1. In the (Uncapacitated) Facility Location problem, we are given as input a set \( V \) of terminals, \( \{1, 2, \ldots, n\} \), together with the opening cost \( f_i \) of each terminal \( i \) and distance \( c_{ij} \) between each pair of terminals \( i \) and \( j \) which forms a metric; that is, \( c_{ii} = 0, \ c_{ij} = c_{ji}, \) and \( c_{ij} + c_{jk} \geq c_{ik} \). The goal is to find a subset \( F \subseteq V \) of terminals to open as facilities such that the following cost is minimized:
\[
    \text{cost}(F) = \sum_{i \in F} f_i + \sum_{j \in V} c(j, F),
\]
where \( c(j, F) = \min_{i \in F} c_{ij} \) is the connection cost of terminal \( j \) and \( i(j) := \arg \min_{i \in F} c_{ij} \) is the facility that terminal \( j \) connects to.

In the class, we learned a 6-approximation using linear programming and rounding: We first obtained an optimal LP solution \( x_{ij} \) and \( y_i \); and converted \((x, y)\) into another feasible LP solution \((x', y')\) such that the LP objective increases by at most twice, and if \( x_{ij} > 0 \), then \( c_{ij} \leq 2\Delta_j \) where \( \Delta_j := \sum_{i \in V} c_{ij} x_{ij} \); and repeatedly picked a terminal \( j \) as an open facility and removed close terminals.

The question is regarding the last step: We picked a terminal \( j \) with the smallest \( \Delta_j \) and opened the facility \( i(j) \) in the \( j \)'s ball \( B_j := \{i \mid c_{ij} \leq 2\Delta_j\} \) with the smallest opening cost. Here, we assigned to \( i(j) \) not only \( j \), but also all terminals \( j' \) such that \( B_j \cap B_{j'} \neq \emptyset \). Then we recursed on the remaining terminals with all such assigned terminals \( j' \) removed.

But we may try the following alternate algorithm: Pick a terminal \( j \) with the smallest \( \Delta_j \) and open the facility \( i(j) \) in the \( j \)'s ball \( B_j := \{i \mid c_{ij} \leq 2\Delta_j\} \) with the smallest opening cost. Assign to \( i(j) \) only the terminals in \( B_j \). Then recurse on the remaining terminals with all the assigned terminals removed.

Explain why this modified algorithm fails to yield an \( O(1) \)-approximation. For simplicity, assume that you are given \((x', y')\), and argue why this change fails.

2. Assume that you have a fair coin that yields either a head or tail, each with probability 1/2. Let \( n > 0 \) be a parameter. For simplicity, assume that \( \log n \) and \( n \) are both integers. Repeat flipping the coin \( k \) times sequentially. What is the (asymptotically) minimum \( k \) where you observe at least one head with probability at least \( 1 - 1/n \)? Also, what is the minimum \( k \) where you observe at least \( \log n \) heads with probability at least \( 1 - 1/n \)?

3. We are given \( n \) items where each item \( j \in \{1, 2, \ldots, n\} \) is associated with a \( d \)-dimensional vector \( v_j = (v_{j1}, v_{j2}, \ldots, v_{jd}) \) with no negative entries, and has cost \( c_j \). The goal is to find a subset \( S \) of items of minimum cost such that \( \sum_{j \in S} v_j \geq (1, 1, \ldots, 1) \); the inequality must hold for each entry. Give a randomized algorithm that gives an \( O(\log d) \)-approximation with probability at least 1/2. Can you improve your approximation ratio to \( O(\log d/\log \log d) \)?

(Caution: You will have to cap \( v_j \)'s entries at 1. Do you see why?)