CSE 135: Introduction to Theory of Computation
Nondeterministic Finite Automata (cont')

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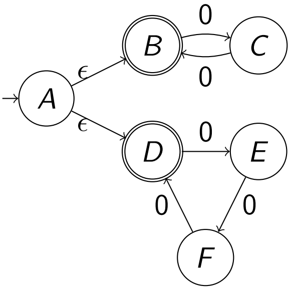
Example II



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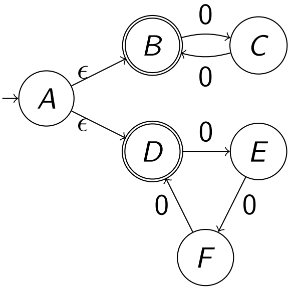
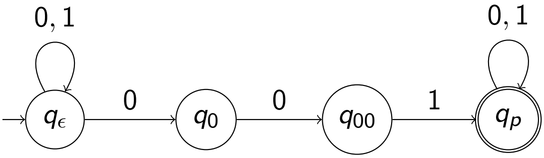


NFA accepting strings of 0s where the length is either a multiple 2 or 3

The NFA "guesses" at the begining whether it will see a multiple of 2 or 3, and then confirms that the guess was correct.

Example III



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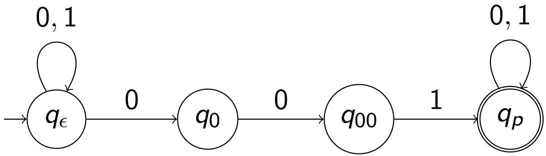

NFA accepting strings with 001 as substring

At some point the NFA "guesses" that the pattern 001 is starting and then checks to confirm the guess.

Nondeterministic Finite Automata (NFA)

Formal Definition

Definition

A nondeterministic finite automaton (NFA) is $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- δ :
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

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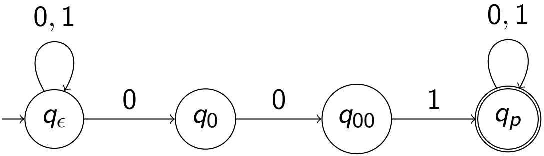
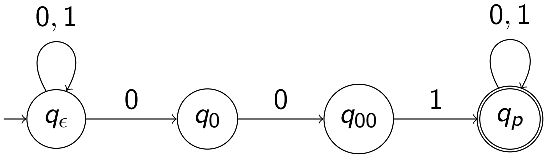
- Q is the finite set of states
- Σ is the finite alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the powerset of Q
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

Example of NFA



Transition Diagram of NFA

Example of NFA



Transition Diagram of NFA

Formally, the NFA is $M_{001} = (\{q_\epsilon, q_0, q_{00}, q_p\}, \{0, 1\}, \delta, q_\epsilon, \{q_p\})$ where δ is given by

$$\delta(q_\epsilon, 0) = \{q_\epsilon, q_0\} \qquad \delta(q_\epsilon, 1) = \{q_\epsilon\} \qquad \delta(q_0, 0) = \{q_{00}\}$$
$$\delta(q_{00}, 1) = \{q_p\} \qquad \delta(q_p, 0) = \{q_p\} \qquad \delta(q_p, 1) = \{q_p\}$$

δ is $\emptyset$ in all other cases.

Computation

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and states $q_1, q_2 \in Q$, we say $q_1 \xrightarrow{w}_M q_2$ if there is one thread of computation on input w from state q_1 that ends in q_2.

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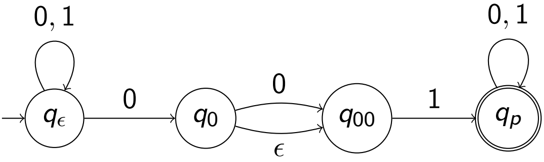
- $r_0 = q_1$,
- for each i, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = q_2$, and
- $w = x_1 x_2 x_3 \cdots x_k$

Example Computation



$q_\epsilon \xrightarrow{0100}_M q_p$ because taking $r_0 = q_\epsilon$, $r_1 = q_0$, $r_2 = q_{00}$, $r_3 = q_p$, $r_4 = q_p$, $r_5 = q_p$, and $x_1 = 0$, $x_2 = \epsilon$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, we have

- $x_1 x_2 \cdots x_5 = 0\epsilon100 = 0100$
- $r_{i+1} \in \delta(r_i, x_{i+1})$

Defining $\hat{\Delta}$

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and state $q_1 \in Q$, we say $\hat{\Delta}(q_1, w)$ to denote states of all the active threads of computation on input w from q_1.

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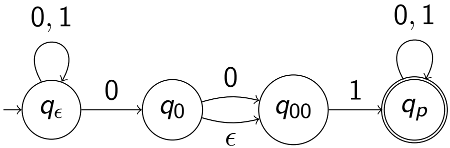
$$\hat{\Delta}(q_1, w) = \{q \in Q \mid q_1 \xrightarrow{w}_M q\}$$

Example



Example NFA

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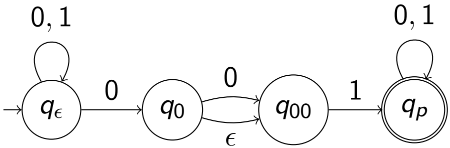

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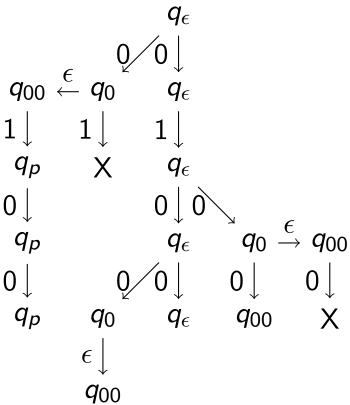
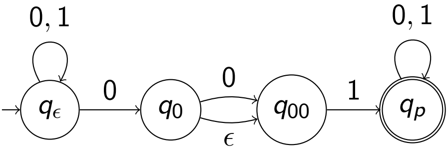


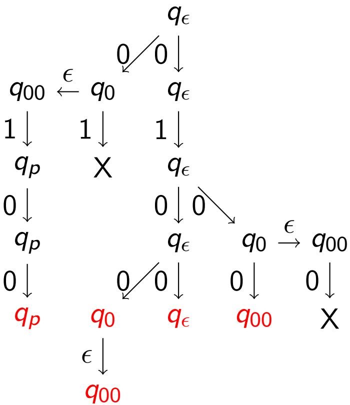
Computation on 0100

Example



Example NFA

$$\hat{\Delta}(q_\epsilon, 0100) = \{q_0, q_p, q_{00}, q_\epsilon\}$$



Computation on 0100

Acceptance/Recognition

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For an NFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say M **accepts** w iff $\hat{\Delta}(q_0, w) \cap F \neq \emptyset$

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Definition

The **language accepted or recognized** by NFA M over alphabet Σ is $L(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$. A language L is said to be **accepted/recognized** by M if $L = L(M)$.

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However, the following proposition about DFAs continues to hold for NFAs

- For NFA M, strings u and v, and state q,

$$\hat{\Delta}(q, uv) = \bigcup_{q' \in \hat{\Delta}(q,u)} \hat{\Delta}(q', v)$$

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- You follow the same methodology as for DFAs, like identifying what needs to be remembered
- But now, the machine can "guess" at certain steps

Back to the Future

Problem

For $\Sigma = \{0, 1, 2, \#\}$, let

$$L = \{w\#c \mid w \in \{0,1,2\}^*,\ c \in \{0,1,2\},\ \text{and } c \text{ occurs in } w\}$$

So $1011\#0 \in L$ but $1011\#2 \notin L$. Design an NFA recognizing L.

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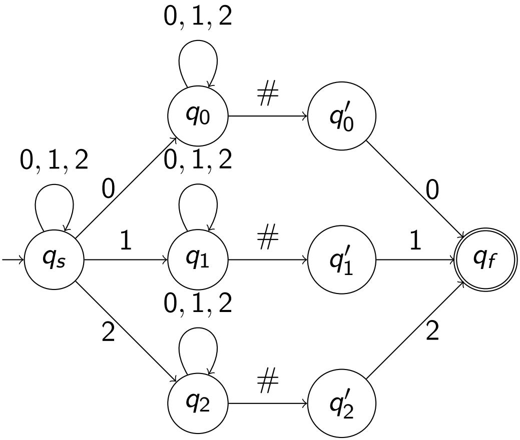
Solution

- Read symbols of w, i.e., portion of input before $\#$ is seen
- Guess at some point that current symbol in w is going to be the same as 'c'; store this symbol in the state
- Read the rest of w
- On reading $\#$, check that the symbol immediately after is the one stored, and that the input ends immediately after that.

Back to the Future

The Automaton



$$L(M) = \{w\#c \mid c \text{ occurs in } w\}$$

Halving a Language

Definition

For a language L, define $\frac{1}{2}L$ as follows.

$$\frac{1}{2}L = \{x \mid \exists y.\ |x| = |y| \text{ and } xy \in L\}$$

In other words, $\frac{1}{2}L$ consists of the first halves of strings in L

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If $L = \{001, 0000, 01, 110010\}$ then $\frac{1}{2}L = \{00, 0, 110\}$.

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- "Run" M on input x; let M be in state q_i after reading all of x
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How do you guess a string y of equal length to x using finite memory? Seems to require remembering the length of x!

Fixing the Idea

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- If we don't first execute M on x, how do we know the state q_i from which we have to execute y from? Guess it! And then check that running M on x does indeed end in q_i, your guessed state.

New Algorithm

On input x, NFA N

1. Guess state q_i and place "left finger" on (initial state of M) q_0 and "right finger" on q_i
2. As characters of x are read, N moves the left finger along transitions dictated by x and simultaneously moves the right finger along nondeterministically chosen transitions labelled by some symbol

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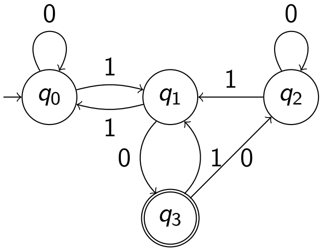
Things to remember: initial guess for right finger, and positions of left and right finger.

Algorithm on Example

$100010 \in L$ and so $x = 100 \in \frac{1}{2}L$



DFA M

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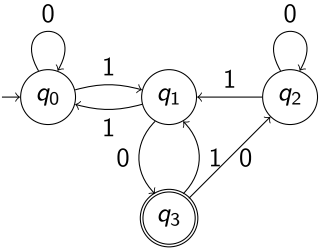

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String Read	Left Finger	Right Finger
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10	q_3	q_1
100	q_2	q_3

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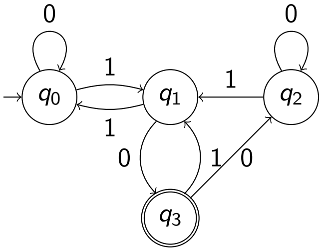

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100	$\boxed{q_2}$		$\boxed{q_3}$

$\uparrow$
accept?

Formal Construction of NFA N

States and Initial State

Given $M = (Q, \Sigma, \delta, q_0, F)$ recognizing L define $N = (Q', \Sigma, \delta', q'_0, F')$ that recognizes $\frac{1}{2}L$

- $Q' = Q \times Q \times Q \cup \{s\}$, where $s \notin Q$
  - s is a new start state
  - Other states are of the form $\langle$left finger, initial guess, right finger$\rangle$; "initial guess" records the initial guess for the right finger
- $q'_0 = s$

Formal Construction of NFA N

Transitions and Final States

- Transitions

  $$\delta'(s, \epsilon) = \{\langle q_0, q_i, q_i\rangle \mid q_i \in Q\}$$

  "Guess" the state q_i that the input will lead to

  $$\delta'(\langle q_i, q_j, q_k\rangle, a) = \{\langle q_l, q_j, q_m\rangle \mid \delta(q_i, a) = q_l, \\ \exists b \in \Sigma.\ \delta(q_k, b) = q_m\}$$

  b is the guess for the next symbol of y and initial guess does not change

- $F' = \{\langle q_i, q_i, q_j\rangle \mid q_i \in Q, q_j \in F\}$