Homework set #3
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This set covers chapters 12–15 of the book *Numerical Optimization* by Nocedal and Wright, 2nd ed.

From the book, the following exercises: 12.5–6, 12.14, 12.16, 12.18–22, 13.1, 13.5, 14.1, 14.10. In addition, the exercises below. There are no Matlab programming exercises. However, you may find it useful to plot the objective function and constraints with `fcontour`.

III.1. KKT conditions and trust regions. Apply the KKT conditions (th. 12.1) to the problem

$$\min_{\mathbf{p}\in\mathbb{R}^n} f + \mathbf{g}^T\mathbf{p} + \frac{1}{2}\mathbf{p}^T\mathbf{B}\mathbf{p} \text{ s.t. } \|\mathbf{p}\|_2 \le \Delta.$$

Relate your results to theorem 4.1 (book, p. 70).

III.2. Nonsmooth objective function and linear constraints. Write a linear program in standard form (eq. (13.1)) to find a point $\mathbf{x} \in \mathbb{R}^2$ satisfying $2x_1 + x_2 \le 10$, $\mathbf{x} \ge \mathbf{0}$ that minimises $|x_1 - 2x_2| + |-3x_1 - x_2|$. Use the KKT conditions to show that $\mathbf{x} = \binom{0}{0}$ is a solution.

III.3. Parameter-dependent problem. Determine the range of values for the parameter $a \in \mathbb{R}$ such that $\mathbf{x} = \binom{4}{3}$ is the optimal solution to $\max_{\mathbf{x}\in\mathbb{R}^2} ax_1 + x_2$ subject to $x_1^2 + x_2^2 \le 25,\ x_1 - x_2 \le 1,\ \mathbf{x} \ge \mathbf{0}$.

III.4. Quadratic-programming problem. Given the constrained optimisation problem

$$\min_{\mathbf{x}\in\mathbb{R}^3} (x_1+1)^2 + x_2^2 + x_3^2 \text{ subject to } x_3 = 0,\ x_1 \ge 0,\ x_1 + x_2 \ge 2$$

1. Solve it by writing the KKT and second-order conditions.
2. Solve the KKT system that results if we assume that the equality constraint is active and:
   (i) No inequality constraints are active.
   (ii) Exactly one inequality constraint is active (2 cases).
   (iii) All inequality constraints are active.

   Verify the solution corresponds to one of the cases in (ii).

III.5. Duality. Consider the LP

$$\min_{\mathbf{x}\in\mathbb{R}^2} x_1 + 2x_2 \text{ s.t. } x_1 + x_2 = 1,\ x_1 \ge \frac{1}{2},\ x_2 \ge 0.$$

Write the dual, solve both (primal and dual), draw them and show that the solutions agree and have the same objective value (strong duality). Pick a point that is primal-dual feasible and show that weak duality holds for it. Show that the Wolfe dual is equivalent to the dual.

III.6. Interior-point methods. Consider the LP

$$\min_{x\in\mathbb{R}} x \text{ s.t. } x \ge 0.$$

1. Write the KKT conditions and find the solution.
2. Determine the central path $\mathcal{C}$ and draw it.
3. Assuming the complementarity conditions equal $\sigma\mu$, write the function $\mathbf{F}$, its Jacobian $\mathbf{J}$, compute the full Newton step ($\alpha = 1$) and show it jumps directly to the central path from any initial point that is strictly feasible.
4. Assuming we always take full steps ($\alpha_k = 1\ \forall k$) starting from a point on the central path and with $\sigma_k = \sigma\ \forall k$, determine whether the interior-point method converges to the solution in the following cases: $\sigma \in (0,1)$, $\sigma = 0$, $\sigma = 1$. Determine the convergence rate, and how many iterations are required to reduce μ_k below $\epsilon\mu_0$.