CSE100 Algorithm Design and Analysis
Spring semester 2013

Homework set #3
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Total possible marks: 100. Homeworks must be solved individually. This set covers chapters 15–16 of the textbook *Introduction to Algorithms*, 3rd. ed., by Cormen et al.

Note: for all numerical exercises, give only the final result. However, give a brief explanation of how you do the computation. For example, for exercise 2.2 about the matrix-chain multiplication, the caption of fig. 15.5 in the textbook illustrates how entry $m[2,5]$ is computed.

Exercise 1: segmented least squares problem (35 points). In the usual least squares problem, we are given n points in the plane, $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n) \in \mathbb{R}^2$, and we want to find a line $y = ax+b$ that minimizes the sum of squared errors

$$E(a,b) = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

(see left panel in the figure). The optimal line is given by $a = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2}$ and $b = \overline{y} - a\overline{x}$, where we define the following moments:

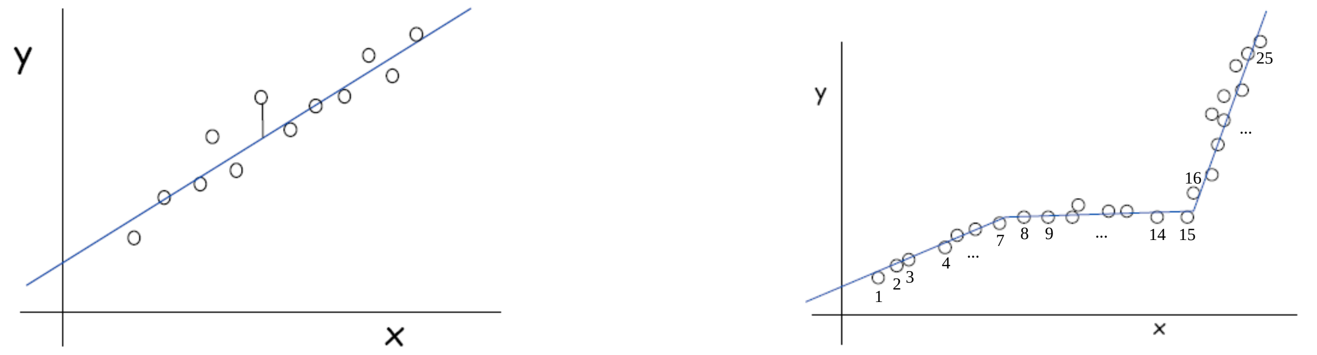
$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad \overline{x^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^2, \quad \overline{xy} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i.$$

For the case $n = 1$, of the many lines with zero error we take $a = 0$ and $b = y_1$ for definiteness.

1. (2 points) For the least-squares problem on n points, give the run time as a function of n (using order notation) of evaluating $E(a,b)$ (for given values of a, b).

In the *segmented least squares problem*, the points are assumed to lie roughly on a sequence of several line segments (rather than a single segment). We are given n points in the plane, $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$, with $x_1 < x_2 < \cdots < x_n$, and we want to find a sequence of lines that minimizes a tradeoff function $\mathcal{E} + cL$, where $1 \le L < n$ is the number of segments, $\mathcal{E} = \sum_{l=1}^{L} E_l$ is the sum of the sums of squared errors within each segment, and $c > 0$ is a given constant. Each segment consists of a consecutive sequence of points, and each point belongs to one and only one segment. This allows us to achieve a good fit of the segments to the points ($\mathcal{E}$), but also to have as few segments as possible (L). For example, for the right panel in the figure (for a certain c value), the optimal solution consists of $L = 3$ segments consisting of points 1–7, 8–15 and 16–25, respectively, and it has a cost $\mathcal{E} + cL$ equal to

$$E_1 + E_2 + E_3 + 3c = \sum_{i=1}^{7} (y_i - (a_1x_i + b_1))^2 + \sum_{i=8}^{15} (y_i - (a_2x_i + b_2))^2 + \sum_{i=16}^{25} (y_i - (a_3x_i + b_3))^2 + 3c.$$



2. (3 points) For the segmented least-squares problem, give a lower bound on the run time of the brute-force approach that examines all possible solutions (i.e., all possible segmentations).

Solve the segmented least squares problem using dynamic programming:

3. (10 points) Write the value of an optimal solution as a recursive expression using the value of optimally solved subproblems. *Hint*: the optimal substructure is very similar to the rod-cutting problem.

4. (10 points) Write pseudocode that implements this solution top-down using memoization, and prints the solution. Give its run time.

5. (10 points) Write pseudocode that implements this solution bottom-up by filling in a table in order, and prints the solution. Give its run time.

Note: one of the lab assignments will ask you to implement the segmented least squares dynamic programming algorithm.

Exercise 2: matrix-chain multiplication (25 points).

1. (7 points) (Exercise 15.2-2 in the textbook.) Give a recursive algorithm MATRIX-CHAIN-MULTIPLY(A, s, i, j) that actually performs the optimal matrix-chain multiplication, given the sequence of matrices $\langle A_1, A_2, \ldots, A_n \rangle$, the s table computed by procedure MATRIX-CHAIN-ORDER, and the indices i and j. (The initial call would be MATRIX-CHAIN-MULTIPLY$(A, s, 1, n)$.) Assume you can use a function MATRIX-MULTIPLY(A, B) that returns a matrix $C = AB$.

2. (10 points) Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $\langle 20, 10, 5, 30, 10 \rangle$. Fill in the m and s tables as in fig. 15.5 in the textbook.

3. (8 points) Give the result obtained using a greedy algorithm that picks the smallest local cost $p_{i-1}p_kp_j$ over $i \leq k < j$. Comment on whether this greedy algorithm always solves correctly the matrix-chain multiplication problem or not.

Exercise 3: longest common subsequence (15 points).

1. (12 points) (Exercise 15.4-3 in the book.) Give a memoized version of LCS-LENGTH (see p. 394) that runs in $O(mn)$ time.

2. (3 points) Which version is faster, top-down using memoization or bottom-up by filling in the table? *Hint*: consider which subproblems are computed.

Exercise 4: Huffman codes (25 points). The following table gives the frequencies of each character in a file with 1 000 characters.

α	β	γ	δ	ϵ	ζ	η	θ
50	10	80	150	5	30	660	15

1. (2 points) How many bits do we need to store the file if using a fixed-length code?

2. (8 points) Show the code built by the Huffman algorithm, both as a tree and as a list (character, codeword).
3. (2 points) How many bits do we need to store the file with this Huffman code?

Assume now that all the characters have the same frequency (= 125).

4. (8 points) Show the code built by the Huffman algorithm, both as a tree and as a list (character,codeword).
5. (2 points) How many bits do we need to store the file with this Huffman code?
6. (3 points) Generalizing from this example, what can you say about the code built by the Huffman algorithm for an alphabet with $n = 2^b$ characters each of which occurs with equal frequency f? Explain your answer.

Bonus exercise: longest common subsequence (15 points). Construct the matrices c and b as in fig. 15.8 in the textbook for the words CHEMISTRY and ENORMITY, and give a longest common subsequence for them.