EECS 275 Matrix Computation

Ming-Hsuan Yang

Electrical Engineering and Computer Science
University of California at Merced
Merced, CA 95344
http://faculty.ucmerced.edu/mhyang

Lecture 3

Overview

- Basic definition: matrix norm, range, rank, null space, matrix inverse
- Elementary analytical ad topological properties

Reading

- Chapter 1-3 *Numerical Linear Algebra* by Trefethen and Bau
- Chapter 2 of *Matrix Computations* by Gene Golub and Charles Van Loan
- Chapter 5 of *Matrix Analysis and Applied Linear Algebra* by Carl Meyer
- Chapter 2 of *Optimization by Vector Space Methods* by David Luenberger
- Chapter 3 and Chapter 4 of *Matrix Algebra From a Statistician's Perspective* by David Harville

Matrix norm

- Matrix form: a function, $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a matrix norm if the following properties hold:

$$\begin{array}{ll} f(A) \geq 0 & A \in \mathbb{R}^{m\times n} \\ f(A+B) \leq f(A) + f(B) & A, B \in \mathbb{R}^{m\times n} \\ f(\alpha A) = |\alpha| f(A) & \alpha \in \mathbb{R}, A \in R^{m\times n} \end{array}$$

- Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = \sum_{i=1}^{m} \|A(i,:)\|_2^2 = \sum_{j=1}^{n} \|A(:,j)\|_2^2 = \sqrt{\mathrm{tr}(A^\top A)}$$

- The Frobenius norm suggests

$$\|A\mathbf{x}\|_2^2 = \sum_{i=1}^{n} |A(i,:)\mathbf{x}|^2 \leq \sum_{i=1}^{n} \|A(i,:)\|_2^2 \|\mathbf{x}\|_2^2 = \|A\|_F^2 \|\mathbf{x}\|_2^2$$

$$\|A\mathbf{x}\|_2 \leq \|A\|_F \|\mathbf{x}\|_2$$

- For matrices A and B

$$\|AB\|_F \leq \|A\|_F \|B\|_F$$

Matrix norm (cont'd)

- p-norms:

$$\|A\|_p = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$$

  Note that matrix p-norms are defined in terms of vector p-norms.

- It is clear that $\|A\|_p$ is the p-norm of the largest vector obtained by applying A to a unit p-norm vector

$$\|A\|_p = \sup_{\mathbf{x}\neq\mathbf{0}} \left\| A\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_p}\right) \right\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$$

- When A is non-singular,

$$\min_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p = \frac{1}{\|A^{-1}\|_p}$$

- Frobenius norm and p-norms define families norms that

$$\|AB\|_p \leq \|A\|_p\|B\|_p \quad A\in\mathbb{R}^{m\times n}, B\in\mathbb{R}^{n\times q}$$

- For every $A\in\mathbb{R}^{m\times n}$ and $\mathbf{x}\in\mathbb{R}^n$, we have $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$

Matrix norm (cont'd)

- Not all matrix norms satisfy the sub-multiplicative property

$$\|AB\| \leq \|A\|\|B\|$$

- For example, if $\|A\|_\Delta = \max |a_{ij}|$, and

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

then $\|AB\|_\Delta > \|A\|_\Delta \|B\|_\Delta$

Matrix norm (cont'd)

- More generally, for any vector norm $\|\cdot\|_\alpha$ on $\mathbb{R}^n$ and $\|\cdot\|_\beta$ on $\mathbb{R}^m$, we have $\|A\mathbf{x}\|_\beta \leq \|A\|_{\alpha,\beta}\|\mathbf{x}\|_\alpha$ where $\|A\|_{\alpha,\beta}$ is a matrix norm defined by

$$\|A\|_{\alpha,\beta} = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_\beta}{\|\mathbf{x}\|_\alpha}$$

- We say that $\|\cdot\|_{\alpha,\beta}$ is subordinate to the vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$
- Since the set $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\alpha = 1\}$ is compact and $\|\cdot\|_\beta$ is continuous, it follows that

$$\|A\|_{\alpha,\beta} = \max_{\|\mathbf{x}\|_\alpha=1} \|A\mathbf{x}\|_\beta = \|A\mathbf{x}^*\|_\beta$$

for some $\mathbf{x}^* \in \mathbb{R}^n$ having unit α-norm

Matrix norm properties

- For $A \in \mathbb{R}^{m\times n}$, the Frobenius and p-norms satisfy certain important properties

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$$

$$\max_{i,j} |a_{ij}| \leq \|A\|_2 \leq \sqrt{mn}\max_{i,j} |a_{ij}|$$

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$$

$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$$

$$\|A\|_2 \leq \sqrt{\|A\|_1\|A\|_\infty}$$

$$\|A(i_1 : i_2, j_1 : j_2)\|_p \leq \|A\|_p$$

Example

- Given a matrix,

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 6 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$

- $\|A\|_1 = \max(3+2+0, 5+6+2, 7+4+8) = 19$ (maximum absolute column sum)
- $\|A\|_\infty = \max(3+5+7, 2+6+4, 0+2+8) = 15$ (maximum absolute row sum)
- $\|A\|_F = 14.3875$

Matrix 2-norm

Theorem

If $A \in \mathbb{R}^{m \times n}$*, then there exists a unit 2-norm* $\mathbf{z} \in \mathbb{R}^n$ *such that* $A^\top A\mathbf{z} = \mu^2 \mathbf{z}$ *where* $\mu = \|A\|_2$*.*

Proof.

Suppose $\mathbf{z} \in R^n$ is a unit vector such that $\|A\mathbf{z}\|_2 = \|A\|_2$. Since $\mathbf{z}$ maximizes the function

$$g(\mathbf{x}) = \frac{1}{2}\frac{\|A\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} = \frac{1}{2}\frac{\mathbf{x}^\top A^\top A\mathbf{x}}{\mathbf{x}^\top \mathbf{x}},$$

it follows that with by setting gradient $\nabla g(\mathbf{z}) = \mathbf{0}$,

$$\frac{\partial g(\mathbf{z})}{\partial z_i} = \left[\ (\mathbf{z}^\top \mathbf{z}) \sum_{j=1}^n (A^\top A)_{ij} z_j - (\mathbf{z}^\top A^\top A\mathbf{z}) z_i\ \right] / (\mathbf{z}^\top \mathbf{z})^2, \forall i$$

In vector notation, $A^\top A\mathbf{z} = (\mathbf{z}^\top A^\top A\mathbf{z})\mathbf{z}$. The theorem follows by setting $\mu = \|A\mathbf{z}\|_2$. □

Matrix 2-norm (cont'd)

- It implies that $\|A\|_2^2$ is a zero of the polynomial $p(\lambda) = \det(A^\top A - \lambda I)$, i.e.,

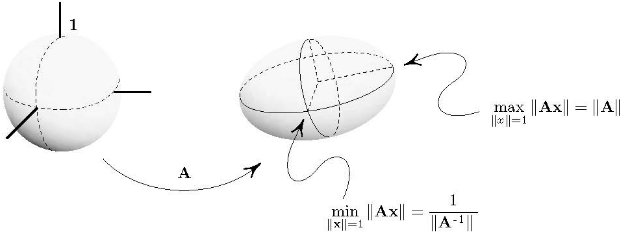
$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 = \sqrt{\lambda_{max}}$$

  where λ_{max} is the largest eigenvalue.
- $\|A\|_2$ is the square root of the largest eigenvalue of $A^\top A$.
- When A is non-singular

$$\|A^{-1}\|_2 = \frac{1}{\min_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2} = \frac{1}{\sqrt{\lambda_{min}}}$$

  where λ_{min} is the smallest eigenvalue of $A^\top A$

Matrix 2-norm (cont'd)



- $\|A\|$ represents the maximum extent to which a vector on the unit sphere can be stretched by A.
- $\frac{1}{\|A^{-1}\|}$ measures the extent to which a non-singular matrix A can shrink vectors on the unit sphere.

Matrix 2-norm (cont'd)

- Computation of matrix 2-norm is iterative and more complicated than that of the matrix 1-norm or ∞-norm.
- The order of magnitude of $\|A\|_2$ can be computed easily.

Corollary

If $A \in \mathbb{R}^{m\times n}$, then $\|A\|_2 \le \sqrt{\|A\|_1\|A\|_\infty}$

Proof.

If $\mathbf{z} \neq \mathbf{0}$ is such that $A^\top A\mathbf{z} = \mu^2\mathbf{z}$ with $\mu = \|A\|_2$, then
$\mu^2\|\mathbf{z}\|_1 = \|A^\top A\mathbf{z}\|_1 \le \|A^\top\|_1\|A\|_1\|\mathbf{z}\|_1 = \|A\|_\infty\|A\|_1\|\mathbf{z}\|_1$ □

Example

- Given the matrix
$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$
and two points $(1, 0)$ and $(0, 1)$
- The amplification factors for 1-norm is 4, and ∞-norm is 3
- The amplification factor for 2-norm is at least $\sqrt{8} \approx 2.8284$ as $(0, 1)$ is mapped to $(2, 2)$. In fact $\|A\|_2 \approx 2.9208$

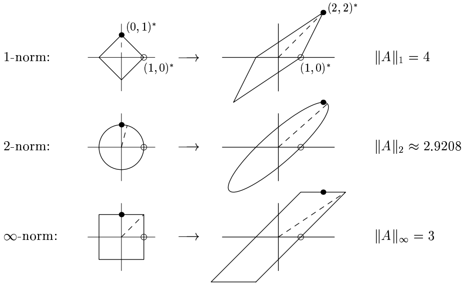


Figure 3.1. *On the left, the unit balls of* $\mathbb{R}^2$ *with respect to* $\|\cdot\|_1$, $\|\cdot\|_2$, *and* $\|\cdot\|_\infty$. *On the right, their images under the matrix* A *of* (3.7). *Dashed lines mark the vectors that are amplified most by* A *in each norm.*

Independence, subspace, and span

- A set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \in \mathbb{R}^m$ is linearly independent if $\sum_{j=1}^{n} \alpha_j \mathbf{a}_j = 0$ implies $\alpha_i = 0$
- Otherwise, nontrivial combination of the $\mathbf{a}_j$ is zero and $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly dependent
- A subspace of $\mathbb{R}^m$ is a subset that is also a vector space
- Given a collection of vectors, $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, the set of all linear combinations of vectors in a subspace referred to as the span of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$

$$\text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \sum_{j=1}^{n} \beta_j \mathbf{a}_j : \beta_j \in \mathbb{R}$$

- If $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is independent and $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, then $\mathbf{b}$ is a unique linear combination of the $\mathbf{a}_j$

Range, null space, and rank

- Range: The range of $A \in \mathbb{R}^{m \times n}$ is

$$\text{ran}(A) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x}, \mathbf{x} \in \mathbb{R}^n\}$$

- Null space:

$$\text{null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

- If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, then

$$\text{ran}(A) = \text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

- Rank: the number of linear independent columns of A.

$$\text{rank}(A) = \dim(\text{ran}(A))$$

- A is rank deficient if $\text{rank}(A) < \min\{m, n\}$.
- If $A \in \mathbb{R}^{m \times n}$, then

$$\dim(\text{null}(A)) + \text{rank}(A) = n$$

Matrix inverse

- The n-by-n identity matrix I_n is defined by the columns

$$I = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

where

$$\mathbf{e}_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k})^\top$$

- If A and X are in $\mathbb{R}^{n\times n}$ and satisfy $AX = I$, then X is the inverse of A and is denoted by A^{-1} (i.e., $AA^{-1} = I$).
- If A^{-1} exists, then A is said to be nonsingular. Otherwise, A is singular.
- Several matrix inverse properties

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ B^{-1} &= A^{-1} - B^{-1}(B-A)A^{-1} \end{aligned}$$

- Sherman-Morrison-Woodbury formula

$$(A + UV^\top)^{-1} = A^{-1} - A^{-1}U(I + V^\top A^{-1}U)^{-1}V^\top A^{-1}$$