Mean of $a + bx = a + bM_x$  \( \text{(i)} \)

Var of $a + bx = b^2 \text{Var}(x)$

\[
\int (a + bx) f(x) \, dx = \int af(x) \, dx + \int bxf(x) \, dx
\]

\[
= a \int f(x) \, dx + b \int xf(x) \, dx
\]

\[
= a + b M_x
\]

Variance:

\[
\int \left[ a + bx - (a + bM_x) \right]^2 f(x) \, dx
\]

\[
= \int b^2x^2 - 2b^2xM_x + b^2(M_x^2) f(x) \, dx
\]

\[
= b^2 \int (x - M_x)^2 f(x) \, dx
\]

\[
= b^2 \sigma_x^2
\]
$f(x, y)$ (generic function)

Example $f(x, y) = x^2 y$

Partial derivative WRT $x$ regard as a constant:

$$\frac{\partial f(x, y)}{\partial x} = 2xy$$

$$\frac{\partial f(x, y)}{\partial y} = x^2$$

Integration:

$$f(x, y) = 2x$$

$$\int \int_{y} 2x \, dx \, dy = \int_{y} x^2 \, dy = \int_{y} x^2 \, dy$$

$$= \frac{x^2 y}{1} = x^2 y$$
2nd and cross derivatives

\[ f(x,y) = x^2y \quad \frac{\partial f(x,y)}{\partial x} = 2xy \]

\[ \frac{\partial^2 f(x,y)}{\partial x^2} = (2xy)_y = 2y \]

\[ \frac{\partial f(x,y)}{\partial y} = x^2 \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad x^2 - 0 \]

\[ \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial (2xy)}{\partial y} = 2x \]

\[ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial (x^2)}{\partial x} = 2x \]
Joint Distributions

We have 2 RVs \( x \) and \( y \), their joint probability distribution determined by \( F(x,y) = P(x \leq x, y \leq y) \).

Densities:

Discrete case (pmf)

Example: \( x \) and \( y \) are independently and identically distributed binomial \((3, \frac{1}{2})\).

<table>
<thead>
<tr>
<th>Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>X</td>
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<td></td>
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<td></td>
</tr>
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<tr>
<td>3</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Marginal probs for $X = P(X = x) = \sum_{y=0}^{3} P(X=x, Y=y)$

Non i.i.d example:

$X \sim \text{Bernoulli}(p = 0.6)$

$Y \sim \text{binomial}$ outcome of that Bernoulli trial + two more

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.192</td>
</tr>
<tr>
<td>3</td>
<td>0.216</td>
<td>0.288</td>
</tr>
</tbody>
</table>

$$P(Y=y \mid X=x) = \frac{P(Y=y, X=x)}{P(X=x)}$$
Continuous bivariate distns

Replace \( \Sigma \) with \( \int \).

Joint density \( f(x, y) \) if the fn is strictly non-neg, piecewise continuous, and

\[
F(x, y) = \int_{x}^{\infty} \int_{y}^{\infty} f(x', y') \, dx' \, dy' = 1.
\]

\[
f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)
\]

Strange example: \( X \sim \text{N}(0, 1) \), \( Y \sim \text{U}(0, 1) \),

\( X \) and \( Y \) are independent.

Result:

\[
f(x, y) = \begin{cases} 
\phi(x) & \text{if } 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
This is circular, but good practice for calculus of two variable.
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) \, dx \, dy = \int_{-\infty}^{\infty} 1 \, dy
\]

\[
= \frac{y'}{1} \bigg|_{0}^{\infty} = 1 - 0 = 1
\]

It's a density!