IV.1. Quadratic programming. Consider the Markowitz model of portfolio optimization (example 16.1 in the book):

\[
\max_{\mathbf{x} \in \mathbb{R}^n} q(\mathbf{x}) = \mathbf{x}^T \mathbf{\mu} - \kappa \mathbf{x}^T \mathbf{G} \mathbf{x} \quad \text{subject to} \quad \mathbf{e}^T \mathbf{x} = 1, \; \mathbf{x} \geq 0
\]

where \( \kappa \geq 0, \; \mathbf{\mu} > 0, \; \mathbf{G} \) is symmetric positive definite and \( \mathbf{e} = (1, \ldots, 1)^T \). W.l.o.g. assume that the largest component of \( \mathbf{\mu} \) is the first \((\mu_1 \geq \mu_i, \; i = 1, \ldots, n)\).

(i) Suppose that \( g_{ii} > g_{jj}, \; i = 1, \ldots, n \). Show that, for a solution to be at one corner of the feasible polytope, namely \( \mathbf{x}^* = (1, 0, \ldots, 0)^T \), the following condition must hold:

\[
\kappa \leq \kappa_a \quad \text{with} \quad \kappa_a = \min_{i=2, \ldots, n} \frac{\mu_1 - \mu_i}{2(g_{11} - g_{ii})}.
\]

Interpret this solution. (Hint: apply the KKT and 2nd-order conditions to \( \mathbf{x}^* \).)

(ii) Suppose that the sum of each column of \( \mathbf{G}^{-1} \) is positive. Show that, for a solution to have only positive components \((\mathbf{x} > 0)\), the following condition must hold:

\[
\kappa > \kappa_c \quad \text{with} \quad \kappa_c = \frac{1}{2} \mathbf{e}^T \mathbf{G}^{-1} \mathbf{x} - \frac{1}{2} \mathbf{e}^T \mathbf{G}^{-1} \mathbf{e} \left( \min_{i=1, \ldots, n} \frac{(\mathbf{G}^{-1} \mathbf{e})_i}{(\mathbf{G}^{-1} \mathbf{e})_i} \right).
\]

What is the solution \( \mathbf{x}^* \)? What is the solution for \( \kappa \to \infty \)?

(iii) Consider a different portfolio optimization problem:

\[
\min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^T \mathbf{G} \mathbf{x} \quad \text{subject to} \quad \mathbf{\mu}^T \mathbf{x} \geq \kappa, \; \mathbf{e}^T \mathbf{x} = 1, \; \mathbf{x} \geq 0
\]

where \( \kappa \geq 0 \) as before. What is the largest \( \kappa \) for which this problem is feasible? (Hint: formulate as an LP \( \kappa = \max \mathbf{\mu}^T \mathbf{x} \) s.t. \( \mathbf{e}^T \mathbf{x} = 1, \; \mathbf{x} \geq 0 \); guess its solution and prove it is a solution using the KKT conditions.)

IV.2. Quadratic-penalty, augmented-Lagrangian, log-barrier and interior-point methods.

(i) Consider the constrained optimization problem \( \min_{\mathbf{x}} \; x_1^2 + x_2^2 \) s.t. \( x_1 + x_2 = 1 \). Find the solution \((\mathbf{x}^*, \lambda^*)\) of this problem using the KKT and second-order conditions. Write the quadratic-penalty function \( Q(\mathbf{x}; \mu) \) and its gradient \( \nabla_x Q(\mathbf{x}; \mu) \) and Hessian \( \nabla^2_{xx} Q(\mathbf{x}; \mu) \). Show that: \( Q(\mathbf{x}; \mu) \) has a single minimiser \( \mathbf{x}_k \) for each \( \mu_k > 0 \); this minimiser tends to the solution \( \mathbf{x}^* \) of the problem as \( \mu_k \to \infty \); the Lagrange multiplier estimate \( \lambda_k \approx -\mu_k c(\mathbf{x}_k) \) (eq. (17.10) in the book) tends to the Lagrange multiplier \( \lambda^* \) at the solution as \( \mu_k \to \infty \); the Hessian of the penalty function at the minimiser becomes progressively more ill-conditioned as \( \mu_k \to \infty \), i.e., \( \text{cond} (\nabla^2_{xx} Q(\mathbf{x}_k; \mu_k)) \to \infty \).

(ii) As in (i) but using the augmented-Lagrangian function \( \mathcal{L}_A(\mathbf{x}, \lambda; \mu) \) where the variable \( \lambda_k \) is updated as \( \lambda_{k+1} \leftarrow \lambda_k - \mu_k c(\mathbf{x}_k) \) (eq. (17.39) in the book). Also, prove that the sequence of iterates \((\mathbf{x}_k, \lambda_k)\) converges to the solution even if \( \mu_k \) is kept at a constant value \( \mu > 0 \) (i.e., we do not drive \( \mu_k \to \infty \)), and give the convergence order. How does it depend on \( \mu \)?

(iii) As in (i) but for the problem \( \min_{\mathbf{x}} \; x_1^2 + x_2^2 \) s.t. \( x_1 \geq 1 \) using the log-barrier function \( P(\mathbf{x}; \mu) \) and where the Lagrange multiplier estimate is \( \lambda_k \approx \frac{\mu_k}{c(\mathbf{x}_k)} \) (eq. (19.47) in the book).

(iv) For the problem in (iii), write the system of perturbed KKT equations \( \mathbf{F}(\ldots) = \mathbf{0} \) for an interior-point method; solve it and find the primal-dual central path; compute the Jacobian \( \mathbf{J} \) of \( \mathbf{F} \) and indicate how to obtain the Newton step; show that \( \mathbf{J} \) does not become progressively more ill-conditioned as we approach the solution.
IV.3. Convex equality-constrained quadratic programming using quadratic-penalty and augmented-Lagrangian methods. Consider the strictly convex equality-constrained QP:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T G x + c^T x \quad \text{subject to} \quad Ax = b$$

where $G$ is symmetric pd and $A_{m \times n}$ has full rank with $m < n$.

(i) Solve the QP, giving an explicit expression for the solution, i.e., the KKT point $(x^*, \lambda^*)$.

(ii) Write the quadratic-penalty function $Q(x; \mu)$, prove it is strictly convex and find its minimizer $x(\mu)$ and its Lagrange multiplier estimates $\lambda(\mu) = -\mu (Ax(\mu) - b)$ explicitly. Prove $x(\mu) \xrightarrow{\mu \to \infty} x^*$ and $\lambda(\mu) \xrightarrow{\mu \to \infty} \lambda^*$. Hint: use the Sherman-Morrison-Woodbury formula.

(iii) Write the augmented-Lagrangian function $L_A(x, \lambda; \mu)$, prove it is strictly convex and find its minimizer $x(\lambda; \mu)$ and the update for $\lambda$ explicitly. Prove $x(\lambda; \mu) \xrightarrow{\mu \to \infty} x^*$ and $\lambda(\mu) \xrightarrow{\mu \to \infty} \lambda^*$.

Now we keep the penalty parameter fixed to a value $\mu > 0$. Prove that iterating the update for $\lambda$ also converges to the solution, and determine the convergence order. Hint: $\|Mu\| \leq \|M\| \|u\|$, using the $L_2$ norm for vectors and matrices, i.e., $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$ and $\|M\| = \text{largest singular value of } M$.

IV.4. Generalized spectral problem. Prove the following theorem. Consider the optimisation problem

$$\max_X \text{tr} \left( XAX^T \right) \quad \text{s.t.} \quad XBX^T = I$$

where $X \in \mathbb{R}^{L \times N}$, $L < N$, $A, B \in \mathbb{R}^{N \times N}$, $A$ is symmetric and $B$ is symmetric positive definite. Let $C = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ have spectral representation $C = UAU^T$ and assume its eigenvalues are distinct though not necessarily positive. Then, the solution of (1) is unique and given by $X = U_L^TB^{-\frac{1}{2}}$, where $U_L = (u_1, \ldots, u_L)$ are the leading $L$ eigenvectors of $C$. 