Approximation of Transient 1D Conduction in a Finite Domain Using Parametric Fractional Derivatives

A solution to the problem of transient one-dimensional heat conduction in a finite domain is developed through the use of parametric fractional derivatives. The heat diffusion equation is rewritten as anomalous diffusion, and both analytical and numerical solutions for the evolution of the dimensionless temperature profile are obtained. For large slab thicknesses, the results using fractional order derivatives match the semi-infinite domain solution for Fourier numbers, \( F_o \in [0, 1/16] \). For thinner slabs, the fractional order solution matches the results obtained using the integral transform method and Green’s function solution for finite domains. A correlation is obtained to display the variation of the fractional order \( p \) as a function of dimensionless time \( (F_o) \) and slab thickness \( (\xi) \) at the boundary \( \xi = 0 \). [DOI: 10.1115/1.4003544]

Keywords: heat conduction, finite domain, fractional derivatives, anomalous diffusion

1 Introduction

The classic one-dimensional transient heat conduction problem [1–7] is well-described by a parabolic partial differential equation on time and space, and its mathematical model is based on two fundamental conditions: (1) the conduction characteristics of the medium are well characterized by nearly constant thermal diffusivity \( \alpha \), and (2) the time scale for temperature variations within the medium is much larger than the time scale of phonon and/or electronic transport through the lattice or molecular structure of the medium. When the first condition is not met, a nonlinear version of the diffusion equation is required [8–10], and when the second condition is not satisfied, a hyperbolic diffusion equation is more appropriate [11–14]. However, when both of the conditions are met, the parabolic heat diffusion equation \( \alpha \frac{\partial^2 T}{\partial t^2} = \frac{\partial T}{\partial t} \) plus suitable initial and boundary conditions form a well-posed boundary value problem for which a host of analytical and numerical solutions has been developed [5,15].

Analytical solutions of the diffusion equation occupy a special place in the methods of Mathematical Physics, because a large number of solution methods are applicable, including Green’s function solutions, separation of variables, integral transformations, series solutions, among others [15–19]. Transient heat conduction problems have been solved using Green’s functions since the second part of the 20th century [5,6]. During the 1980s, error-estimated methods were introduced by Ozisik [15,20], Beck [16], Beck et al. [7], and Cotta [21] for boundary value problems. In this work, we endeavor to expand a less-known mathematical method beyond its original range of applications. The method in question is the direct fractional-calculus solution applied to a one-dimensional finite domain undergoing transient conduction. The semi-infinite version of this problem, where one of the boundaries is unaffected by changes in the other boundary, yields a direct solution for which the dimensionless heat flux is equal to the dimensionless half-derivative of temperature with respect to time evaluated at the boundary located at the origin of the coordinate system [22]. In the present work, we explore the parametric order of this derivative (which is 1/2 for the limiting semi-infinite problem) for a range of slab thicknesses.

The integral transform technique is shown by Mikhailov and co-workers [22,28] with three examples solving nonlinear models with prescribed accuracy and lumped-differential formulations. Recent applications of integral transform include natural and forced convection, external flows, porous cavities, and contaminant dispersion in rivers [29–32]. The integral transform method has been used not only to model physical processes in nature but also as a benchmark for the development of other solution methodologies.

Although fractional derivatives have been considered since the late 17th century [33,34], practical applications of fractional calculus did not appear in the literature until the book by Oldham and Spanier [22]. Since then, many applications using fractional order derivatives have been discussed in the literature. Comprehensive reviews of the methods employed can be found in Refs. [22,35–37]. Oldham and Spanier [22] explored diffusion applications of the differintegral concept. A comparative analysis between the traditional integer differential model and fractional order differential equations is presented by Podlubny [35]. Kilbas et al. [37] provided a fractional model for the superdiffusion equation. Coimbra and co-workers [34,38,39] generalized the concept of a variable order operator with well-posed initial conditions (lower limit of integration) and applied this operator to a number of problems in dynamics and viscoplasticity. Other applications of fractional order modeling for a variety of topics can be found in literature [40–46]. Recently, Aoki et al. [47] showed that time-dependent temperatures in a transient, conductive thermal system can be approximately modeled by a fractional order differential equation. Coimbra and Rangel [48], and later Khalponov and Zak-
iev [49] showed that the diffusion equation can be represented in the form of the product of two factors. Therefore, the one-dimensional heat diffusion equation is rewritten using the exponential rule [22] as follows (anomalous diffusion equation):

\[ \alpha^\mu \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial \theta^2} \]  

(1)

where \( \mu \) represents the order of the fractional derivative with respect to time. Three distinct methods are used as benchmarks for solving the initial-boundary value problem described above: For the finite domain, the integral transform technique [20,25] and Green’s function solution [16] are utilized; and for the semi-infinite domain, the similarity solution is used for comparison [50,51]. The fractional order solution has the capability to match the results obtained with these three analytical methodologies.

2 Description of the Problem

The heat diffusion equation plus initial and boundary conditions describe transient heat conduction in a homogeneous body. Considering a finite slab of thickness \( H \), the distribution of the temperature \( T \) along the domain is a function of \( x \) and \( t \) only [18]. The plane at \( x=0 \) is exposed to a constant heat flux \( q'' \) (Neumann boundary condition), and a Dirichlet boundary condition is imposed at \( x=H \) with the same value of the initial temperature \( T_i \), as shown in Fig. 1. Thus, the transient boundary value problem is formulated as \( \frac{\partial T}{\partial x} = \alpha (\frac{\partial^2 T}{\partial x^2}) \), with \( \alpha \) being the thermal diffusivity, and where the initial and boundary conditions are given by

\[ T(x,0) = T_i, \quad \left. \frac{\partial T}{\partial x} \right|_{x=0} = q'', \quad T(x=H,t) = T_i \]  

(2)

The nondimensional heat diffusion equation becomes

\[ \frac{\partial \theta}{\partial F_o} = \frac{\partial^2 \theta}{\partial \xi^2} \]  

(3)

where the dimensionless initial and boundary conditions take the form

\[ \theta(\xi,0) = 0 \]  

(4)

\[ \left. \frac{\partial \theta}{\partial \xi} \right|_{\xi=0} = -1 \]  

(5)

\[ \theta(1,F_o) = 0 \]  

(6)

The normalized temperature \( \theta \) and spatial coordinate \( \xi \) are defined as \( \theta(\xi,F_o) = (T(x,t) - T_i)k/q''H, \xi = x/H, \) and the Fourier number \( F_o \) is expressed as \( F_o = \alpha t/H^2 \). The objective of this study is to obtain an approximation of this problem using the anomalous diffusion equation (Eq. (1)). Results are obtained for a range of values of the slab thickness. A parametric analysis of the fractional derivatives at the boundary condition is also performed using a variable order operator [34,52].

3 Analysis With Integral Transform

The transient solution of \( \theta \) from the diffusion differential equation is a function of both dimensionless space and time. Known analytical methods of solution for the heat equation include, among others, similarity transformation, integral transform, and Laplace transforms [5,15,21,50,51].

The integral transform technique is especially attractive for transient heat conduction problems in that it has no inversion difficulties because both the integral transform and the inversion formula are defined at the onset of the problem [20,23]. In this study the integral transform technique is utilized as a benchmark. The integral transform, and inversion formula applied to Eq. (3) are normalized as follows:

\[ \bar{\theta}(\beta_m,F_o) = \int_{\xi=0}^1 K(\beta_m,\xi') \theta(\xi',F_o) d\xi' \]  

(7)

\[ \theta(\xi,F_o) = \sum_{m=1}^{\infty} K(\beta_m,\xi) \bar{\theta}(\beta_m,F_o) \]  

(8)

with dimensionless boundary conditions given in Eqs. (4)–(6). The kernel for this transformation is

\[ K(\beta_m,\xi) = \sqrt{\pi} \cos \beta_m \xi \]  

(9)

which corresponds to a second kind boundary condition at \( \xi=0 \) and a first kind boundary condition at \( \xi=1 \) [20]. The eigenvalues can be calculated from the following equation:

\[ \cos \beta h = 0 \Rightarrow \beta_m = \frac{(2m-1)\pi}{2h} \]  

(10)

where \( h = h/H \), \( h \) is an arbitrarily chosen slab size (\( h \leq H \)), and \( m = 1,2,3, \ldots, n \).

Consequently, the following analytical solution is obtained [20]:

\[ \theta(\xi,F_o) = \frac{2}{\lambda} \sum_{m=1}^{\infty} \cos \beta_m \xi \left[ 1 - e^{-\beta_m^2 F_o} \right] \]  

(11)

For large slab thicknesses, the solution given in Eq. (11) matches the results obtained with Eq. (12) for the semi-infinite domain, as seen in Fig. 2. The results match for \( F_o = 0 [0,1/16] \). A Fourier number equal to 1/16=0.0625 corresponds to the dimensionless time when an arbitrarily large slab thickness \( H \) is equal to the penetration distance in the semi-infinite domain solution that, in this study, corresponds to \( \lambda = 1 \). Results for \( \lambda < 1 \) represent thinner slabs, which are also shown in Fig. 2 for \( F_o = 0.0625 \).

\[ \theta(\xi,F_o) = \sqrt{\frac{4F_o}{\pi}} e^{-\xi^2/4F_o} - \xi \text{erfc} \left( \frac{\xi}{2\sqrt{4F_o}} \right) \]  

(12)

Since the semi-infinite solution cannot predict finite slabs cases, the integral transform results are also compared against the values obtained using finite differences. For small values of \( \lambda \), the dimensionless temperature reaches steady state indicated by a linear temperature drop across the slab.

4 Slab Analysis Using Fractional Derivatives

4.1 Semi-Infinite Domain Case. The one-dimensional normalized diffusion equation, Eq. (3), can be rewritten by using the exponential rule as
Equation (13) is thus no longer valid. The larger the error of symmetry so that Eq. (21) is thus no longer valid.

4.2 Perturbed Fractional Derivative Approximation. The semi-infinite domain model and the half-derivative solution cannot calculate a temperature profile when a finite domain is imposed since that formulation has one less boundary condition. A generalized expression of Eq. (20) is developed in this section to handle finite-domain cases with fractional derivatives. An approximate solution of the diffusion equation given in Eq. (3) is introduced by using exponential rule and the perturbed fractional order derivative $D_\alpha^\mu f(x)=\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f'(t)\,dt$, where $\mu$ is a small number. The larger the value of the perturbation $\varepsilon$, the larger the error of symmetry so that Eq. (21) is thus no longer valid.

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**Table 1. Summary of dimensionless temperature predictions by Green’s function solution and fractional order approximation**

<table>
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<tr>
<th>$\theta_{GF}$</th>
<th>$\theta_{FO}$</th>
<th>Error (%)</th>
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The boundary condition given in Eq. 4

\[
\frac{\partial}{\partial \xi} - \frac{d^{(1/2-\varepsilon)}}{d\xi^{(1/2-\varepsilon)}} \left( \frac{\partial \theta}{\partial \xi} + \frac{d^{(1/2+\varepsilon)}}{d\xi^{(1/2+\varepsilon)}} \theta \right) = 0
\]

(21)

This approximation is valid when

\[
\left( \frac{\partial}{\partial \xi} - \frac{d^{(1/2-\varepsilon)}}{d\xi^{(1/2-\varepsilon)}} \right) \approx \left( \frac{\partial}{\partial \xi} - \frac{d^{(1/2+\varepsilon)}}{d\xi^{(1/2+\varepsilon)}} \right) \frac{\partial \theta}{\partial \xi}
\]

(22)

The above condition indicates an error of symmetry that is defined as

\[
E = -\frac{\frac{\partial}{\partial \xi} \frac{d^{(1/2+\varepsilon)}}{d\xi^{(1/2+\varepsilon)}} \frac{\partial ^{1/2-\varepsilon}}{d\xi^{1/2-\varepsilon}} \theta}{\frac{\partial}{\partial \xi} \frac{d^{(1/2+\varepsilon)}}{d\xi^{(1/2+\varepsilon)}} \frac{\partial ^{1/2-\varepsilon}}{d\xi^{1/2-\varepsilon}} \theta}
\]

(23)

The magnitude of this error is analyzed after the solution with perturbed half-order derivative is obtained.

By applying Laplace transform to the second term on the LHS of Eq. (21) an ordinary differential equation is obtained.

\[
\frac{d\theta}{d\xi} = -\xi^{(1/2+\varepsilon)} \theta
\]

(24)

\[
\Theta = Ce^{-\xi^{(1/2+\varepsilon)}}
\]

(25)

After taking the derivative with respect to \( \zeta \) and applying the boundary condition given in Eq. (5), the expression for \( \Theta \) takes the form \( \Theta = e^{-\xi^{(1/2+\varepsilon)}} \).

Finally, applying inverse Laplace transform to \( \Theta \), an expression for \( \theta \) is obtained as follows:

\[
\theta(\zeta,F_o) = \frac{2}{\pi} \int_0^{F_o} \int_0^\infty e^{-F_y \sqrt{\zeta}} \cos(\sqrt{\theta^{(1/2+\varepsilon)}}) d\gamma dF_y
\]

(26)

This equation is integrated numerically and verified for \( \varepsilon = 0 \), as shown in Fig. 4(a), where the results are in agreement with the results obtained with the integral transform and semi-infinite solutions for \( \lambda = 1 \) and for the \( F_o \) value calculated when the penetration distance is equal to the slab with a large thickness. Also, the results obtained with Eq. (26) match the integral transform solution for several values of \( \lambda \) provided steady state conditions are not reached. The comparison of the results is given in Figs. 4(b)–4(d). As \( \lambda \) decreases, \( p \) decreases with \( \varepsilon \) becoming negative.

Figure 5 shows the change in the absolute value of the perturbation \( |\varepsilon| \) as a function of the dimensionless distance \( \zeta \) for several \( \lambda \). The value of \( |\varepsilon| \) varies as \( \zeta \) is increased for a given value of \( \lambda \). The perturbation remains equal to zero from \( \zeta = 0 \) up to a point of departure from where it becomes increasingly higher. The plot is shown for \( F_o = 0.0625 \) but a similar behavior is observed for other Fourier numbers.

The increase in the absolute value of the perturbation \( \varepsilon \) augments the error of symmetry given in Eq. (23). To calculate these errors, the derivatives of \( \theta \) with respect to \( F_o \) and \( \zeta \) are obtained using Eq. (11) where the fractional derivatives in Eq. (23) are computed using Coimbra’s operator [34,52]

\[
\frac{\partial^p \theta(F_o,\zeta)}{\partial F_o^p} = \frac{1}{\Gamma(1-p)} \int_0^{F_o} (F_o - \sigma)^{p-1} \frac{\partial^p \theta(\sigma)}{\partial F_o^p} d\sigma
\]

valid for \( 0 \leq p(\lambda,F_o) \leq 1 \). The values of the error for the considered perturbations are plotted in Fig. 6 for \( F_o = 0.0625 \).
As the Fourier number increases, this condition has to be satisfied for the energy transport. A correlation, given in Eq. 29, is used to analyze the change in temperature as a function of time at the boundary. Derivatives with respect to time are no longer governing the order of the derivative $p$ as a function of time at the boundary condition given in Eq. 29. Figure 7 displays the behavior of the fractional derivative of $\theta$ with respect to Fourier number $p$ for several values of $\lambda$ and for $F_o = 0.0625$. Similar plots can be obtained for other Fourier numbers. For values of $\lambda$ close to unity, solutions to Eq. 28 exist. However, at this Fourier number, there is no solution for smaller values of $\lambda$, which represent thin slabs. The reason is that at this Fourier number, thin slabs have reached steady state so that integer or fractional derivatives with respect to time are no longer governing the energy transport. A correlation, given in Eq. 29, was obtained to show the parametric behavior of the fractional order $p$ as a function of $\lambda$ and $F_o$.

\[
p = C_1 \left( \frac{1}{4 \lambda} \right)^{9/2} - C_2 \left( \frac{1}{4 \lambda} \right)^{8/2} + 0.5
\]

where the coefficients $C_1$ and $C_2$ change with Fourier number as follows: $C_1 = 209.29F_o^{2.5} + 1.7498F_o - 0.1823$ and $C_2 = 12.757F_o - 0.5555$. Figure 8 shows the fitting curves for several values of $F_o$.

5 Conclusions

A solution based on fractional calculus is developed for the one-dimensional transient heat conduction problem as a contribution to the understanding of the unsteady heat transfer in a finite domain. This approximation captures the effect of the variable order derivative in the transfer of heat. First, a case of 1D unsteady heat diffusion in finite domain was introduced in the description of the problem, where Neumann and Dirichlet boundary conditions were imposed at the left and right sides of a slab, respectively. Then, a solution of the problem case using integral transform technique was presented in Sec. 3, and it was utilized as a benchmark. The dimensionless form of the integral transform solution is able to describe heat transfer in several dimensionless slab sizes $\lambda$. The fractional order solutions are also compared against Green’s function solution. An analysis for the slab using fractional derivatives is presented, starting with a half-derivative approximation that behaves as the well known semi-infinite domain solution. Using the Laplace transform technique, half-derivative and perturbed half-derivative solutions are obtained. For large slab thicknesses the fractional-calculus solution matches the semi-infinite domain analytical solution for Fourier number $F_o = 0$. For thinner slabs, a perturbed numerical approximation is obtained that compares well with results obtained using the integral transform solution for finite domains. As expected, the fractional solutions break down as the transient temperature profiles approach the steady regime. Finally, a correlation is obtained to show the behavior of the fractional order $p$ of the derivative with respect to Fourier number $F_o$ at $\xi = 0$. The fractional order derivative is expressed as a function of dimensionless parameters $\lambda$ and $F_o$. The fractional order method described in this paper has applications to a variety of heat conduction problems, and it is particularly suitable for physical problems presenting anomalous diffusion behavior.

Nomenclature

- $\text{erfc}(\xi, F_o)$ = complementary error function
- $E$ = error of symmetry
- $F_o$ = Fourier’s number
- $k$ = thermal conductivity (W m$^{-1}$ K$^{-1}$)
- $K$ = transformation kernel
- $h$ = characteristic slab length (m)
- $H$ = characteristic slab length for limiting case (m)
- $p$ = fractional order derivative
- $q''$ = heat flux (W m$^{-2}$)
- $s$ = transform variable
- $T$ = temperature (K)
- $T_i$ = initial temperature (K)
- $t$ = time (s)
- $x$ = regular coordinate
- $\mathcal{L}$ = Laplace transform
- $\mathcal{L}^{-1}$ = inverse Laplace transform

Greek Letters

- $\alpha$ = thermal diffusivity (m$^2$ s$^{-1}$)
- $\beta_m$ = eigenvalues
\[ \gamma = \text{variable} \]
\[ \varepsilon = \text{perturbation} \]
\[ \xi = \text{dimensionless transformed coordinate} \]
\[ \theta = \text{normalized temperature} \]
\[ \bar{\theta} = \text{normalized transformed temperature} \]
\[ \lambda = \text{normalized slab length} \]
\[ \Gamma = \text{Gamma function} \]
\[ \Theta = \text{Laplace transform temperature} \]

**Subscripts and Superscripts**

- \( i \) = initial condition
- \( m \) = integer counter

**References**


