

This set covers chapters 12–15 of the book *Numerical Optimization* by Nocedal and Wright, 2nd ed.

From the book, the following exercises: 12.5–6, 12.14, 12.16, 12.18–22, 13.1, 13.5, 14.1, 14.10. In addition, the exercises below. There are no Matlab programming exercises. However, you may find it useful to plot the objective function and constraints with `fcontour`.

III.1. KKT conditions and trust regions. Apply the KKT conditions (th. 12.1) to the problem

$$\min_{\mathbf{p} \in \mathbb{R}^n} f + \mathbf{g}^T \mathbf{p} + \frac{1}{2} \mathbf{p}^T \mathbf{B} \mathbf{p} \text{ s.t. } \|\mathbf{p}\|_2 \leq \Delta.$$

Relate your results to theorem 4.1 (book, p. 70).

III.2. Nonsmooth objective function and linear constraints. Write a linear program in standard form (eq. (13.1)) to find a point $\mathbf{x} \in \mathbb{R}^2$ satisfying $2x_1 + x_2 \leq 10$, $\mathbf{x} \geq \mathbf{0}$ that minimises $|x_1 - 2x_2| + |-3x_1 - x_2|$. Use the KKT conditions to show that $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution.

III.3. Parameter-dependent problem. Determine the range of values for the parameter $a \in \mathbb{R}$ such that $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is the optimal solution to $\max_{\mathbf{x} \in \mathbb{R}^2} ax_1 + x_2$ subject to $x_1^2 + x_2^2 \leq 25$, $x_1 - x_2 \leq 1$, $\mathbf{x} \geq \mathbf{0}$.

III.4. Quadratic-programming problem. Given the constrained optimisation problem

$$\min_{\mathbf{x} \in \mathbb{R}^3} (x_1 + 1)^2 + x_2^2 + x_3^2 \text{ subject to } x_3 = 0, x_1 \geq 0, x_1 + x_2 \geq 2$$

1. Solve it by writing the KKT and second-order conditions.
2. Solve the KKT system that results if we assume that the equality constraint is active and:
 - (i) No inequality constraints are active.
 - (ii) Exactly one inequality constraint is active (2 cases).
 - (iii) All inequality constraints are active.

Verify the solution corresponds to one of the cases in (ii).

III.5. Duality. Consider the LP

$$\min_{\mathbf{x} \in \mathbb{R}^2} x_1 + 2x_2 \text{ s.t. } x_1 + x_2 = 1, x_1 \geq \frac{1}{2}, x_2 \geq 0.$$

Write the dual, solve both (primal and dual), draw them and show that the solutions agree and have the same objective value (strong duality). Pick a point that is primal-dual feasible and show that weak duality holds for it. Show that the Wolfe dual is equivalent to the dual.

III.6. Interior-point methods. Consider the LP

$$\min_{x \in \mathbb{R}} x \text{ s.t. } x \geq 0.$$

1. Write the KKT conditions and find the solution.
2. Determine the central path $\mathcal{C}$ and draw it.
3. Assuming the complementarity conditions equal $\sigma\mu$, write the function $\mathbf{F}$, its Jacobian $\mathbf{J}$, compute the full Newton step ($\alpha = 1$) and show it jumps directly to the central path from any initial point that is strictly feasible.
4. Assuming we always take full steps ($\alpha_k = 1 \ \forall k$) starting from a point on the central path and with $\sigma_k = \sigma \ \forall k$, determine whether the interior-point method converges to the solution in the following cases: $\sigma \in (0, 1)$, $\sigma = 0$, $\sigma = 1$. Determine the convergence rate, and how many iterations are required to reduce μ_k below $\epsilon\mu_0$.