NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS

Numer. Linear Algebra Appl. 2000; **00**:1–6 *Prepared using* nlaauth.cls *[Version: 2002/09/18 v1.02]*

A simplified pivoting strategy for symmetric tridiagonal matrices

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SUMMARY

The pivoting strategy of Bunch and Marcia for solving systems involving symmetric indefinite tridiagonal matrices uses two different methods for solving 2×2 systems when a 2×2 pivot is chosen. In this paper, we eliminate this need for two methods by adding another criterion for choosing a 1×1 pivot. We demonstrate that all the results from the Bunch and Marcia pivoting strategy still hold. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: symmetric indefinite factorization, tridiagonal matrices, normwise backward stability

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Contract/grant sponsor: National Library of Medicine; contract/grant number: 5T15LM007359-03

Received

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ADDENDUM

We consider a pivoting strategy that simplifies the one proposed in [1] for solving systems involving symmetric tridiagonal matrices. This addendum is meant to follow it directly. Thus, we use the same notations as in [1], and references to equations, lemmas, and algorithms are made without explicitly referring to the paper. Here, we denote the pivot size for the simplified strategy by s_S. The proposed simplified strategy is as follows:

Algorithm A1. *(Simplified pivoting strategy).*

$\alpha = (\sqrt{5}-1)/2 \approx 0.62$

$\Delta = \alpha_1\alpha_2 - \beta_2^2$

if $|\alpha_1\alpha_2| \geq \alpha\beta_2^2$ or $|\Delta| \leq \alpha|\alpha_1\beta_3|$ or $|\beta_2\Delta| \leq \alpha|\alpha_1^2\beta_3|$

$s_S = 1$

else

$s_S = 2$

end

Algorithm A1 differs from the alternative pivoting strategy (Algorithm 3.1) in that the criterion $|\alpha_1\alpha_2| \geq \alpha\beta_2^2$ is added for choosing a 1×1 pivot. This added criterion eliminates having to solve the 2×2 system in Algorithm 3.1 in two different ways. Specifically, if a 2×2 pivot is chosen in Algorithm A1, then $|\alpha_1\alpha_2| \leq \alpha\beta_2^2$, Thus if Algorithm 3.2 is used to solve the 2×2 system, then the explicit inverse (6) is used automatically. The added criterion also eliminates the need for a 2×2 pivot if T is positive definite by the following:

Property A1. *If T is positive definite, then the LBL^T factorization using Algorithm A1 reduces to the LDL^T factorization.*

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Proof. By induction on n. The case $n = 1$ is trivial. Thus we assume that for a symmetric positive definite tridiagonal matrix of size $(n-1)\times(n-1)$, Property A1 holds. If T is positive definite, then $\alpha_1\alpha_2 \geq \beta_2^2$. Since $\alpha_1\alpha_2 \geq \alpha\beta_2^2$, a 1×1 pivot is chosen in Algorithm A1 and the first diagonal entry in B is a 1×1 block.. The Schur complement is a symmetric positive-definite tridiagonal matrix of size $(n-1)\times(n-1)$. By induction, the LBL^T factorization of this matrix using Algorithm A1 reduces to the LDL^T factorization. Thus every element in B is a 1×1 diagonal block. □

The simplified pivoting strategy can be related to the original Bunch strategy in the following way.

Lemma A2. *If $s_S = 1$, then $s_B = 1$.*

Proof. Suppose $s_S = 1$. If $|\alpha_1\alpha_2| \geq \alpha\beta_2^2$, then $\sigma|\alpha_1| \geq \alpha\beta_2^2$ trivially. Thus $s_B = 1$. Otherwise, $|\Delta| \leq \alpha|\alpha_1\beta_3|$ or $|\beta_2\Delta| \leq \alpha|\alpha_1^2\beta_3|$. Thus $s_A = 1$ and consequently $s_B = 1$ by Lemma 3. □

Lemma A3. *If $s_S = 2$ and $s_B = 1$, then $s'_B = 1$.*

Proof. It is clear that if $s_S = 2$, then $s_A = 2$. Thus $s'_B = 1$ by Lemma 4.□

We now demonstrate that the bound on the growth factor for this simplified pivoting strategy is the same as that for the Bunch pivoting strategy. If $s_S = 1$, then $s_B = 1$ by Lemma A3, and therefore, $|\beta_2^2/\alpha_1| \leq \sigma/\alpha$. Thus

$$|\tilde{\alpha}_2| = \left|\ \alpha_2 - \frac{\beta_2^2}{\alpha_1}\ \right| \leq \sigma + \frac{\sigma}{\alpha}.$$

If $s_S = 2$, then $s_A = 2$. Thus

$$|\tilde{\alpha}_3| \leq \sigma + \frac{\sigma}{\alpha}.$$

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Thus, the growth factor ρ_n for this pivoting strategy satisfies $\rho_n \leq 2 + \alpha \approx 2.62$.

To show the stability of the LBL^T factorization using Algorithm A1, we must show that $|F|$ and $|G|$ in Section 3.4 are bounded by T in some norm. For $|F|$, if $s_S = 1$, then $\|F\|_\infty = |\beta_2| \leq \sigma$. If $s_S = 2$, then $s_A = 2$ and $\|F\|_\infty \leq (4\alpha + 5)\sigma$ by Section 3.4. For $|G|$, if $s_S = 1$, then $s_B = 1$ by Lemma A2. Thus $\|G\|_\infty \leq \sigma/\alpha$. If $s_S = 2$, then $s_A = 2$, and $\|G\|_\infty \leq (7\alpha + 11)\sigma$. Thus

$$\||L||B||L^T|\|_M \leq 16 \times 2.62\|T\|_M < 42\|T\|_M.$$

We conclude by modifying Theorem 7 to demonstrate the normwise backwards stability of solving symmetric tridiagonal matrices using the LBL^T factorization whose pivoting is described in Algorithm A1.

Theorem A4. *Let the LBL^T factorization with the pivoting strategy of Algorithm A1 be applied to a symmetric tridiagonal matrix $T \in \Re^{n\times n}$ to yield the computed factorization $T \approx \hat{L}\hat{B}\hat{L}^T$, and let $\hat{x}$ be the computed solution to $Tx = b$ obtained using the factorization. Assume that all linear systems $Ey = f$ involving 2×2 pivots E are solved using the explicit inverse (6). Then*

$$T + \Delta T_1 = \hat{L}\hat{B}\hat{L}^T, \quad (T + \Delta T_2)\hat{x} = b,$$

where

$$\|\Delta T_i\|_M \leq cu\|T\|_M + O(u^2), \quad i = 1, 2,$$

where c is a constant.

REFERENCES

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