Total possible marks: 100. Homeworks must be solved individually. This set covers chapters 6–12 of the textbook Introduction to Algorithms, 3rd. ed., by Cormen et al.

Exercise 1: heaps (5 points). (Exercise 6.5-8 in the book.) The operation `Heap-Delete(A, i)` deletes the item in node `i` from heap `A`. Give an implementation of `Heap-Delete` that runs in \( O(\lg n) \) time for an \( n \)-element max-heap.

Exercise 2: quicksort (21 points). (Combination of exercises 7.1-2, 7.2-2 and problem 7.2 in the book.) Consider an array `A` with \( n \) elements, and refer to the pseudocode for the algorithms `Partition` (p. 171), `Quicksort` (p. 171) and `Randomized-Quicksort` (p. 179).

1. (3 points) What value of \( q \) does `Partition` return when all elements in the array \( A[p..r] \) have the same value? Modify `Partition` so that \( q = \lfloor (p + r)/2 \rfloor \) when all elements in the array \( A[p..r] \) have the same value. Hint: this can be done by adding a short piece of pseudocode just before the `return` statement in `Partition`. You may give just this piece of pseudocode as the solution (no need to write the rest).

2. (3 points) What is the running time of `Quicksort` when all elements of array `A` have the same value? Give the recurrence explicitly.

3. (2 points) What is the running time of `Randomized-Quicksort` when all elements of array `A` have the same value? Explain.

4. (7 points) The `Partition(A, p, r)` procedure returns an index \( q \) such that each element of \( A[p..q-1] \) is less than or equal to \( A[q] \) and each element of \( A[q+1..r] \) is greater than \( A[q] \). Modify the `Partition` procedure to produce a procedure `Partition'`(A, p, r), which permutes the elements of \( A[p..r] \) and returns two indices \( q \) and \( t \), where \( p \leq q \leq t \leq r \), such that

   - all elements of \( A[q..t] \) are equal,
   - each element of \( A[p..q-1] \) is less than \( A[q] \), and
   - each element of \( A[t+1..r] \) is greater than \( A[q] \).

Like `Partition`, your `Partition'` procedure should take \( \Theta(r - p) \) time. Explain briefly how your `Partition'`(A, p, r) works.

5. (3 points) Give a loop invariant for `Partition'` that would allow one to prove its correctness. You don’t have to prove correctness, just state the invariant.

6. (3 points) Modify the `Randomized-Quicksort` procedure to call `Partition'`, and name the new procedure `Randomized-Quicksort'`. Then modify the `Quicksort` procedure to produce a procedure `Quicksort'`(A, p, r) that calls `Randomized-Partition'` and recurses only on partitions of elements not known to be equal to each other.
Exercise 3: decision tree for comparison sorts (15 points). Consider the Selection-Sort(A) algorithm (exercise 2.2-2 in the book), using the following pseudocode:

\begin{verbatim}
Selection-Sort(A)
1  n = A.length
2  for j = 1 to n - 1
3      smallest = j
4      for i = j + 1 to n
5          if A[i] < A[smallest]
6              smallest = i
7      exchange A[j] with A[smallest]
\end{verbatim}

1. (5 points) Draw the decision tree corresponding to Selection-Sort when running on an array of \( n = 3 \) elements \( A = \langle a_1, a_2, a_3 \rangle \) as in fig. 8.1 (keep “\( \leq \)” on the left child and “\( > \)” on the right child).

2. (2 points) Mark the execution path followed for the array \( A = \langle 6, 4, 2 \rangle \), as in fig. 8.1.

3. (2 points) For the tree you drew, what is the depth for the best and worst cases? Comment on the result.

4. (3 points) For a tree corresponding to an array with \( n \) elements, what would be the depth for the best and worst cases? Comment on the result.

5. (3 points) For the tree you drew, how many leaves does it have? Compare this with \( n! \) and comment on the result.

Exercise 4: hash tables (27 points). Consider a hash table with \( m = 11 \) slots and using the hash function \( h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor \) where \( A = (\sqrt{5} - 1)/2 \) and \( k \) is a natural number. Consider the keys \( k = 3, 6, 14, 71, 74, 60, 51, 7, 1 \) (in that order).

1. (3 points) Give \( h(k) \) for each of those keys.

Now, consider inserting those keys in the order given above into the hash table. Show the final table in these two cases:

2. (12 points) Chaining using as hash function \( h(k) \).

3. (12 points) Open addressing using linear probing and the same hash function \( h(k) \).

Exercise 5: hash tables (10 points). We have a hash table \( T_1 \) that uses chaining to resolve collisions; it has \( m \) slots and contains currently \( n \) elements. We have another hash table \( T_2 \) that uses open addressing; it contains currently \( n \) elements and occupies exactly the same amount of memory as \( T_1 \). Now, we execute a search for a key \( k \) which is in neither table. On average, in which of the two tables does the search take fewer operations? \textit{Hint:} assume that pointers and keys occupy each one word of memory, and that an unsuccessful search requires on average \( 1 + \alpha(T_1) \) operations for \( T_1 \) and \( \frac{1}{1-\alpha(T_2)} \) operations for \( T_2 \), where \( \alpha \) is the load factor.
Exercise 6: binary search trees (22 points).

1. (10 points) Starting from an empty binary search tree, draw the final tree resulting from the insertion of the following keys: 14, 10, 20, 6, 15, 2, 5, 30, 12, 9 (in that order).

2. (2 points) Do the inorder tree walk, printing the resulting keys.

3. (10 points) Starting from the tree obtained in the former question, draw the tree resulting from the removal of the following keys: 10, 12 (in that order).

Bonus exercise: bucket sort (20 points).  (Exercise 8.4-4 in the book.) We are given $n$ points in the unit circle, $p_i = (x_i, y_i)$, such that $0 < x_i^2 + y_i^2 \leq 1$ for $i = 1, 2, \ldots, n$. Suppose that the points are uniformly distributed; that is, the probability of finding a point in any region of the circle is proportional to the area of that region. Design an algorithm with an average-case running time of $\Theta(n)$ to sort the $n$ points by their distances $d_i = \sqrt{x_i^2 + y_i^2}$ from the origin.  

*Hint:* design the bucket sizes in BUCKET-SORT to reflect the uniform distribution of the points in the unit circle.