EECS260 Optimization	Homework set $\#1$
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This set covers chapters 1–4 and A of the book Numerical Optimization by Nocedal and Wright, 2nd ed.

From the book, the following exercises: 2.2–2.3, 2.6–2.9, 2.13–2.15, 3.1, 3.3–3.4, 4.1, 4.10. In addition, the exercises below. The Matlab programming exercises are 3.1 and **I.4–I.9**. For exercises 2.2 and 2.9, plot your results with Matlab (for your own information); for exercise 3.1, estimate the convergence rate with the **convseq** function of exercise **I.4**.

I.1. Extended Rosenbrock function. This is an extension of exercise 2.1. Consider the function $f(\mathbf{x}) = a(x_2 - x_1^2)^2 + (1 - x_1)^2$. Compute the gradient $\nabla f(\mathbf{x})$ and the Hessian $\nabla^2 f(\mathbf{x})$. Find and classify (as maxima, minima and saddle points) the stationary points of f. Compute the condition number of the Hessian at the stationary points. Plot the contours of f in the rectangle $[-2, 2] \times [-1, 3]$ for a = 4, 10, 0, -1, -4 (you may need to select the contours manually to view the stationary point).

Repeat for the extended Rosenbrock function in n variables (where n is even):

$$f(\mathbf{x}) = \sum_{i=1}^{\lfloor n/2 \rfloor} \left(a(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2 \right).$$

I.2. Steepest descent & Newton directions in quadratic forms. Consider the function $f(\mathbf{x}) = \frac{1}{2}(x_1^2 - x_2^2)$.

- Sketch the contours of f around its stationary point.
- Compute the steepest descent direction and the Newton direction at \mathbf{x} . Verify the steepest descent direction is a descent direction for any \mathbf{x} . For what points \mathbf{x} is the Newton direction a descent direction?
- Compute explicitly \mathbf{x}_{k+1} given \mathbf{x}_0 for the steepest descent method using constant step size $\alpha_k = \alpha > 0$. What does it tend to for $k \gg 1$? Study the geometry of the problem for $\alpha < 1$, $\alpha = 0$ and $\alpha > 1$. Test the following starting points \mathbf{x}_0 : $\binom{0}{1}$, $\binom{1}{1}$, $\binom{2}{1}$, $\binom{0}{0}$. What happens with $\mathbf{x}_0 = \mathbf{0}$? What happens if using an exact line search?

Repeat for the function $f(\mathbf{x}) = x_1^2 + 2x_2^2$.

I.3. Convergence rate. Prove that the convergence of the sequences $(\frac{1}{k})$, (2^{-k}) , (k^{-k}) and (2^{-2^k}) is sublinear, linear, superlinear and quadratic, respectively. Tabulate them for a few values of k.

I.4. Empirical convergence rate. Consider a sequence $\mathbf{x}_0, \ldots, \mathbf{x}_K \in \mathbb{R}^n$ produced by an optimization method where the optimizer is \mathbf{x}^* . We can empirically estimate the rate of the method by fitting a line to the consecutive distances: $\log d_{k+1} = a + b \log d_k$ where $d_k = ||\mathbf{x}_k - \mathbf{x}^*||$; b will be the order of the method and $M = e^a$ the rate constant. Write a Matlab function convseq that takes as input a $(K + 1) \times n$ matrix \mathbf{X} (containing the sequence, rowwise) and a $1 \times n$ vector \mathbf{x}^* and plots the K pairs of consecutive log-distances and the least-squares line, and gives the value of the order b and the constant M. Apply it to the sequences of the previous exercise. What happens with the sub- and superlinear cases?

I.5. Coordinate descent. Program in Matlab the coordinate descent method using backtracking line search. Test it as in exercise 3.1. Estimate the convergence rate with the convseq function of exercise **I.4**.

I.6. Exact line search in quadratic forms. Program in Matlab the steepest descent method using exact line search for a quadratic function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x} + \mathbf{b}^T \mathbf{x}$ (use the result of exercise 3.3). Test it for $\mathbf{x} \in \mathbb{R}^2$ with matrices \mathbf{A} having condition numbers $\kappa(\mathbf{A}) = 2$, 10, 100 and plot the sequence of iterates as in fig. 3.7. Estimate the convergence rate with convseq.

Repeat but for the coordinate descent method.

I.7. Floating-point computations. Compare the true result with the numerical Matlab answer in the following computations.

- 1. Cancellation:
 - (a) Let x = 1. Evaluate in Matlab the following expressions: y x, $\sqrt{(y x)^2}$, $\sqrt{y^2 + x^2 2xy}$ when $y = 1 + \exp/2$, $y = 1 + \exp$, $y = 1 + \sqrt{\exp}$, $y = 1 + 10^3 \sqrt{\exp}$.
 - (b) Consider the equation $ax^2 + bx + c = 0$ with a = 1, $b = 2(1 + \epsilon)$, $c = 1 + 2\epsilon$. Compute in Matlab the two roots using the formula $x_{\pm} = (-b \pm \sqrt{b^2 4ac})/2a$ and then compute their difference $d = x_+ x_-$. Tabulate the relative error in d for $\epsilon = 10^0, 10^{-1}, 10^{-2}, \ldots, 10^{-17}$.
- 2. Ill-conditioning:
 - (a) Compute the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} = \begin{pmatrix} 1+\epsilon & 1\\ 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2+\epsilon\\ 2 \end{pmatrix}$, for $\epsilon \in \{\text{5eps}, \text{2eps}, \text{eps}, \text{eps}, \text{eps}, 2eps \}$.

I.8. Non-isolated strict local minimisers. Plot $x^4 \cos \frac{1}{x} + 2x^4$ around x = 0.

I.9. Modified Newton's method. Consider Newton's method with a Hessian modification (algorithm 3.2 in p. 48 of the book) $\mathbf{B}_k = \nabla^2 f(\mathbf{x}_k) + \lambda_k \mathbf{I}$, so that the search direction is $\mathbf{p}_k = -\mathbf{B}_k^{-1} \nabla f(\mathbf{x}_k)$, and has as extreme cases the pure Newton step for $\lambda = 0$ and the steepest descent direction for $\lambda \to \infty$. Assume that $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x}$ and $\mathbf{x}_k = (1, -3)$. Plot the contours of f, the gradient at \mathbf{x}_k and the search direction \mathbf{p}_k as a function of $\lambda \ge 0$ for the following three cases: $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$; $\mathbf{A} = \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$. What is the relation with theorem 4.1 (p. 70 in the book) and with exercise 4.1?