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A pivoting strategy for symmetric tridiagonal matrices

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SUMMARY

The LBL^T factorization of Bunch for solving linear systems involving a symmetric indefinite tridiagonal matrix T is a stable, efficient method. It computes a unit lower triangular matrix L and a block 1×1 and 2×2 matrix B such that $T = LBL^T$. Choosing the pivot size requires knowing *a priori* the largest element σ of T in magnitude. In some applications, it is required to factor T as it is formed without necessarily knowing σ. In this paper, we present a modification of the Bunch algorithm that can satisfy this requirement. We demonstrate that this modification exhibits the same bound on the growth factor as the Bunch algorithm and is likewise normwise backward stable.

KEY WORDS: symmetric indefinite factorization, tridiagonal matrices, normwise backward stability

1. INTRODUCTION

Linear systems involving symmetric tridiagonal matrices can be solved in various ways. Gaussian elimination with partial pivoting is a stable method for solving the linear system $Tx = b$, where $T \in \Re^{n\times n}$ is symmetric and tridiagonal and x and $b \in \Re^n$. However, this method does not take advantage of the symmetry property or sparsity structure of T. If T is positive definite, the Cholesky factorization $T = RR^T$, where $R \in \Re^{n\times n}$ is lower triangular, can be easily computed, and the linear system can be solved via the triangular systems $R^T y = b$ and $Rx = y$, with $y \in \Re^n$. A slightly more efficient method is to use the LDL^T factorization of T, where $T = LDL^T$ for some unit lower triangular matrix L and some diagonal matrix D with positive entries. However, both of these factorizations are unstable or may not exist when T is not positive definite. The *block* LDL^T, also known as LBL^T, factorizations with the various pivoting strategies (e.g., [2, 4, 5]) are stable methods for solving linear systems with symmetric indefinite matrices. These methods compute the factorization $P^T TP = LBL^T$, where P is a permutation matrix, L is unit lower triangular, and B is block diagonal with 1×1 and 2×2

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blocks. The row and column interchanges can create fill-in, thereby destroying the sparsity structure of T in the Schur complement. The LBL^T factorization of Bunch [3] for tridiagonal matrices does not permute any row or column and preserves the tridiagonal structure in the Schur complement. This method does not suffer from the disadvantages of the other methods: it does not create fill-in and is shown to be normwise backward stable [8]. It is also easily implemented. This paper focuses on a variation of the Bunch pivoting strategy for the LBL^T factorization of symmetric indefinite tridiagonal matrices.

In the LBL^T factorization of Bunch, a 1×1 pivot is chosen if the leading $(1, 1)$ element is sufficiently large relative to the sub-diagonal $(1, 2)$ element (see Algorithm 2.1). This pivoting strategy involves determining the largest element in magnitude in the matrix T. Thus, the full matrix must be known *a priori*. Whereas the LDL^T factorization can be computed as T is formed, i.e., only the k-th diagonal and sub-diagonal elements are needed at the k-th step of the factorization, the LBL^T factorization of Bunch for indefinite matrices requires that the whole matrix be known initially. In some applications, it is desired to form the LBL^T factorization as T is formed. For example, when the Lanczos method is applied to solve a linear system involving a symmetric indefinite matrix, one must be able to factor the resulting indefinite tridiagonal matrix T_k at each iteration k. In this situation, the LBL^T factorization of Bunch cannot be applied.

In this paper, we present an alternative pivoting strategy that is closely related to the Bunch pivoting strategy. We show that the block structure of both pivoting strategies are similar and that both algorithms exhibit the same bound on the growth factor. We demonstrate that an LBL^T factorization using this alternative pivoting strategy is normwise backward stable using arguments similar to Higham's proof [8] of the stability of the Bunch LBL^T factorization. The paper is organized as follows. In Section 2, we discuss the Bunch pivoting strategy and some of its properties. In Section 3, we introduce an alternative pivoting strategy and present a proof of its stability. We summarize the paper in Section 4.

Notation. We will denote the size of the pivot for the Bunch and alternative pivoting strategy by s_B and s_A, respectively. For an exact value x, we denote the corresponding computed value by $\hat{x}$.

2. THE PIVOTING STRATEGY OF BUNCH

Let $T \in \Re^{n \times n}$ be a symmetric tridiagonal matrix with $\alpha_i, i = 1, \ldots, n$, on the diagonal and $\beta_j, j = 2, \ldots, n$, on the off-diagonal:

$$T = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & \beta_3 & \ddots & \vdots \\ 0 & \beta_3 & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_n \\ 0 & \cdots & 0 & \beta_n & \alpha_n \end{bmatrix}.$$

Denote the largest element of T in magnitude by σ, and partition T as

$$T = \begin{array}{c} \\ s \\ n-s \end{array} \begin{array}{c} \begin{array}{cc} s & n-s \end{array} \\ \left[\begin{array}{cc} B_1 & T_{21}^T \\ T_{21} & T_{22} \end{array} \right] \end{array} . \tag{1}$$

The computation of the LBL^T factorization involves choosing the dimension ($s = 1$ or 2) of the pivot B_1 at each stage. If B_1 is singular for both choices of s, then $\alpha_1 = 0$ and $\beta_2 = 0$, which implies that $T_{21} = 0$. Therefore the first row and column of T are in diagonal form, and the algorithm can proceed to the following stage. Thus B_1 can be assumed to be nonsingular. Then

$$T = \begin{bmatrix} I_s & 0 \\ T_{21}B_1^{-1} & I_{n-s} \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & T_{22} - T_{21}B_1^{-1}T_{21}^T \end{bmatrix} \begin{bmatrix} I_s & B_1^{-1}T_{21}^T \\ 0 & I_{n-s} \end{bmatrix} . \tag{2}$$

Let $S = T_{22} - T_{21}B_1^{-1}T_{21}^T \in \Re^{(n-s)\times(n-s)}$ be the Schur complement of B_1 in T. If $s = 1$, then $T_{21} = \beta_2 e_1$, where the unit vector $e_1 \in \Re^{n-1}$, and

$$S = T_{22} - (\beta_2^2/\alpha_1)e_1e_1^T.$$

The rank-one matrix $e_1e_1^T$ is nonzero only in the $(1,1)$ entry. Thus, S differs from T_{22} only in the leading entry, which is given by

$$\tilde{\alpha}_2 = \alpha_2 - (\beta_2^2/\alpha_1) = \Delta/\alpha_1,$$

where $\Delta = \alpha_1\alpha_2 - \beta_2^2$, which is the determinant of the leading 2×2 block of T. If $s = 2$, then the matrix $T_{21} \in \Re^{(n-2)\times 2}$ can be written as $T_{21} = \beta_3 e_1 e_2^T$ with the unit vectors $e_1 \in \Re^{n-2}$ and $e_2 \in \Re^2$. Then

$$S \;=\; T_{22} - \frac{1}{\Delta}\left(\beta_3 e_1 e_2^T\right) \begin{bmatrix} \alpha_2 & -\beta_2 \\ -\beta_2 & \alpha_1 \end{bmatrix} \left(\beta_3 e_2 e_1^T\right) \;=\; T_{22} - \left(\frac{\alpha_1\beta_3^2}{\Delta}\right) e_1e_1^T.$$

Again, S differs from T_{22} only in the $(1,1)$ entry, which is given by

$$\tilde{\alpha}_3 = \alpha_3 - (\alpha_1\beta_3^2/\Delta).$$

In both choices of pivot size, the Schur complement differs from T_{22} only in the $(1,1)$ entry, and, therefore, its tridiagonal structure is preserved. Thus, the LBL^T factorization can then be applied recursively.

The algorithm for determining the size s_B of the pivot B_1 using the Bunch pivoting strategy at each stage can be described sufficiently in the first stage of the factorization.

Algorithm 2.1. *(Bunch's pivoting strategy).*

$\sigma = \max\{|\alpha_i|, |\beta_j| \colon i, j = 2\colon n\}$
$\alpha = (\sqrt{5}-1)/2 \approx 0.62$
if $|\alpha_1|\sigma \geq \alpha\beta_2^2$
  $s_B = 1$
else
  $s_B = 2$
end

The constant α is a root of the equation $\alpha^2 + \alpha - 1 = 0$ and is chosen to equate the maximal element growth for both pivot sizes. A recursive application of Algorithm 2.1 yields a factorization $T = LBL^T$, where L is unit lower triangular and B is block diagonal with 1×1 and 2×2 blocks. Using Algorithm 2.1, we have the following properties for the Bunch pivoting strategy.

Property 1. *If $s_B = 2$, then the determinant $\varDelta$ of the B_1 satisfies $\varDelta \leq (\alpha - 1)\beta_2^2 < 0$. Thus,*

$$|\varDelta| \geq (1 - \alpha)\beta_2^2.$$

This property implies that Algorithm 2.1 will choose a 2×2 pivot only when its determinant is bounded away from zero.

Property 2. *The growth factor ρ_n of the LBL^T factorization with the Bunch pivoting strategy satisfies*

$$\rho_n \leq \frac{1}{2}(\sqrt{5} + 3) \approx 2.62.$$

Although the growth factor is bounded, it does not imply that the LBL^T factorization is stable the way it does for Gaussian elimination (see [7]).

3. ALTERNATIVE PIVOTING STRATEGY

A new pivoting strategy for symmetric tridiagonal matrices was motivated by the need to form the factors without having the full matrix. In other words, it is desired to factor T *as* its elements are computed. Although stable, efficient, and easily implemented, Bunch's pivoting strategy cannot be used for such a factorization because the largest element σ in T must be known *a priori*.

3.1. Algorithm

The pivot size at each step is chosen by minimizing the entry values in magnitude in the matrix L. Let $L_1 = T_{21}B_1^{-1}$ in Equation (2). If a 1×1 pivot is used, then the $(1,1)$ element of L_1 is β_2/α_1. If a 2×2 pivot is used, then the $(1,1)$ and $(1,2)$ elements of L_1 are $-\beta_2\beta_3/\varDelta$ and $\alpha_1\beta_3/\varDelta$ respectively. With elements of L_1 for a 2×2 pivot scaled by the constant α from the Bunch pivoting strategy, a 1×1 pivot is chosen if

$$\frac{|\beta_2|}{|\alpha_1|} \leq \max \alpha \left\{ \frac{|\beta_2\beta_3|}{|\varDelta|}, \frac{|\alpha_1\beta_3|}{|\varDelta|} \right\}, \tag{3}$$

and a 2×2 pivot is chosen otherwise. The choice of pivot size is summarized as follows:

Algorithm 3.1. *(Alternative pivoting strategy).*
$\alpha = (\sqrt{5} - 1)/2 \approx 0.62$
$\varDelta = \alpha_1\alpha_2 - \beta_2^2$
if $|\varDelta| \leq \alpha|\alpha_1\beta_3|$ or $|\beta_2\varDelta| \leq \alpha|\alpha_1^2\beta_3|$
$s_A = 1$
else
$s_A = 2$
end

Intuitively, Algorithm 3.1 chooses a 1×1 pivot if α_1 is sufficiently large relative to the determinant of the 2×2 pivot, i.e., a 1×1 pivot is chosen if a 2×2 pivot is relatively closer to being singular than α_1 is to zero. Like the Bunch pivoting strategy, Algorithm 3.1 avoids small $(1, 1)$ pivots and nearly singular 2×2 pivots. Scaling by α in (3) provides two properties that relate the two pivoting strategies.

Lemma 3. *If $s_A = 1$, then $s_B = 1$.*

Proof. If $|\alpha_1| \geq |\beta_2|$, then

$$|\alpha_1|\sigma \geq |\alpha_1|^2 \geq |\beta_2|^2 \geq \alpha|\beta_2|^2.$$

Thus $s_B = 1$. Otherwise, if $|\alpha_1| < |\beta_2|$, then $|\frac{\alpha_1}{\beta_2}||\alpha_1\beta_3| \leq |\alpha_1\beta_3|$. If $s_A = 1$ with $|\beta_2\Delta| \leq \alpha|\alpha_1^2\beta_3|$, then

$$|\Delta| \leq \alpha \left| \frac{\alpha_1^2}{\beta_2}\beta_3 \right| \leq \alpha|\alpha_1\beta_3|.$$

Thus $|\Delta| \leq \alpha|\alpha_1\beta_3|$ if $s_A = 1$. Now,

$$\alpha|\alpha_1||\beta_3| \geq |\Delta| = |\alpha_1\alpha_2 - \beta_2^2| \geq \beta_2^2 - |\alpha_1||\alpha_2|.$$

Thus,

$$\beta_2^2 \leq |\alpha_1|\big(\alpha|\beta_3| + |\alpha_2|\big) \leq |\alpha_1|\sigma(\alpha + 1).$$

Since $1/(\alpha + 1) = \alpha$, then $|\alpha_1|\sigma \geq \alpha\beta_2^2$. Therefore, $s_B = 1$. □

This lemma implies that whenever our pivoting strategy chooses a 1×1 pivot, the Bunch pivoting strategy chooses a 1×1 pivot as well. Lemma 3 also implies that if the Bunch pivoting strategy chooses a 2×2 block, the proposed pivoting strategy will choose a 2×2 block. The converse of Lemma 3 is not true, however, since for $\alpha_1 = \alpha_2 = 2$, $\beta_2 = 1$, and $\beta_3 = 0$, the pivot size for Algorithm 3.1 is $s_A = 2$ while $s_B = 1$.

In the case where pivots of different sizes are chosen, i.e., $s_A = 2$ and $s_B = 1$, we have the following lemma. Let s'_B be the pivot size in the subsequent step in the Bunch algorithm.

Lemma 4. *If $s_A = 2$ and $s_B = 1$, then $s'_B = 1$.*

Proof. Since $s_A = 2$, then $|\Delta| \geq \alpha|\alpha_1\beta_3|$. Recall that the Schur complement S differs from T_{22} in (1) only in the $(1, 1)$ position and that the entry in that position is $\tilde{\alpha}_2 = \Delta/\alpha_1$. Thus

$$|\tilde{\alpha}_2|\sigma = |\Delta/\alpha_1|\sigma \geq \alpha|\beta_3|\sigma \geq \alpha\beta_3^2.$$

Thus, $s'_B = 1$. □

These two lemmas imply that the block structure of B_A, the block-diagonal matrix from using the alternative pivoting strategy, and B_B, the block-diagonal matrix from using the Bunch pivoting strategy, are indeed similar. (In particular, there exists a unit lower triangular matrix L_B such that $B_A = L_B B_B L_B^T$.) The difference arises only when the Bunch pivoting strategy chooses two 1×1 blocks and Algorithm 3.1 chooses one 2×2 block. The Schur complement resulting from two 1×1 pivots is identical to the Schur complement resulting from one 2×2 pivot [5].

3.2. Growth factor

We have seen that the Schur complement S differs from T_{22} only in the $(1,1)$ element. Thus we need only examine the possible element growth in this position.

If $s_A = 1$, then $s_B = 1$ by Lemma 3 and therefore, $|\beta_2^2/\alpha_1| \le \sigma/\alpha$. Thus

$$|\tilde{\alpha}_2| = \left| \alpha_2 - \frac{\beta_2^2}{\alpha_1} \right| \le \sigma + \frac{\sigma}{\alpha}.$$

If $s_A = 2$, then $|\Delta| \ge \alpha|\alpha_1\beta_3|$ and

$$|\tilde{\alpha}_3| = \left| \alpha_3 - \frac{\alpha_1\beta_3^2}{\Delta} \right| \le \sigma + \frac{\sigma}{\alpha}.$$

The $(1,1)$ element does not affect the bounds on subsequent Schur complement $(1,1)$ elements, and therefore the growth is not cumulative. Thus, the growth factor ρ_n for this pivoting strategy satisfies

$$\rho_n = \frac{\max_{i,j,k}|S_{i,j}^{(k)}|}{\max_{i,j}|T_{i,j}|} \le 1 + \frac{1}{\alpha} = 2 + \alpha \approx 2.62,$$

which is the same bound on the growth factor as for the Bunch pivoting strategy. It can be shown that this bound on the growth factor is tight, just as it is in the Bunch algorithm.

3.3. Error Analysis

The error analysis presented in this section is similar to those of Higham in [7] and [8]. In this section, we introduce a method for solving 2×2 linear systems to show that a general result from [7] for an LBL^T factorization is applicable to Algorithm 3.1. The usual model of floating point arithmetic

$$fl(x \text{ op } y) = (x \text{ op } y)(1+\delta), \quad |\delta| \le u, \quad \text{op } = +, *, /,$$

is used, where u is the unit roundoff. The constant

$$\gamma_n \equiv \frac{nu}{1-nu},$$

is defined with the assumption that $nu < 1$. Note that for $c > 1$, $c\gamma_n \le \gamma_{cn}$.

Higham proves the following general result in [7]. Absolute values of matrices and inequalities between matrices are to be interpreted componentwise.

Theorem 5. *[Higham] Let block LDL^T factorization with any pivoting strategy be applied to a symmetric matrix $A \in \Re^{n\times n}$ to yield the computed factorization $PAP^T \approx \hat{L}\hat{B}\hat{L}^T$, where P is a permutation matrix and $\hat{B}$ has diagonal blocks of dimensions 1 or 2. Let $\hat{x}$ be the computed solution to $Ax = b$ obtained using the factorization. Assume that for all linear systems $Ey = f$ involving 2×2 pivots E the computed solution $\hat{y}$ satisfies*

$$(E + \Delta E)\hat{y} = f, \quad |\Delta E| \le (cu + O(u^2))|E|, \tag{4}$$

where c is a constant. Then

$$P(A + \Delta A_1)P^T = \hat{L}\hat{B}\hat{L}^T, \quad (A + \Delta A_2)\hat{x} = b,$$

where

$$|\Delta A_i| \le p(n)u\left(|A| + P^T|\hat{L}||\hat{B}||\hat{L}^T|P\right) + O(u^2), \quad i = 1, 2,$$

with p a linear polynomial.

If the matrix A in Theorem 5 has a fixed bandwidth (i.e., independent of n), then the polynomial $p(n)$ is of degree zero. In the case of the proposed pivoting strategy, since T is tridiagonal, $p(n)$ can be set to some constant c. Also, since row and column interchanges are not used in the LBL^T factorization, the permutation matrix $P = I$. We now discuss how Condition (4) is satisfied. For simplicity of notation, we let $E = B_1$ of size $s = 2$ in the partition of T in (1).

Given a 2×2 block B_1, we solve the system $B_1 y = f$ using the following algorithm:

Algorithm 3.2. *(Solving the 2×2 systems in Algorithm 3.1).*

if $|\alpha_1\alpha_2| \ge \alpha|\beta_2^2|$
    Use $B_1 = L_1 D_1 L_1^T$ to solve $B_1 y = f$.
else
    Use explicit inverse.
end

The following theorem shows that using Algorithm 3.2, Condition (4) is satisfied.

Theorem 6. *If a 2×2 linear system $Ey = b$ is solved using Algorithm 3.2, then the computed solution $\hat{y}$ satisfies Condition (4).*

Proof. If $|\alpha_1\alpha_2| \ge \alpha|\beta_2^2|$, then $s_B = 1$ when the Bunch pivoting strategy is applied to solve the 2×2 system. Thus the LDL^T factorization is stable even if $|\alpha_1| \le |\beta_2|$. For a 2×2 system, the backward error result

$$(E + \Delta E)\hat{y} = f, \quad |\Delta E| \le (7u + O(u^2))|\hat{L}_1||\hat{D}_1||\hat{L}_1| \tag{5}$$

can be easily shown using a proof similar to ([9], Theorem 9.4). Since $|\hat{L}_1||\hat{D}_1||\hat{L}_1| = |L_1||D_1||L_1^T| + O(u)$, then ΔE must satisfy the inequality in (5) with the exact factors on the right hand side. Now

$$|L_1||D_1||L_1^T| = \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & \frac{|\beta_2^2|}{|\alpha_1|} + \left|\alpha_2 - \frac{\beta_2^2}{\alpha_1}\right| \end{bmatrix} \le \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & \left(\frac{2}{\alpha} + 1\right)|\alpha_2| \end{bmatrix} \le (\sqrt{5} + 2)|E|.$$

Thus, if $|\alpha_1\alpha_2| \ge \alpha|\beta_2^2|$ and Algorithm 3.2 is used to solve $Ey = b$, then Condition (4) holds with $c = 7(\sqrt{5} + 2)$.

To show Condition (4) holds using an explicit inverse when $|\alpha_1\alpha_2| \le \alpha|\beta_2^2|$, we give an argument similar to that in [7]. Solving the linear system $Ey = b$ using an explicit inverse formula for E gives

$$y = \frac{1}{\beta_2\left(\frac{\alpha_1}{\beta_2}\cdot\frac{\alpha_2}{\beta_2} - 1\right)} \begin{bmatrix} \frac{\alpha_2}{\beta_2} & -1 \\ -1 & \frac{\alpha_1}{\beta_2} \end{bmatrix} b, \tag{6}$$

as done in LAPACK [1] and LINPACK [6]. A potential source of instability for this formula is the term

$$\mu = \left(\frac{\alpha_1}{\beta_2} \cdot \frac{\alpha_2}{\beta_2} - 1 \right), \tag{7}$$

whose computed value might be arbitrarily small. Thus, we must show that the relative error in μ is bounded away from 0. Using the notation θ_4 and δ_4 as in [7], we have

$$\hat{\mu} = \frac{\alpha_1}{\beta_2} \cdot \frac{\alpha_2}{\beta_2}(1+\theta_4) - (1+\delta_4), \quad |\theta_4| \le \gamma_4.$$

Now, since $|\alpha_1\alpha_2| \le \alpha|\beta_2^2|$,

$$\frac{|\alpha_1\alpha_2|}{\beta_2^2} \le \alpha,$$

and (7) implies that $|\mu| \ge 1-\alpha$. Thus

$$|\mu - \hat{\mu}| \le \gamma_4 \left(\frac{|\alpha_1\alpha_2|}{\beta_2^2} + 1 \right) \le \gamma_4(\alpha+1) \le \gamma_4 \left(\frac{1+\alpha}{1-\alpha} \right) |\mu| \le 5\gamma_4|\mu|.$$

Let Z be the 2×2 matrix in (6). Then $y = (\beta_2\mu)^{-1}Zb$. Thus we have the backward error result

$$\hat{y} = (\beta_2\mu)^{-1}\big(Z + \Delta Z\big)b, \quad |\Delta Z| \le \gamma_{50}|Z|.$$

Thus $b - E\hat{y} = -E((\beta_2\mu)^{-1}\Delta Z)b$ so that

$$\begin{aligned} |b - E\hat{y}| &\le \gamma_{50}|E||E^{-1}||b| \\ &\le \gamma_{50}|E||E^{-1}||E||y|. \end{aligned}$$

Now,

$$\begin{aligned} |E||E^{-1}||E| &\le \frac{1}{(1-\alpha)} \begin{bmatrix} \frac{|\alpha_1\alpha_2|}{\beta_2^2} + 1 & 2\frac{|\alpha_1|}{|\beta_2|} \\ 2\frac{|\alpha_2|}{|\beta_2|} & \frac{|\alpha_1\alpha_2|}{\beta_2^2} + 1 \end{bmatrix} \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & |\alpha_2| \end{bmatrix} \\ &\le \frac{1}{(1-\alpha)} \begin{bmatrix} \alpha+1 & 2\frac{|\alpha_1|}{|\beta_2|} \\ 2\frac{|\alpha_2|}{|\beta_2|} & \alpha+1 \end{bmatrix} \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & |\alpha_2| \end{bmatrix} \\ &= \frac{1}{(1-\alpha)} \begin{bmatrix} (\alpha+1)|\alpha_1| + 2|\alpha_1| & (\alpha+1)|\beta_2| + 2\frac{|\alpha_1\alpha_2|}{|\beta_2|} \\ 2\frac{|\alpha_1\alpha_2|}{|\beta_2|} + (\alpha+1)|\beta_2| & 2|\alpha_2| + (\alpha+1)|\alpha_2| \end{bmatrix} \\ &\le \frac{3+\alpha}{(1-\alpha)} \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & |\alpha_2| \end{bmatrix} \\ &\le 10|E|. \end{aligned}$$

Thus

$$|b - E\hat{y}| \le \gamma_{500}|E||y| \le \gamma_{500}|E|(|\hat{y}| + O(u)).$$

By the Oettli-Prager Theorem [10], ([8] Theorem 7.3),

$$(E + \Delta E)\hat{y} = b, \quad |\Delta E| \le \gamma_{500}|E| + O(u^2).$$

□

We have demonstrated that Condition (4) is satisfied when Algorithm 3.2 is used to solve linear systems involving 2×2 pivots. Therefore, Theorem 5 holds for an LBL^T factorization using Algorithm 3.1 as a pivoting strategy.

3.4. Normwise analysis

To show the stability of the LBL^T factorization using Algorithm 3.1, we must show that $|\hat{L}||\hat{B}||\hat{L}^T|$ is suitably bounded by T in some norm. Since $|\hat{L}||\hat{B}||\hat{L}^T| = |L||B||L^T| + O(u)$, it is sufficient to bound the product $|L||B||L^T|$ of the exact factors. We write

$$\begin{aligned} |L||B||L^T| &= \begin{bmatrix} I & \\ |L_{21}| & |L_S| \end{bmatrix} \begin{bmatrix} |B_1| & \\ & |B_S| \end{bmatrix} \begin{bmatrix} I & |L_{21}^T| \\ & |L_S^T| \end{bmatrix} \\ &= \begin{bmatrix} |B_1| & |B_1||L_{21}^T| \\ |L_{21}||B_1| & |L_{21}||B_1||L_{21}^T| + |L_S||B_S||L_S^T| \end{bmatrix}. \end{aligned}$$

Let $F = |L_{21}||B_1|$. If $s_A = 1$, then $F \in \Re^{n-1}$ with $\|F\|_\infty = |\beta_2| \le \sigma$. If $s_A = 2$, then $F \in \Re^{(n-2)\times 2}$, which is all zeros except for the first row given by

$$\begin{bmatrix} \dfrac{|\beta_2\beta_3|}{|\Delta|} & \dfrac{|\alpha_1\beta_3|}{|\Delta|} \end{bmatrix} \begin{bmatrix} |\alpha_1| & |\beta_2| \\ |\beta_2| & |\alpha_2| \end{bmatrix} = \begin{bmatrix} \dfrac{2|\alpha_1\beta_2\beta_3|}{|\Delta|} & \dfrac{|\beta_2^2\beta_3|}{|\Delta|} + \dfrac{|\alpha_1\alpha_2\beta_3|}{\Delta|} \end{bmatrix}.$$

Thus

$$\|F\|_\infty = \frac{2|\alpha_1\beta_2\beta_3|}{|\Delta|} + \frac{|\alpha_1\alpha_2\beta_3|}{|\Delta|} + \frac{|\beta_2^2\beta_3|}{|\Delta|}.$$

Since $s_A = 2$, $|\Delta| \ge \alpha|\alpha_1\beta_3|$, and therefore

$$\begin{aligned} \|F\|_\infty &\le \frac{2}{\alpha}|\beta_2| + \frac{|\alpha_1\alpha_2\beta_3|}{|\Delta|} + \frac{|\beta_2^2\beta_3|}{|\Delta|} \\ &\le \frac{2}{\alpha}|\beta_2| + \frac{1}{\alpha}|\alpha_2| + \frac{|\beta_2^2|}{\alpha|\alpha_1|}. \end{aligned} \tag{8}$$

If $|\alpha_1|\sigma \ge \alpha\beta_2^2$, i.e., $s_B = 1$, then

$$\begin{aligned} \|F\|_\infty &\le \frac{2}{\alpha}|\beta_2| + \frac{1}{\alpha}|\alpha_2| + \frac{1}{\alpha^2}\sigma \\ &\le (4\alpha + 5)\sigma. \end{aligned}$$

Otherwise, $s_B = 2$, and therefore $|\alpha_1\alpha_2| \le \alpha\beta_2^2$. By Property 1, $|\Delta| \ge (1-\alpha)\beta_2^2$. Thus from (8),

$$\begin{aligned} \|F\|_\infty &\le \frac{2}{\alpha}|\beta_2| + \frac{\alpha|\beta_2^2\beta_3|}{|\Delta|} + \frac{|\beta_2^2\beta_3|}{|\Delta|} \\ &\le \frac{2}{\alpha}|\beta_2| + \frac{\alpha}{1-\alpha}|\beta_3| + \frac{1}{1-\alpha}|\beta_3| \\ &\le (4\alpha + 5)\sigma. \end{aligned}$$

Therefore, for both pivot sizes, $\|F\|_\infty \leq 8\sigma$.

Now let $G = |L_{21}||B_1||L_{21}^T|$. If $s_A = 1$, then $\|G\|_\infty = \beta_2^2/|\alpha_1|$. By Lemma 3, $s_B = 1$ and $|\alpha_1|\sigma \geq \alpha\beta_2^2$. Therefore, $\|G\|_\infty \leq \sigma/\alpha$. If $s_A = 2$, then $|\Delta| \geq \alpha|\alpha_1\beta_3|$ and

$$\begin{aligned} \|G\|_\infty &= \frac{3|\alpha_1\beta_2^2\beta_3^2|}{|\Delta|^2} + \frac{|\alpha_1^2\alpha_2\beta_3^2|}{|\Delta|^2} \\ &\leq \frac{3|\beta_2^2\beta_3|}{\alpha|\Delta|} + \frac{|\alpha_1\alpha_2\beta_3|}{\alpha|\Delta|} \\ &\leq \frac{3|\beta_2^2|}{\alpha^2|\alpha_1|} + \frac{|\alpha_2|}{\alpha^2}. \end{aligned} \tag{9}$$

If $|\alpha_1|\sigma \geq \alpha\beta_2^2$, i.e., $s_B = 1$, then

$$\|G\|_\infty \leq \frac{3|\beta_2^2|}{\alpha^2|\alpha_1|} + \frac{|\alpha_2|}{\alpha^2} \leq \frac{3}{\alpha^3}\sigma + \frac{\sigma}{\alpha^2} = (7\alpha + 11)\sigma.$$

Otherwise, $s_B = 2$. Using Property 1, i.e., $|\Delta| \geq (1-\alpha)\beta_2^2$, and the inequality $|\Delta| \geq \alpha|\alpha_1\beta_3|$, we get from (9),

$$\|G\|_\infty \leq \frac{3|\beta_2^2\beta_3|}{\alpha|\Delta|} + \frac{|\alpha_1\alpha_2\beta_3|}{\alpha|\Delta|} \leq \frac{3}{\alpha(1-\alpha)}|\beta_3| + \frac{1}{\alpha^2}|\alpha_2| \leq (7\alpha + 11)\sigma.$$

Thus, for both pivot sizes, $\|G\|_\infty \leq 16\sigma$. Note that the bounds for $\|F\|_\infty$ and $\|G\|_\infty$ are the same as those in Higham's analysis in [8].

Now the matrices L_S and B_S are the LBL^T factors of the Schur complement S of B_1 in T. Now every Schur complement satisfies

$$\|S\|_M \leq \rho_n\|T\|_M,$$

where

$$\|A\|_M = \max_{i,j} |a_{ij}|.$$

From Section 3.2, the growth factor for this pivoting strategy satisfies $\rho_n \leq 2.62$. Using the bounds for $\|F\|_\infty$, $\|G\|_\infty$, and $\|S\|_\infty$ recursively, we obtain the bound

$$\||L||B||L^T|\|_M \leq 16 \times 2.62\|T\|_M < 42\|T\|_M.$$

The following result summarizes the stability of the LBL^T factorization using the pivoting strategy in Algorithm 3.1.

Theorem 7. *Let LBL^T factorization with the pivoting strategy of Algorithm 3.1 be applied to a symmetric tridiagonal matrix $T \in \Re^{n\times n}$ to yield the computed factorization $T \approx \hat{L}\hat{B}\hat{L}^T$, and let $\hat{x}$ be the computed solution to $Tx = b$ obtained using the factorization. Assume that all linear systems $Ey = f$ involving 2×2 pivots E are solved using Algorithm 3.2. Then*

$$T + \Delta T_1 = \hat{L}\hat{B}\hat{L}^T, \quad (T + \Delta T_2)\hat{x} = b,$$

where

$$\|\Delta T_i\|_M \leq cu\|T\|_M + O(u^2), \quad i = 1, 2,$$

where c is a constant.

4. CONCLUSION

We presented a normwise backward stable LBL^T factorization based on the Bunch algorithm for factoring a symmetric tridiagonal matrix T and solving a linear system $Tx = b$. Lemmas 3 and 4 showed that the proposed strategy for choosing the size of the pivots and the Bunch pivoting strategy are related. We showed that the strategies have the same bound on the growth factor, and using arguments similar to Higham's in [8] to demonstrate the stability of the Bunch factorization, we demonstrated the stability of the proposed algorithm as well. The key difference between the two strategies, however, is that the proposed algorithm does not need the largest entry in magnitude of the matrix to determine the pivot size.

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