EECS 279: Approximation Algorithms — Release: 10-14-2014
Homework 4-A — Due: 3:00pm, Thursday, 10-23-2014

Make your solutions concise and formal. Your goal is to convince me that you know the solutions. You are highly encouraged to typeset your solutions in Latex. See the course website for further homework policies. Please write down your collaborators names and references if any.

1. In the (Uncapacitated) Facility Location problem, we are given as input a set V of terminals, $\{1, 2, ..., n\}$, together with the opening cost f_i of each terminal i and distance c_{ij} between each pair of terminals i and j which forms a metric; that is, $c_{ii} = 0$, $c_{ij} = c_{ji}$, and $c_{ij} + c_{jk} \geq c_{ik}$. The goal is to find a subset $F \subseteq V$ of terminals to open as facilities such that the following cost is minimized:
$$cost(F) = \sum_{i \in F} f_i + \sum_{j \in V} c(j, F),$$
where $c(j, F) = \min_{i \in F} c_{ij}$ is the connection cost of terminal j and $i(j) := \arg\min_{i \in F} c_{ij}$ is the facility that terminal j connects to.

   In the class, we learned a 6-approximation using linear programming and rounding: We first obtained an optimal LP solution x_{ij} and y_i; and converted (x, y) into another feasible LP solution (x', y') such that the LP objective increases by at most twice, and if $x'_{ij} > 0$, then $c_{ij} \leq 2\Delta_j$ where $\Delta_j := \sum_{i \in V} c_{ij} x_{ij}$; and repeatedly picked a terminal j as an open facility and removed close terminals.

   The question is regarding the last step: We picked a terminal j with the smallest Δ_j and opened the facility $i(j)$ in the j's ball $B_j := \{i \mid c_{ij} \leq 2\Delta_j\}$ with the smallest opening cost. Here, we assigned to $i(j)$ not only j, but also all terminals j' such that $B_{j'} \cap B_j \neq \emptyset$. Then we recursed on the remaining terminals with all such assigned terminals j' removed.

   But we may try the following altenertive algorithm: Pick a terminal j with the smallest Δ_j and open the facility $i(j)$ in the j's ball $B_j := \{i \mid c_{ij} \leq 2\Delta_j\}$ with the smallest opening cost. Assign to $i(j)$ only the terminals in B_j. Then recurse on the remaining terminals with all the assigned terminals removed.

   Explain why this modified algorithm fails to yield an $O(1)$-approximation. For simplicity, assume that you are given (x', y'), and argue why this change fails.

2. Assume that you have a fair coin that yields either a head or tail, each with probability 1/2. Let $n > 0$ be a parameter. For simplicity, assume that $\log n$ and n are both integers. Repeat flipping the coin k times sequentially. What is the (asymptotically) minimum k where you observe at least one head with probability at least $1 - 1/n$? Also, what is the minimum k where you observe at least $\log n$ heads with probability at least $1 - 1/n$?

3. We are given n items where each item $j \in \{1, 2, ..., n\}$ is associated with a d-dimensional vector $v_j = (v_{j1}, v_{j2}, ..., v_{jd})$ with no negative entries, and has cost c_j. The goal is to find a subset S of items of minimum cost such that $\sum_{j \in S} v_j \geq (1, 1, ..., 1)$; the inequality must hold for each entry. Give a randomized algorithm that gives a $O(\log d)$-approximation with probability at least 1/2. Can you improve your approximation ratio to $O(\log d / \log\log d)$?

   (Caution: You will have to cap v_j's entries at 1. Do you see why?)