Make your solutions concise and formal. Your goal is to convince me that you know the solutions. You are highly encouraged to typeset your solutions in Latex. See the course website for further homework policies.

1. In class we saw a $(1-1/e)$-approximation for the maximum k-coverage problem using a greedy algorithm. In the maximum coverage problem, you are given a universe U of n elements and a collections $\mathcal{S}$ of subsets of U such that $\cup_{S_i \in \mathcal{S}} S_i = U$. The goal is to select a set $X \subseteq \mathcal{S}$ where $|X| = k$ to maximize $|\cup_{S_i \in X} S_i|$. The greedy algorithm we considered iteratively selects sets such that at any point in time it chooses the set that includes the maximum number of uncovered elements.

   Consider the following modification to the greedy algorithm, which we will call the ϵ-greedy algorithm that is parameterized by a constant $0 \leq \epsilon < 1$. This algorithm, like greedy, starts with a empty set and greedily builds it solution by adding sets one at a time to the solution. Let X_i denote ϵ-greedy's solution after adding i sets (note $X_0 = \emptyset$). Now rather than adding the set in each step that covers the most number of uncovered elements, it can add any set that covers almost as many elements as the best possible set. That is, in the ith set, ϵ-greedy adds any set S_j such that $|S_j \setminus \cup_{S_a \in X_{i-1}} S_a| \geq (1-\epsilon) \max_{j'} |S_{j'} \setminus \cup_{S_a \in X_{i-1}} S_a|$. Note that if $\epsilon = 0$ then this is exactly the greedy algorithm discussed in class. Prove the best approximation ratio possible for ϵ-greedy.

2. In the Metric-TSP problem discussed in class, we are given an undirected graph $G = (V, E)$ where for each $e \in E$ there is a cost $c(e) \geq 0$. The metric property ensures that $c(uw) \leq c(uv) + c(vw)$ for any $u, v, w \in V$ and that the graph is a complete graph. The goal is to find a Hamiltonian cycle of minimum length in G.

   Consider the following algorithm which greedily builds a Hamiltonian Cycle. Let u_i be the ith vertex the cycle visits and set u_1 to be any arbitrary vertex. In the ith iteration of the algorithm, the next vertex chosen to visit is the closest vertex to u_i that is not in $\cup_{j \leq i} u_j$. That is, the closest unvisited vertex. The algorithm completes when all vertices are visited and the last visited vertext v_n is connected to the first vertex v_1. Prove that this algorithm gives a $O(\log n)$-approximation for the Metric-TSP problem where $n = |V|$. (Hint: consider the greedy Steiner tree algorithm discussed in class.)

3. In the well known knapsack problem you are given a set of n items where each item i has size s_i and profit p_i. The goal is to find a maximum profit subset of items that can fit in a knapsack of size B.

   In the *multiple identical* knapsack problem there are m knapsacks each of size B that the items can be packed in. Say that we know of an algorithm that can solve the single (classic) knapsack problem optimally in polynomial time. Using this, there is a natural greedy strategy for the multiple knapsack problem: Pack the first knapsack using the exact algorithm, remove the packed items, and recursively fill the remaining knapsacks.

   (a) Prove that this greedy strategy does not give an exact solution.

(b) Prove that this algorithm gives an $O(1)$-approximation for the *multiple identical* knapsack problem.

(c) **Extra Credit:** Prove that this algorithm gives an $O(1)$-approximation for the *multiple* knapsack problem where knapsacks have different sizes and knapsacks are packed in decreasing order of size.