EECS 279: Approximation Algorithms — Release: 09-30-2014

Homework 3-A — Due: 3:00pm, Thursday, 10-09-2014

Make your solutions concise and formal. Your goal is to convince me that you know the solutions. You are highly encouraged to typeset your solutions in Latex. See the course website for further homework policies. Please write down your collaborators names and references if any.

1. In the Bin Packing problem, we are given as input a set of n items where item i has size $s_i \in (0,1]$. Our goal is to pack in the minimum number of bins of size 1.

   Consider the following integer programming for the Bin Packing Problem. For notational convenience, let $[n] := \{1, 2, ..., n\}$.

$$\min \sum_{b \in [n]} x_b$$

$$\sum_{i \in [n]} s_i \cdot x_{b,i} \leq x_b \qquad \forall b \in [n]$$

$$\sum_{b \in [n]} x_{b,i} \geq 1 \qquad \forall i \in [n]$$

$$x_b \in \{0,1\} \qquad \forall b \in [n]$$

$$x_{b,i} \in \{0,1\} \qquad \forall b, i \in [n]$$

   Here, variable $x_b = 1$ if and only if the bth bin is opened. Likewise, y_{bi} is an indicator variable that is 1 if and only if item i is assigned to the bth bin, otherwise 0. The first constraint says that if the bth bin is opened, then items of up to size 1 can be packed into the bin. The second constraint says that every item much be packed into a bin. We can obtain a valid linear program relaxation by replacing $x_b, x_{b,i} \in \{0,1\}$ with $x_b, x_{b,i} \in [0,1]$.

   Show that for any $\epsilon > 0$, this linear program relaxation has an integrality gap of $2 - \epsilon$.

2. In the Maximum Independent Set problem (MIS), we are given as input an undirected graph $G(V,E)$, and asked to find a maximum size independent set. For simplicity, we assume that the given graph G is connected. A subset of vertices $V' \subseteq V$ is said to be independent if no edge $e \in E$ has both end points in V'. The size of an independent set V' is defined as the number of vertices in V'. Consider the following linear program relaxation for MIS.

$$\max \sum_{v \in V} x_v$$

$$x_u + x_v \leq 1 \qquad \forall e = (u,v) \in E$$

$$x_u \geq 0 \qquad \forall u \in V$$

   Note that this relaxation is obtained by relaxing the constraint $x_u \in \{0,1\}$ to $x_u \in [0,1]$, and lifting the redundant condition $x_u \leq 1$. (If the graph is not connected, we can't remove $x_u \leq 1$. Do you see why?)

   Give a dual of the above LP.

3. We revisit the variant of the bin packing problem: There is a set U of n items where each item has a size s_i and a weight w_i. Unlike the bin packing problem discussed in class, now an item can be rejected. If an item i is rejected the algorithm has to pay a cost of w_i. All items not rejected must be packed into bins of size 1. The goal is to find a set of items S and pack them into m bins to minimize $m + \sum_{i \in U \setminus S} w_i$.

   Give an LP for this problem, and an $O(1)$-approximaiton using LP and rounding.

4. In the class, we saw the following IP for the Min Cut. Recall that c_{ij} is edge (i,j)'s cost/weight, and s and t are the source and sink vertices, respectively.

$$\begin{aligned}
\min \sum_{(i,j) \in E} c_{ij} d_{ij} & \\
d_{ij} - p_i + p_j \geq 0 & \qquad \forall (i,j) \in E \\
p_s - p_t \geq 1 & \\
d_{ij} \in \{0,1\} & \qquad \forall (i,j) \in E \\
p_i \in \{0,1\} & \qquad \forall i \in V
\end{aligned}$$

   We relaxed the last two constraints into $d_{ij} \in [0,1]$ and $p_i \in [0,1]$, and claimed that we can without loss of generality remove $d_{ij} \leq 1$ and $p_i \leq 1$. Explain why concisely.