CSE100 Algorithm Design and Analysis — Homework set #1

Fall semester 2020 — Miguel Á. Carreira-Perpiñán

Total possible marks: 100. This set covers chapters 1–4 of the textbook *Introduction to Algorithms*, 3rd. ed., by Cormen et al.

Exercise 1: asymptotic behavior (20 points).

1. (10 points) Assume you have two computers, C_A and C_B, capable of performing 10^6 and 10^8 operations per second, respectively. Both computers run a set of algorithms whose precise complexities $f(n)$ are given below. Determine the size n^* of the biggest input that can be processed in 1 second for each computer, as in the example.

$f(n)$	n^* for C_A	n^* for C_B
$\sqrt[3]{n}$	10^{18}	10^{24}
$10n+10$		
$\lg\lg n$		
$n\log_3 n + n$		
n^2		
$n * 2^n$		
2^n		
n^n		
$n!$		
4^n		

The precise complexity tells you how many operations are performed to solve an instance of size n. Assume each operation takes the same time and that the input sizes are natural numbers $1, 2, 3, \dots$

2. (10 points) (Exercise 3.1-1 in the textbook.) Let f and g be asymptotically nonnegative functions of $n = 1, 2, 3 \dots$ (that is, $f(n) \geq 0$ and $g(n) \geq 0$ if n is sufficiently large). Using the basic definition of Θ-notation, prove that $\max(f, g) = \Theta(f + g)$. *Hint*: find values for the constants c_1, c_2, n_0 in the definition of $\Theta(\cdot)$ and show it holds.

Exercise 2: (34 points). The following functions compute the factorial $n!$ of a natural number n using iteration and recursion, respectively:

```
ITERATIVE-FACTORIAL(n)
f = 1
for i = 2 to n
    f = i * f
return f
```

```
RECURSIVE-FACTORIAL(n)
if n == 1
    return 1
return n * RECURSIVE-FACTORIAL(n - 1)
```

1. (10 points) Write a recursive function to compute the factorial using divide and conquer as in Merge-Sort. That is, assuming $n \geq m$, write a function Factorial(n, m) that computes $n(n-1)(n-2)\cdots(m+1)m$. The top-level call will be Factorial$(n, 1)$.

2. (6 points) Prove the algorithm is correct, i.e., that Factorial$(n, 1) = n!$. *Hint*: use induction to prove that Factorial(n, m) is correct.

3. (6 points) Write the runtime $T(n)$ of Factorial$(n, 1)$ as a recurrence.

4. (6 points) Solve the recurrence and obtain $T(n)$ in asymptotic notation.
5. (6 points) Is the runtime $T(n)$ better than for ITERATIVE-FACTORIAL(n) or RECURSIVE-FACTORIAL(n)? Explain.

Exercise 3: sorting algorithms and runtime (10 points). (Exercise 2.3-6 in the textbook.) Observe that the **while** loop of lines 5–7 of the INSERTION-SORT procedure in Section 2.1 of the book uses a linear search to scan (backward) through the sorted subarray $A[1\,..\,j-1]$. Can we use a binary search (see Exercise 2.3-5) instead to improve the overall worst-case running time of insertion sort to $\Theta(n \lg n)$?

Exercise 4: recurrent equations (36 points). (This is based on textbook problems 4.1 and 4.3.) Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$.

1. (6 points) $T(n) = T(\sqrt{n}) + 1$
2. (6 points) $T(n) = 2T(n/4) + 1$
3. (6 points) $T(n) = 7T(n/2) + \Theta(n^2)$
4. (6 points) $T(n) = 2T(2n) + n^2$
5. (6 points) $T(n) = T(n-1) + n$, base case: $T(0) = 0$
6. (6 points) $T(n) = 3T(n/4) + n \log n$

Justify your answers. If you use the master theorem, specify which case and show that its hypotheses are satisfied. If you use induction then you should prove induction hypothesis.

Bonus exercise: recurrent equations (10 points). Consider the recurrence

$$T(n) = aT(n/b) + f(n).$$

Case 3 of the master theorem requires that $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant $\epsilon > 0$ and that $f(n)$ satisfies the regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n. Prove that the regularity condition is always satisfied if $f(n) = n^k$. (This means we don't need to check it when f is a polynomial.)

Bonus exercise: recurrent equations (30 points). Consider the following particular type of recurrence:

$$T(n) = \begin{cases} 1, & n = 1 \\ aT(n/b) + n^c, & n > 1 \end{cases}$$

where $a \geq 1$ and $b > 1$ are integers, and $k = \log_b n$ is integer (that is, we can only pick sizes n of the form $n = b^k$ where $k \geq 0$ is an integer). Use mathematical induction to prove that

$$T(n) = \begin{cases} n^c(1 + \log_b n), & \text{if } c = \log_b a \\ \frac{\frac{a}{b^c} n^{\log_b a} - n^c}{\frac{a}{b^c} - 1}, & \text{if } c \neq \log_b a. \end{cases}$$

Hint: as a simpler example, see exercise 2.3-3.

Consequently, prove the master theorem for this recurrence, that is, prove that

$$T(n) = \begin{cases} \Theta(n^{\log_b a}), & \text{if } c < \log_b a \\ \Theta(n^c \lg n), & \text{if } c = \log_b a \\ \Theta(n^c), & \text{if } c > \log_b a. \end{cases}$$