point that current symbol in $w$ is going to be the same as ‘c’; store this symbol in the state
- Read the rest of $w$
- On reading $\#$, check that the symbol immediately after is the one stored, and that the input ends immediately after that.
Back to the Future
The Automaton

\[ L(M) = \{w \# c \mid c \text{ occurs in } w \} \]
Halving a Language

Definition
For a language $L$, define $\frac{1}{2}L$ as follows.

$$\frac{1}{2}L = \{x \mid \exists y. |x| = |y| \text{ and } xy \in L\}$$

In other words, $\frac{1}{2}L$ consists of the first halves of strings in $L$. 
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Example
If $L = \{001, 0000, 01, 110010\}$ then $\frac{1}{2}L =$ 

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Example
If $L = \{001, 0000, 01, 110010\}$ then $\frac{1}{2}L = \{00, 0, 110\}$. 
Proposition

If $L$ is recognized by a DFA $M$ then there is a NFA $N$ such that $L(N) = \frac{1}{2}L$. 

Proof Idea

On input $x$, need to check if $x$ is the first half of some string $w = xy$ that is accepted by $M$. 

Run $M$ on input $x$; let $M$ be in state $q_i$ after reading all of $x$. 

Guess a string $y$ such that $|y| = |x|$. 

Check if $M$ reaches a final state on reading $y$ from $q_i$. 

How do you guess a string $y$ of equal length to $x$ using finite memory? Seems to require remembering the length of $x$!
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Recognizing Halves of Regular Languages

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- How do you guess a string $y$ of equal length to $x$ using finite memory? Guess one symbol of $y$ as you read one symbol of $x$!
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- If we don’t first execute $M$ on $x$, how do we know the state $q_i$ from which we have to execute $y$ from?
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- If we don’t first execute \( M \) on \( x \), how do we know the state \( q_i \) from which we have to execute \( y \) from? Guess it! And then check that running \( M \) on \( x \) does indeed end in \( q_i \), your guessed state.
New Algorithm

On input $x$, NFA $N$

1. Guess state $q_i$ and place “left finger” on (initial state of $M$) $q_0$ and “right finger” on $q_i$

2. As characters of $x$ are read, $N$ moves the left finger along transitions dictated by $x$ and simultaneously moves the right finger along nondeterministically chosen transitions labelled by some symbol.
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Things to remember: initial guess for right finger, and positions of left and right finger.
Algorithm on Example

100010 \in L \text{ and so } x = 100 \in \frac{1}{2} L
Algorithm on Example

100010 ∈ L and so x = 100 ∈ \( \frac{1}{2} L \)
NFA N execution on x = 100 is

<table>
<thead>
<tr>
<th>String Read</th>
<th>Left Finger</th>
<th>Right Finger</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon )</td>
<td>( q_0 )</td>
<td>( q_2 )</td>
</tr>
<tr>
<td>1</td>
<td>( q_1 )</td>
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</tr>
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Algorithm on Example

DFA $M$

100010 $\in L$ and so $x = 100 \in \frac{1}{2} L$

NFA $N$ execution on $x = 100$ is

String Read | Left Finger | Right Finger
-------------|-------------|-----------------|
$\epsilon$    | $q_0$       | $q_2$           |
1             | $q_1$       | $q_2$           |
10            | $q_3$       | $q_1$           |
100           | $q_2$       | accept?         |
Formal Construction of NFA $N$

States and Initial State

Given $M = (Q, \Sigma, \delta, q_0, F)$ recognizing $L$ define $N = (Q', \Sigma, \delta', q'_0, F')$ that recognizes $\frac{1}{2}L$

- $Q' = Q \times Q \times Q \cup \{s\}$, where $s \notin Q$
  - $s$ is a new start state
  - Other states are of the form $\langle$left finger, initial guess, right finger$\rangle$; “initial guess” records the initial guess for the right finger

- $q'_0 = s$
Formal Construction of NFA $\mathcal{N}$

Transitions and Final States

- **Transitions**

  $$\delta'(s, \epsilon) = \{ \langle q_0, q_i, q_i \rangle \mid q_i \in Q \}$$

  “Guess” the state $q_i$ that the input will lead to

  $$\delta'(\langle q_i, q_j, q_k \rangle, a) = \{ \langle q_l, q_j, q_m \rangle \mid \delta(q_i, a) = q_l, \exists b \in \Sigma. \delta(q_k, b) = q_m \}$$

  $b$ is the guess for the next symbol of $y$ and initial guess does not change

- **$F'$**

  $$F' = \{ \langle q_i, q_i, q_j \rangle \mid q_i \in Q, q_j \in F \}$$