Make your solutions concise and formal. Your goal is to convince me that you know the solutions. You are highly encouraged to typeset your solutions in Latex. See the course website for further homework policies. Please write down your collaborators names and references if any.

1. Consider the maximum k converge problem where there is a university U of n items and a collection $\mathcal{S}$ of subsets of the universe. The goal is to select k sets that cover the most number of unique elements in U. Adapt the LP given in class for the set cover problem to the maximum converge problem. Show that the expected value of the solution computed with randomized rounding is $(1-\frac{1}{e})$-approximation.

2. Consider the problem of minimizing makespan on *unrelated* machines. In this problem there are m machines and n jobs. Each job j has processing time $p_{i,j}$ on machine i. The processing times of the jobs between the machines need not be correlated and could be very small on some machines and huge on others. The goal is to assign all jobs to the machines to minimize makespan. That is, if S_i is the set of jobs assigned to machine i, the goal is to find an assignment that minimizes $\max_i \sum_{j\in S_i} p_{i,j}$.

   Say that you have a guess of the optimal makespan M^*. Let $x_{i,j}$ be a variable indicating if job j is assigned to machine i. We have the following LP relaxation for the problem, that we can use to determine if M^* is a good guess of the optimal solution. That is, by checking if the following LP has a feasible solution, we can tell if M^* is a good guess. Note that this LP has no objective function, and we are only interested in if there exists a feasible solution to the LP.

$$\begin{aligned}
\sum_{j=1}^{n} p_{i,j}x_{i,j} &\le M^* && \forall i\in[m]\\
\sum_{i=1}^{m} x_{i,j} &= 1 && \forall j\in[n]\\
x_{i,j} &= 0 && \forall i,j \text{ such that } p_{i,j} > M^*\\
x_{i,j} &\in [0,1] && \forall i,j
\end{aligned}$$

   The first constraint says that the maximum load of jobs assigned to a machine cannot exceed M^*. The second constraint enforces each job to be assigned to a machine. The third constraint says that a job cannot be assigned to a machine where its processing time exceeds M^*. We can do a binary search on M^* to determine the minimum value M^* can have where the LP is feasible.

   (a) Show that if there is an assignment with makespan no greater than M^*, there exists a feasible integral solution to the above LP.

   (b) For this LP solution, show the randomized rounding that assigns each job j to machine i with probability $x_{i,j}$ independently yields an $O(\log m)$-approximation with probability at least $1 - 1/m^c$ for some $c > 1$.

(c) Consider an alternative linear programming relaxation where there is no third constraint and the objective is to minimize M^*. Show that this LP has an integrality gap of $\Omega(m)$.