Thermal-Electric Modeling of Graphite: Analysis of Charge Carrier Densities and Joule Heating of Intrinsic Graphite Rods

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Graphite is a versatile material with many applications. Its intrinsic density has been extensively studied and quantified experimentally and theoretically. Experimental data suggests that the electron and hole densities are roughly equal, but the degree of inequality or the magnitude of the difference has not been quantified before for intrinsic graphite under biased (applied voltage) conditions. In this paper, a steady-state one-dimensional drift-diffusion thermal-electric model is developed to explore and analyze the nature of the electron, hole and intrinsic carrier densities of graphite rods under an externally applied voltage. The maximum difference between hole and electron densities as a function of length and applied voltage are numerically determined, showing that the maximum absolute difference between hole and electron densities decreases as the ratio of length over electric field increases. Moreover, it is observed that the difference between hole and electron densities varies with position. The model is further utilized for the analysis of the dependency of charge carrier transport, heat transport and temperature under a variety of operating conditions. The results show that higher applied voltages lead to higher current densities and higher current densities result in higher power dissipation by Joule heating, leading to higher lattice temperatures and voltage gradients along the length of the graphite rod. Dimensionless and scaling analysis showed that in the range of operating conditions, convective and radiation losses were negligible, with the lattice temperature increasing significantly when Joule heating was high.

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Graphite and graphite-derived materials, such as graphene, play a key role in many engineering applications. In the past, the electronic properties of graphite have been studied experimentally and theoretically to a great extent. For example, Wallace\textsuperscript{1} applied the band theory of solids to analyze the physical properties of graphite, including its electrical and thermal conductivities. He also noted the high anisotropy of electrical conductivity along the parallel and perpendicular directions of the layer planes. In addition, Wallace found graphite to have a zero gap between the conduction and valence bands. As a consequence, graphite is sometimes referred to as a zero-gap semiconductor.

Kinchin\textsuperscript{2} performed pioneering work in analyzing the electronic properties of polycrystalline and single crystal graphite at high temperatures. He found the number of free electrons to be about $6 \times 10^{24}$ m$^{-3}$ at room temperature. He also estimated the ratio of the mobilities of holes to electrons to be about 0.80. In addition, Kinchin calculated the resistivity, the thermal mean free path, and the product of the effective mass and the velocity of free electrons as a function of temperature. Kinchin’s findings served a key role in later investigations, and his findings were found to agree fairly well with later results obtained by other researchers.

McClure\textsuperscript{3,4} performed more work related to the band structure of graphite, where he estimated the band overlap of graphite to be approximately 0.03 eV.\textsuperscript{3} In later investigations, McClure\textsuperscript{5} and Soule\textsuperscript{6,7} performed independent additional studies which included quantification of the electron/hole densities, carrier mobilities, and carrier effective masses at low and room temperatures. McClure\textsuperscript{5} found the electron/hole densities to be about $7 \times 10^{24}$ m$^{-3}$, and the mobility to be about 1 V/(m$^2$ s) at room temperature. Soule\textsuperscript{6} further determined that the magnitude of the electron/hole density varied from $5 \times 10^{24}$ to $2 \times 10^{24}$ m$^{-3}$, and the ratio of the electron/hole mobility varied from 1.1 to 0.79, for temperatures in the range between 298 K and 4.2 K. In addition, Soule\textsuperscript{6,7} calculated the effective masses for electrons and holes to be roughly $0.03m_e$ and $0.06m_e$, respectively, along the layer planes at a temperature of 4.2 K.

Further work on the electronic properties of graphite was carried out by Klein\textsuperscript{8,9} Klein described graphite as a semimetal, and he proposed a model which assumes parabolic bands, known as the Simple Two Band (STB) Model to describe the Fermi energy/carrier densities
of graphite at high temperatures. This model is based on a carrier effective mass, an average carrier mobility, and an overlap energy. The different parameter values obtained by Klein agree well with previous publications. Yet Klein provided simple expressions to determine the values of these parameters and their variation with temperature. For instance, he provided an expression to calculate the value of the intrinsic density of graphite and its variation with temperature, also indicating that the electron and hole densities are roughly equal to each other.\textsuperscript{9} It is common in the literature to consider the assumption that electron and hole densities are almost equal but, currently, there is no clear quantification of the extent of the equality/inequality amongst the electron, hole and intrinsic densities for intrinsic graphite under biased conditions.

The electronic properties of graphite and related materials such as graphene, are essential for the proper modeling of heating effects due to current flow through these materials. Joule heating is an important physical mechanism that affects many engineering devices. Uncontrolled Joule heating causes thermal instabilities that can lead to thermal breakdown, which occurs when the heat generated inside the device is greater than the heat dissipated to the environment. Breakdown of a device happens when the material reaches its glass or melting temperature, which leads to irreversible changes in the material properties and device failure. The electrical conductivity is a key property that can help enhance or prevent the thermal instabilities leading to thermal breakdown. When the electrical conductivity depends positively on the temperature, an increase in temperature leads to an increase in electrical conductivity, and if not controlled, this process leads to an almost instantaneous breakdown.\textsuperscript{10} The mechanisms leading to breakdown of dielectric materials were thoroughly investigated and led to the development of the theory of breakdown in solids.\textsuperscript{11,12} This theory was divided into three areas: thermal, intrinsic and avalanche.\textsuperscript{11}, where special attention was given to thin films in the investigations and development of the theory.

Of the various types of breakdown, thermal breakdown gained great attention in the analysis of polymer insulators.\textsuperscript{10,13–15} Hikita et al.\textsuperscript{13,14} performed experimental analyses of thermal breakdown of thin polymer films, as well as numerical simulations of steady-state breakdown of thicker films based on the thermal breakdown theory. They found a nonlinear dependence of the current density with applied voltage, and that the electric strength decreased with temperature. Noskov et al.\textsuperscript{10} utilized a transient electro-thermal model to analyze the thermal instability of a thin polymer film assuming an exponential temperature
dependence of the electrical conductivity. They found that the temperature began increasing exponentially reaching values of over 1000 °C in just over 50 microseconds, comparable to the times reported by Neff et al.\textsuperscript{15}

Other studies analyzing thermal instabilities were performed for graphite brushes\textsuperscript{16} and microbolometers.\textsuperscript{17} Yune et al.\textsuperscript{16} carried out numerical simulations to analyze the behavior of thermal instabilities in graphite brushes. They reported steady-state temperatures in the range of 37 to over 3000 °C for corresponding current densities of 0.32 to 4.06 MA/m\textsuperscript{2}, produced with applied voltages of 0.25 to 3.14 V. Brandao et al.\textsuperscript{17} analyzed various characteristics of the thermal stability and runaway in microbolometers. In addition to determining the stability conditions, they also reported that steady-state was achieved in the order of milliseconds under current biased conditions, and within seconds under voltage biased conditions. Steady-state temperatures near 2000 and 4000 °C were reached in two cases when thermal runaway occurred.

More recently, the development of micro and nanodevices has led to ever increasing power (current) densities that require high heat dissipation rates. For example, carbon nanotubes and graphene nanoribbons have been reported to sustain current densities in the order of $10^9$ A/cm\textsuperscript{2} and $10^8$ A/cm\textsuperscript{2}, respectively.\textsuperscript{18,19} Pop et al.\textsuperscript{18} investigated Joule heating and breakdown of carbon nanotubes numerically and experimentally. They determined a nearly linear dependence of the breakdown voltage with length. They also found that the current tends to flatten out at higher voltages due to Joule heating, and a similar trend was found by Kuroda et al.\textsuperscript{20} Durkan et al.\textsuperscript{19} analyzed breakdown of graphene devices due to Joule heating under current stressing conditions. In addition to measuring the current-voltage characteristics, they estimated resistivity values for the devices. Graphene transistors have also been subject to studies analyzing Joule heating. Several studies have determined the appearance of hot spots under different biased conditions. It has been determined that hot spots appear at locations with minimum carrier densities, which also correspond to large gradients in electric field.\textsuperscript{21–23} Modeling of these devices has also enabled the estimation of carrier densities, temperature, field, and electrical resistance. Other studies have also shown the impact of Joule heating in organic semiconductors,\textsuperscript{24} for which transient simulations using a drift-diffusion model coupled with the heat equation have been used to analyze the dynamic and steady-state behavior of the temperature, current and capacitance.

One of the main purposes of this paper is to numerically quantify the difference amongst
electron, hole and intrinsic densities of graphite under biased conditions. In order to perform this analysis, the authors developed a one-dimensional steady-state drift-diffusion thermal-electric model. Once the difference between densities is determined, the developed model is used for the purpose of analyzing Joule heating in graphite rods, which is relevant to many engineering devices, such as electrical brushes,\textsuperscript{16} polymer insulators,\textsuperscript{13–15} microbolometers,\textsuperscript{17} carbon nanotubes,\textsuperscript{18,20} and graphene transistors.\textsuperscript{21,22}

The objectives of this paper are summarized as follows: (1) to show that the assumption of equality of the electron, hole and thermal densities does not lead to a consistent solution; (2) to show that the consistent solution of the drift-diffusion model in fact leads to small differences among the electron, hole and thermal densities; (3) quantify the maximum absolute difference between electrons and holes; (4) calculate the variation of the difference between hole and electron densities with position; and (5) analyze the effects of Joule heating on the characteristics of charge and heat transport in graphite rods. A relevant description of a novel application of this effect is also provided.

II. DESCRIPTION OF THE PROBLEM AND MATHEMATICAL MODEL

A. Description of the problem

In this study, a drift-diffusion steady-state one-dimensional mathematical model is developed to describe the charge carrier transport and lattice temperature distribution in a graphite rod of length $L$ and diameter $D$, with $L \geq D$. Voltage drop, temperature changes, and transport of electrons and holes vary only along the $x$-direction, as shown in Fig. 1. A DC voltage, $V_{app}$, is applied on the left end of the rod while the right end is grounded. Due to the electric field, charge carriers, i.e., electrons and holes, move across the rod producing a total current density, $J$, which generates localized Joule heating on the rod. Consequently, due to Joule heating, the lattice temperature rises, resulting in thermal generation of electrons and holes. Both ends of the rod have a fixed temperature equal to $T_\infty$. The rod is exposed to heat losses by natural convection to the surrounding air and radiation exchange with the surrounding surfaces, where the surfaces and the air are at the same temperature $T_\infty$. 
FIG. 1. Schematic: Graphite rod under an applied voltage exposed to cooling by free convection and radiation.

B. Modeling equations

The steady-state one-dimensional mathematical model is based on the drift-diffusion approximation that is derived from the general Boltzmann Transport Equation (BTE).\textsuperscript{25,26} The set of equations to be solved is composed of the following: the potential distribution described by Gauss' Law in Eq. (1a), the continuity equations for electrons and holes, Eqs. (1b) and (1c), the momentum equations for electrons and holes in Eqs. (1d) and (1e), and the equation for the lattice temperature (1f)

\[
\frac{d^2 V(x)}{dx^2} = -\frac{e}{\epsilon_r \epsilon_0} (n_h(x) - n_e(x)), \quad (1a)
\]

\[
\frac{d(-en_e(x)u_e(x))}{dx} = -e g_{e,net}(x), \quad (1b)
\]

\[
\frac{d(en_h(x)u_h(x))}{dx} = e g_{h,net}(x), \quad (1c)
\]

\[
u_e(x) = \mu_e(T_L) \frac{dV(x)}{dx} - \frac{\mu_e(T_L) k_B T_e}{e} \frac{1}{n_e(x)} \frac{dn_e(x)}{dx}, \quad (1d)
\]

\[
u_h(x) = -\mu_h(T_L) \frac{dV(x)}{dx} - \frac{\mu_h(T_L) k_B T_e}{e} \frac{1}{n_h(x)} \frac{dn_h(x)}{dx}, \quad (1e)
\]

\[
\frac{d}{dx} \left( k_e(T_L) \frac{dT_L(x)}{dx} \right) = \frac{4h T_{tot}}{D} (T_L(x) - T_\infty) + e (-n_e(x)u_e(x) + n_h(x)u_h(x)) \frac{dV(x)}{dx}, \quad (1f)
\]

where \(V(x)\) is the electric potential, \(n_e(x)\) is the electron density, \(n_h(x)\) is the hole density, \(u_e(x)\) is the electron velocity, \(u_h(x)\) is the hole velocity, \(T_L(x)\) is the lattice temperature, \(T_\infty\) is the ambient temperature, and \(g_{e,net}(x)\) and \(g_{h,net}(x)\) are the net generation rates of electrons and holes, respectively. Lattice temperature changes only occur along the x-
direction. This approximation is valid because the Biot number is lower than 0.1.\cite{27} It is also assumed that the electrons and holes are in thermal equilibrium,\cite{28,29} for which the temperature will be referred to as the carrier temperature, $T_c$. It is further assumed that $T_c$ is constant throughout the rod, but it can be set to different values. This is a standard approximation made in drift-diffusion models.\cite{29} In the section related to the effects of Joule heating, it will be shown that under the assumptions and operating conditions relevant to this study, the value of $T_c$ does not have a significant effect on the results.

The physical parameters used in the model are the electron charge, $e$, the permittivity of free space, $\epsilon_0$, the relative permittivity (dielectric constant) of graphite, $\epsilon_{r,g}$, Boltzmann constant, $k_B$, the mobility of electrons, $\mu_e(T_L)$, the mobility of holes, $\mu_h(T_L)$, and the thermal conductivity of graphite, $k_L(T_L)$, where the carrier mobilities and the thermal conductivity depend on the lattice temperature. The total heat transfer coefficient is defined as $h_{Tot} = h_{conv} + h_{rad}$, where $h_{conv}$ is the convection coefficient and $h_{rad} = \varepsilon \sigma_{SB}(T_{L,ave} + T_{\infty})(T_{L,ave}^2 + T_{\infty}^2)$ is the linearized radiation coefficient; $\varepsilon$ is the emissivity of graphite and $\sigma_{SB}$ is the Steffan-Boltzmann constant.\cite{27}

The current densities for electrons, $J_e(x)$, holes, $J_h(x)$, and the total current density, $J$, are defined as follows:

$$J_e(x) = -en_e(x)u_e(x),$$  \hspace{1cm} (2a)

$$J_h(x) = en_h(x)u_h(x),$$  \hspace{1cm} (2b)

$$J = J_e(x) + J_h(x).$$  \hspace{1cm} (2c)

Global current continuity requires the total current density to be constant, thus the following relationship holds

$$\frac{dJ}{dx} = \frac{d(J_e(x) + J_h(x))}{dx} = 0.$$  \hspace{1cm} (3)

In addition, the electric field is related to the potential as

$$E(x) = -\frac{dV(x)}{dx}.$$  \hspace{1cm} (4)

The net generation rate of electrons is equal to the net generation rate of holes,\cite{30} that is

$$g_{e,net}(x) = g_{e,net}(x) = g_{net}(x).$$  \hspace{1cm} (5)

This means that when an electron gains sufficient energy to jump from the valence band to the conduction band, a hole is created and left behind in the valence band.
Because we do not know the net generation rates explicitly, the following approach is taken for solving the continuity equations for the electron and hole densities. Substituting Eqs. (1d) and (1e) into Eqs. (2), and rearranging, the following expression is obtained:

\[ J = -e \left( n_e(x) \mu_e(T_L) + n_h(x) \mu_h(T_L) \right) \frac{dV(x)}{dx} + k_B T_c \left( \mu_e(T_L) \frac{dn_e(x)}{dx} - \mu_h(T_L) \frac{dn_h(x)}{dx} \right). \]  

(6)

Since intrinsic graphite is being used, the electron and hole mobilities are replaced with an average carrier mobility, \( \mu_{ave}(T_L) \), hereafter called mobility. Thus,

\[ J = 2e \mu_{ave}(T_L) n_{th}(T_L) E(x) - k_B T_c \mu_{ave}(T_L) \frac{d}{dx} \left( n_h(x) - n_e(x) \right), \]  

(7)

where the expression \( 2\mu_{ave}(T_L) n_{th}(T_L) \) has been used in place of \( n_e(x) \mu_e(T_L) + n_h(x) \mu_h(T_L) \); and where \( n_{th}(T_L) \) is the thermal or intrinsic density. Thermal density and intrinsic density refer to the same parameter, and will be used interchangeably in this paper.

Graphite is sometimes referred to as a zero-gap semiconductor\(^1\) or as a semimetal\(^8\) because it does not have a band gap. Instead graphite has a small overlap between the valence band and the conduction band.\(^3,8,9\) The presence of a band overlap indicates that the electron and hole densities are nearly equal to each other (and to the thermal density).\(^8,9\) Based on the two-band model, the thermal density can be approximated as:\(^9\)

\[ n_{th}(T_L) = \frac{16\pi m_{eff}}{h_P^2 c_0} k_B T_L \ln \left[ 1 + \exp \left( \frac{E_{bo}}{2k_B T_L} \right) \right], \]  

(8)

where \( m_{eff} \) is the effective carrier mass, \( c_0 \) is twice the layer spacing, \( h_P \) is Planck’s constant, and \( E_{bo} \) is the band overlap energy.

The first term on the right hand side of Eq. (7) is the drift current density, and the second term is the diffusion current density. Under the conditions used here, and using graphite properties, the drift current density becomes much larger than the diffusion current density for all physically meaningful values of the carrier temperature \( T_c \). Thus, the diffusion term can be safely neglected, and the total current density is fully determined by the drift current density

\[ J \approx 2e \mu_{ave}(T_L) n_{th}(T_L) E(x). \]  

(9)

The group of parameters multiplying the electric field in Eq. (9) is defined as the electrical conductivity, \( \sigma = 2e \mu_{ave} n_{th} \), so the well known expression for current density may be obtained \( J = \sigma E \).
Making use of subsequent expressions, Eqs. (1) can be simplified and recast as follows:

\[
E(x) = \frac{J}{2e\mu_{ave}(T_L)n_{th}(T_L)}, \tag{10a}
\]

\[
0 = \frac{d}{dx} \left( k_L(T_L) \frac{dT_L(x)}{dx} \right) - \frac{4hT_{ Tot}}{D}[T_L(x) - T_\infty] + JE(x), \tag{10b}
\]

\[
-\frac{dV(x)}{dx} = E(x), \tag{10c}
\]

\[
n_h(x) - n_e(x) = \frac{\epsilon_{r,g}\epsilon_0}{e} \frac{dE(x)}{dx}. \tag{10d}
\]

The boundary conditions are as follows: \( V = V_{app} \) at \( x = 0 \), \( V = 0 \) at \( x = L \), and \( T_L = T_\infty \) at \( x = 0, L \), as shown in Fig. 1.

Experimental data suggests that the electron, hole and thermal densities are nearly equal to each other,\(^9\) which may lead us to assume that they are exactly equal. However, in Sec. III, it will be shown that for the biased intrinsic graphite rods, all three carrier densities are nearly equal to each other, but not exactly equal. Moreover, it will be shown that assuming their equality does not lead to a consistent solution. In addition, the maximum absolute difference among the carrier densities is found to be several orders of magnitude smaller than the thermal density.

In the absence of a gate voltage, the small differences among the electron, hole and thermal densities in intrinsic graphite are important because, though these differences are small, they give rise to non-constant electric fields. On the contrary, in graphene transistors, for example, the gate voltage gives rise to large differences among the electron, hole and thermal densities, which results in very large variations in the electric field.\(^{21-23}\) Therefore, the initial small differences in carrier densities become insignificant.

C. Numerical solution

Given an initial value for the total current density, Eqs. (10a) and (10b) are iterated until convergence for temperature is obtained. Once the temperature converges, Eq. (10c) is integrated and the voltage solution at the left boundary is compared to the applied voltage. If the voltages differ by more than the predefined tolerance, the current density value is increased/decreased accordingly; this procedure is a form of the shooting method. Eqs. (10a) through (10c) are then solved iteratively until convergence for both temperature and voltage is achieved. When the final solution is obtained, Eq. (10d) can be used to solve...
For the difference of electron and hole densities along the rod. In addition, one can solve for the net generation rates for electrons, holes and the diffusion current density using Eqs. (1b), (1c), and (7), respectively. In addition, one can use the mass action law, \( n_{th}^2 = n_e n_h \) to calculate the electron/hole density distributions. Since the actual values of \( n_e \) and \( n_h \) will be very close to \( n_{th} \), calculating their difference instead of their actual values gives more insight into the solution of the problem.

Equations (10) were discretized using second order finite difference approximations with a mesh of 101 nodes. Convergence is achieved when the absolute relative error for voltage and the maximum absolute relative error for temperature fall below a tolerance of \( \tau = 1 \times 10^{-5} \), i.e., \( \max|T_{L,new} - T_{L,old}|/T_{L,new} \leq \tau \), and \( |(V_{app} - V(0))/V_{app}| \leq \tau \). The grid size and tolerance were chosen based on grid and tolerance independence tests.

D. Properties of pyrolytic graphite and system parameter values

The properties of pyrolytic graphite are presented in Table I. The experimental data for the temperature dependent mobility,\(^9\) from room temperature to 1000 K, was fitted with the following function in order to use it in the simulations

\[
\mu_{ave}(T_L) = 7211.2T_L^{-1.555}.
\]

(11)

where \( r^2 = 0.9954 \). Similarly, the experimental data for the temperature dependent thermal conductivity,\(^31,32\) was fitted with the following function

\[
k_L(T_L) = 839269.0T_L^{-1.068}.
\]

(12)

where \( r^2 = 0.9988 \).

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twice the layer spacing</td>
<td>( c_0 )</td>
<td>( 0.672 \times 10^{-9} ) m</td>
</tr>
<tr>
<td>Band overlap energy</td>
<td>( E_{bo} )</td>
<td>0.01 eV</td>
</tr>
<tr>
<td>Emissivity</td>
<td>( \varepsilon )</td>
<td>0.8</td>
</tr>
<tr>
<td>Relative permittivity of graphite</td>
<td>( \varepsilon_{r,g} )</td>
<td>13.0</td>
</tr>
<tr>
<td>Effective carrier mass</td>
<td>( m_{eff}/m_e )</td>
<td>0.0125</td>
</tr>
<tr>
<td>Thermal conductivity at ( T_L = 298 ) K</td>
<td>( k_{L,ref} )</td>
<td>1911 W/(m K)</td>
</tr>
<tr>
<td>Ave. carrier mobility at ( T_L = 298 ) K</td>
<td>( \mu_{ave,ref} )</td>
<td>1.02 m²/(V s)</td>
</tr>
</tbody>
</table>
Unless otherwise stated, the system parameter values used for the baseline case and for other analyses are listed in Table II.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
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</thead>
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<tr>
<td>Rod length</td>
<td>$L$</td>
<td>$10^{-2}$ m</td>
</tr>
<tr>
<td>Rod diameter</td>
<td>$D$</td>
<td>$10^{-3}$ m</td>
</tr>
<tr>
<td>Convection heat transfer coefficient</td>
<td>$h_{conv}$</td>
<td>10.0 W/(m$^2$ K)</td>
</tr>
<tr>
<td>Ambient temperature</td>
<td>$T_{\infty}$</td>
<td>298 K</td>
</tr>
</tbody>
</table>

III. RESULTS AND DISCUSSION

In the published literature, several approximations of the intrinsic density of graphite have been proposed based on available experimental data. A popular expression to calculate the intrinsic density of graphite, Eq. (8), has been developed by Klein according to his two-band theory. The fact that there is a band overlap rather than a band gap means that there are nearly equal numbers of electron and holes. As a result, it has been generally assumed that the electron and hole densities are exactly equal. In this section, several cases are solved to demonstrate that although the electron and hole densities are nearly equal to each other and to the thermal density, there exists a small difference among them. In addition, the difference is also bounded to a maximum value dictated by material properties, geometry and operating conditions. The order of magnitude of the maximum absolute difference between holes and electrons is here quantified with the aid of Gauss’ law in the form of a Poisson equation.

A. Exploring the equality of the electron, hole and thermal densities

In this section, the consistency in the solution of the governing equations is analyzed based on the assumption of equality of the electron, hole and thermal densities. From Eq. (3), it is known that the total current density, $J$, is constant. Also, in Eqs. (10a) and (10d), the thermal density $n_{th}(T_L)$ and carrier mobility $\mu_{ave}(T_L)$ are dependent on the lattice temperature $T_L(x)$ and the electric field $E(x)$ is a function of position.

Case 1: $n_e(x) = n_h(x) \approx n_{th}(T_L)$. When $n_e(x) = n_h(x)$, the right side of Eq. (10d)
becomes equal to zero, and Poisson’s equation reduces to Laplace’s equation, \( \frac{d^2V}{dx^2} = 0 \). Solution of the Laplace equation leads to a linear voltage distribution, and consequently to a constant electric field \( E(x) = E_0 = \frac{V_{\text{app}}}{L} \). The thermal density can be obtained from two expressions, i.e., directly calculated from Eq. (8), or by rewriting Eq. (10a) in the form:

\[
n_{\text{th}}^0(x) = \frac{J}{2e\mu_{\text{ave}}(T_L)E_0},
\]

where it has been labeled as \( n_{\text{th}}^0 \) to differentiate from the value of \( n_{\text{th}} \) obtained from Eq. (8).

The results obtained for this case are shown in Fig. 2, where the current density was obtained iteratively by evaluating the thermal density and the mobility at the average lattice temperature, using a constant electric field, i.e., \( J = 2e\mu_{\text{ave}}(T_{L,\text{ave}})n_{\text{th}}(T_{L,\text{ave}})E_0 \). As expected, Fig. 2(a) shows a constant electric field and a linear variation of the voltage due to the assumption of \( n_e(x) = n_h(x) \). However, Fig. 2(b) shows that the values of \( n_{\text{th}} \) and \( n_{\text{th}}^0 \) are not equal, except in a couple of locations where the curves intersect each other, and thus, under this assumption, the system of equations is inconsistent. This solution suggests that the electric field should not be constant; and that the electron, hole, and thermal densities should be approximately equal but not exactly equal.

![Graphs showing results for Case 1: \( n_e(x) = n_h(x) \). Parameter values: \( D = 10^{-3} \) m, \( L = 10^{-2} \) m, \( V_{\text{app}} = 1.0 \) V.](image)

FIG. 2. Results for Case 1: \( n_e(x) = n_h(x) \). Parameter values: \( D = 10^{-3} \) m, \( L = 10^{-2} \) m, \( V_{\text{app}} = 1.0 \) V.

Now that it has been determined that the electron, hole and thermal densities are slightly
different, the maximum absolute difference between electron and hole densities will be quan-
tified in the next section.

### B. Quantification of the maximum absolute difference between hole and electron densities

The results shown in the previous section suggest that while the hole and electron densities are nearly equal to each other and to the thermal density, i.e., \( n_e(x) \approx n_h(x) \approx n_{th}(T_L) \), there is a small difference between the hole and electron densities \( |n_h(x) - n_e(x)| \ll n_{th}(T_L) \).

The main objective in this paper is to numerically quantify the difference between hole and electron densities \( n_h(x) - n_e(x) \), which may vary along the length of the rod. The first step is to quantify the maximum absolute difference, \( \max|n_h(x) - n_e(x)| \), which may occur anywhere along the rod.

To simplify the analysis, the potential and the position parameter \( x \) will be non-dimensionalized as follows: \( V^* = V/V_{app} \) and \( x^* = x/L \). Thus, Poisson’s equation (10d) becomes

\[
\frac{d^2V^*(x^*)}{dx^*2} = \frac{eL^2}{\epsilon_r\epsilon_0V_{app}} \max|n_h(x^*) - n_e(x^*)|. \tag{14}
\]

In order to quantify the maximum difference between hole and electron densities, only the magnitude of this value is considered, which is independent of position. Integrating Eq. (14) analytically from 0 to 1 yields the equation for a parabola

\[
V^*(x^*) = \frac{eL}{2\epsilon_r\epsilon_0E_0} \max|n_h(x^*) - n_e(x^*)| \left[ x^* - x^{*)2 \right] + [1 - x^*], \tag{15}
\]

where \( E_0 = V_{app}/L \).

Typical values for this problem are taken as length \( L = 10^{-2} \) m and electric field of 100 V/m, where the value for the dielectric constant used for graphite is 13.0. Substituting and simplifying, the following expression is obtained:

\[
V^*(x^*) = (6.96 \times 10^{-14} \text{ m}^3) \max|n_h(x^*) - n_e(x^*)| \left[ x^* - x^{*)2 \right] + [1 - x^*]. \tag{16}
\]

Because the units of the electron/hole densities are in \( \text{m}^{-3} \), the units will cancel out upon multiplication. Although integration of the Poisson equation yields a parabolic equation, a nearly linear voltage distribution is expected. This means that the coefficient multiplying the quadratic term should be of order 1, nearly equal to the linear term. In order for
This coefficient to have such magnitude, the maximum absolute difference between holes and electrons, \( \max |n_h(x) - n_e(x)| \), must be of the order of \( 10^{14} \) or lower. The potential distribution using Eq. (16) was plotted for various values of \( \max |n_h(x) - n_e(x)| \) in Fig. 3(a).

It can be seen that as \( \max |n_h(x) - n_e(x)| \) increases the voltage distribution becomes more parabolic than linear, and the peak voltage is higher than the applied voltage. When the peak voltage becomes higher than the applied voltage, it is considered as a non-physical solution. In order to obtain a meaningful physical solution for the given length \( L = 10^{-2} \) m and applied voltage \( V_{app} = 1.0 \) V, the maximum absolute difference, \( \max |n_h(x) - n_e(x)| \), should be in the order of \( 10^{13} \) m\(^{-3} \). As expected, this number is much smaller than the thermal density.

![Graph showing voltage distribution](image)

**FIG. 3.** (a) Voltage distribution for various values of \( \max |n_h(x) - n_e(x)| \) obtained with Eq. (16). Parameter values: \( D = 10^{-3} \) m, \( L = 10^{-2} \) m, \( V_{app} = 1.0 \) V, (\( E_0 = 100 \) V/m); legend: \( \max |n_h(x) - n_e(x)| \) (m\(^{-3} \)). (b) \( \max |n_h(x) - n_e(x)| \) vs. \( L/E_0 \).

Based on this analysis, it can be estimated that in order to obtain a meaningful physical solution for the voltage distribution, the maximum difference between holes and electrons is limited to a certain value. This value is dictated directly by the dielectric constant of the material \( \varepsilon_{r,g} \), the applied voltage, \( V_{app} \), the length \( L \), and indirectly affected by the carrier mobility \( \mu_{ave} \), the effective carrier mass \( m_{eff} \) and diameter \( D \). The mobility, and to a lesser
extent the effective mass, also affect the overall electrical conductivity of the material, and therefore influence the magnitude of the voltage, as seen in Fig. 3(b).

On the other hand, in order to have a slightly non-linear voltage distribution, there needs to be a small difference between the electron and hole densities. Moreover, it is noted that the electron, hole and thermal densities vary along the rod due to a non-uniform temperature distribution; therefore, it is expected that the difference \( n_h(x) - n_e(x) \) also varies along the length of the rod. In conclusion, it can be stated that for a given \( \epsilon_r, g \), knowing that \( n_h(x) \approx n_e(x) \approx n_{th}(T_L) \), it is expected that \( 0 < \max|n_h(x) - n_e(x)| < \xi(L/E_0) \), where \( \xi(L/E_0) \) is a function of \( L/E_0 \).

C. General results for charge and heat transport

The previous analyses showed that indeed the electron, hole and thermal densities are nearly equal, yet slightly different. The order of magnitude of the maximum absolute difference between hole and electron densities was also quantified under different operating conditions. In this section, the governing equations given by Eqs. (10) are solved following the procedure described in Sec. II C.

Case 2 (baseline case): in this case the parameter values from Table II are used together with an applied voltage, \( V_{app} \), equal to 1.0 V to obtain the distribution of the lattice temperature, electric field, electron and hole densities, as well as, the Joule heating effect.

The results are shown in Fig. 4, where the voltage distribution and the electric field are shown in Fig. 4(a). It can be observed that the electric field, dotted-dashed line, is not constant, and its average magnitude is close to the nominal value \( E_0 = V_{app}/L = 100 \text{ V/m} \) throughout the rod. The fact that the electric field is not constant gives rise to a non-linearity in the voltage distribution, which can be seen as a slight distortion on the almost linear solid line.

The thermal density and the difference between the hole and electron densities are shown in Fig. 4(b). Because the thermal density has a nearly linear dependence with respect to temperature, the thermal density curve resembles that of the temperature distribution, shown in Fig. 4(d), reaching a maximum value slightly higher than \( 9 \times 10^{24} \text{ m}^{-3} \). The difference between hole and electron densities is represented by the dashed black curve. It can be seen that the values vary approximately between \(+0.5 \times 10^{13} \text{ m}^{-3}\) and \(-0.5 \times 10^{13} \text{ m}^{-3}\),
displaying a non-symmetric distribution, where the difference becomes zero at the midpoint along the length of the rod. It is also observed that the maximum difference between holes and electrons coincides with the largest gradient of electric field and temperature, which in this case occurs near the boundaries. In addition, there are more holes than electrons on the left half of the rod due to the positive voltage and more electrons than holes on the right half of the rod near the grounded end. Although \( \max|n_h(x) - n_e(x)| \ll n_{th}(T_L) \), the small difference gives rise to the small non-linearity in the voltage and a non-constant electric field. The operating conditions used for this case corresponds to the black circle depicted in Fig. 3(b).

The Joule heating power and heat losses due to convection and radiation, given in a per-unit-length basis, are shown in Fig. 4(c), where the Joule heating power, \( P_{Jh} = JE(x)A_c \), has a magnitude close to \( 1.5 \times 10^4 \) W/m, about three to four orders of magnitude higher than the heat losses, \( q_{cw} = (4h_{tot}(T_L(x) - T_\infty)/D)A_c \). The heat losses are small due to the relatively small surface area of the rods.

The temperature distribution is shown in Fig. 4(d), where it is observed that the maximum temperature reaches 173 °C in the middle of the rod, due to the symmetric boundary conditions. The total current density corresponding to Case 2 is \( 1.74 \times 10^8 \) A/m².

**Case 3:** the parameter values from Table II are used again, but with an applied voltage equal to -2.0 V, in order to analyze the effects of polarity and magnitude of \( V_{app} \). The voltage distribution and the electric field are shown in Fig. 5(a), where it is seen that the electric field has a larger variation than in the previous case, and therefore, the non-linearity of the voltage distribution is more pronounced compared to the baseline case. The thermal density and the difference between the hole and electron densities are shown in Fig. 5(b), where it is observed that the magnitude of the difference increased compared to the baseline case, with a maximum absolute difference between holes and electrons of \( 5 \times 10^{13} \) m\(^{-3} \) for Case 2 compared to \( 2.7 \times 10^{13} \) m\(^{-3} \) for Case 3. Also, because the polarity of the voltage was modified, the variation of the difference between holes and electrons also changed, having more electrons on the left half of the rod and more holes on the right (grounded) half.

The Joule heating power and heat losses are shown in Fig. 5(c), where it is seen that the magnitude of both the Joule heating power and the heat losses increased with voltage. The maximum Joule heating power increased from \( 1.49 \times 10^4 \) W/m for Case 2 to \( 4.86 \times 10^4 \) W/m for Case 3, while the maximum heat losses increased from 8.16 W/m for Case 2 to 85.4
FIG. 4. Results for Case 2 (baseline case) with parameter values: $V_{app} = 1.0 \, \text{V}$, $L = 10^{-2} \, \text{m}$, $D = 10^{-3} \, \text{m}$.

Although the heat losses increased more with voltage, the Joule heating power is still about two to three orders of magnitude higher.

Lastly, the temperature distribution is shown in Fig. 5(d), where the maximum temperature increased from 173 °C for Case 2 to 763 °C for Case 3. The total current density for Case 3 is $-2.42 \times 10^8 \, \text{A/m}^2$, which corresponds to a 39% increase in absolute value.
with respect to Case 2. This result suggests that the Joule heating power (or total current density), does not vary linearly with applied voltage. This will be explored and discussed in more detail in sections III E and III F.

Using Eq. (7) the diffusion current can be computed as it varies with position. For Case 3, using a carrier temperature of $T_c = 298$ K, the diffusion current remains in the range between $10^{-7}$ and $10^{-5}$ A/m$^2$. For this case, $T_c$ would have to be several orders of magnitude higher in order for the diffusion current density to have a significant contribution to the total current density. This result confirms that the total current density can be fully determined by the drift current density. In addition to the results discussed here, one can use the continuity equations (1b) and (1c) to solve for the net generation rate $g_{net}$.

D. Comparison with published results

In order to verify the model developed in this work, the temperature distribution obtained with the present model is compared with the analytical expression for steady-state temperature distribution for graphene obtained by Chandran. In his work, the steady-state temperature distribution is determined from knowledge of the volumetric heat generation, end contact temperature, and ambient temperature. The parameter values used are as follows: $V_{app} = -1.2$ V, $L = 10^{-4}$ m, $D = 10^{-5}$ m, $k_L = 1100$ W/(m K), $h_{Tot} = h_{conv} = 10$ W/(m$^2$ K), with no radiation losses, and where $T_E = T_0 = T_\infty = 298$ K. Chandran defined the total length of the device as $2l$, therefore, $L = 2l = 10^{-4}$ m. The temperature distributions are compared in Fig. 6, where very good agreement between the two models is observed.

E. Parametric studies

The developed model is used in this section to perform several parametric analyses to investigate the effects of applied voltage, rod diameter, rod length and heat transfer coefficient.
FIG. 5. Results for Case 3 with parameter values: $V_{app} = -2.0$ V, $L = 10^{-2}$ m, $D = 10^{-3}$ m.

1. **Non-dimensional analysis**

Equations (10) are non-dimensionalized using the following parameter definitions: $V^* = V/V_{app}$, $n_e^* = n_e/n_{th,ref}$, $n_h^* = n_h/n_{th,ref}$, $x^* = x/L$, and $T_L^* = T_L/T_\infty$, and re-written as follows:
FIG. 6. Comparison of temperature distribution with published expression by Chandran.34 Parameter values: $V_{app} = -1.2$ V, $L = 10^{-4}$ m, $D = 10^{-5}$ m, $k_L = 1100$ W/(m K), $h_{Tot} = h_{conv} = 10$ W/(m² K), no radiation losses, $T_E = T_0 = T_\infty = 298$ K.

\[
J^* = \frac{\mu_{ave}(T_L^*) \, n_{th}(T_L^*)}{\mu_{ave,ref} \, n_{th,ref}} E^*(x^*) ,
\]

(17a)

\[
0 = \frac{d}{dx^*} \left( \frac{k_L(T_L^*)}{k_{L,ref}} \frac{dT_L^*(x^*)}{dx^*} \right) + \frac{1}{\beta} J^* E^*(x^*) - \frac{\gamma}{\beta} \frac{h_{Tot}(T_L^*)}{h_{Tot,ref}} \left[ T_L^*(x^*) - 1 \right] ,
\]

(17b)

\[
E^*(x^*) = -\frac{dV^*(x^*)}{dx^*} ,
\]

(17c)

\[
\frac{d^2 V^*(x^*)}{dx^{*2}} = -\alpha (n^*_e(x^*) - n^*_i(x^*)) ,
\]

(17d)

where, $\alpha = eL^2 n_{th,ref}/V_{app} \epsilon_0 \beta_0$, $\beta = k_{L,ref}T_\infty / J_0 V_{app} L$, $\gamma = 4 h_{Tot,ref} T_\infty L / J_0 V_{app} D$, $J^* = J/J_0$, and $J_0 = 2e \mu_{ave,ref} n_{th,ref} V_{app} / L$.

Equations (17), being in non-dimensional form, allow for a quick comparison of the orders of magnitude of the various terms involved. Let’s focus in the heat equation (17b) for instance. The diffusion term has a coefficient of order one, the Joule heating term has a coefficient equal to $1/\beta$, and the convection plus radiation losses term has a coefficient of $\gamma/\beta$. Both coefficients, $1/\beta$ and $\gamma/\beta$, can be re-written as follows:

\[
\frac{1}{\beta} = \frac{J_0 E_0 L^2}{k_{L,ref} T_\infty} ,
\]

(18)

and

\[
\frac{\gamma}{\beta} = \frac{4 h_{Tot,ref} L^2}{k_{L,ref} D} ,
\]

(19)
FIG. 7. Comparison of the magnitude of the non-dimensional coefficients in the energy equation (17b).

which resembles the well-known Biot number, \( \text{Bi} = hD/k \). A comparison of the three terms can be made by varying the value of \( 1/\beta \) through the applied voltage, and the value of \( \gamma/\beta \) through the length and diameter, keeping the heat transfer coefficient fixed. The results are shown in Fig. 7. Above the diagonal blue dashed line, \( \gamma/\beta > 1/\beta \), which means that the convection and radiation losses dominate, while below the line, \( \gamma/\beta < 1/\beta \), which means that conduction losses through the boundaries dominate. However, the vertical magenta dashed line separates the regions where Joule heating is low and high, left and right, respectively. In other words, to the left of the vertical magenta line, there is low amounts of Joule heating to be removed regardless of which losses dominate. To the right of the vertical magenta line and to the bottom of the diagonal blue curve, most of the Joule heating will be removed by heat conduction through the boundaries. Lastly, to the right of the vertical magenta dashed line and above the blue dashed line most of the Joule heating will be removed by convection plus radiation losses. Moreover, the intersection of both dashed lines means that all three coefficients have an order of magnitude of one, which means that all three terms are equally
important. Consequently, three different ratios of \( L^2/D \) were used, representing the range of interest in our studies. The three cases are represented by the different symbols in the same figure. It can be seen that all three cases fall well below the intersection of the dashed lines. Two major conclusions can be drawn. First, Joule heating is significant for values of \( 1/\beta > \sim 1 \), which corresponds to an applied voltage greater than about 0.54 Volts. Second, all our cases fall well below the intersection point of the dashed lines. This means that heat losses by convection plus radiation are negligible for all our cases when high Joule heating is present. In order for convection plus radiation losses to become significant for the high Joule heating cases, the ratio of \( L^2/D \) has to be about 100 m or larger.

To further explore the extent of the effect of the convection and radiation losses, the ratio of \( h_{\text{Tot}}/D \) was varied over more than three orders of magnitude. It was found that the difference in the maximum lattice temperature reached was less than about 1 °C. In addition, the current density was only affected in the fourth significant digit. Thus, again, convection and radiation losses are insignificant for all operating conditions considered in this paper.

2. **Scaling analysis**

Using scaling analysis, and safely ignoring the convection term, the energy equation (10b) can be written as follows:

\[
\frac{k_L \Delta T_L}{L^2} \sim J E, \tag{20}
\]

or by taking \( \Delta T_L \sim T_{L,\text{max}} \), \( k_L \sim k_L(T_{L,\text{max}}) \), and \( J E \sim P''_{\text{Jh,\text{max}}} \), we get

\[
k_L(T_{L,\text{max}})T_{L,\text{max}} \sim P''_{\text{Jh,\text{max}}} L^2. \tag{21}
\]

The order of magnitude of the two terms can be compared for a given range of applied voltages, similarly as done in the previous subsection. Notice that \( P''_{\text{Jh,\text{max}}} L^2 / k_L(T_{L,\text{max}})T_{L,\text{max}} \equiv 1/\beta \), evaluated at \( T_{L,\text{max}} \) instead of \( T_\infty \). The ratio of these two quantities may be seen as the efficiency at which the input energy, i.e., electrical current, is converted into heat, i.e., high temperature. Figure 8 presents the dependency of this ratio vs. the applied voltage. It can be seen at as the applied voltage increases, the ‘efficiency’ of heating the material
increases. Rather than following a straight line, the efficiency increases in a non-linear fashion. This is due to the fact that because the mobility decreases with temperature, so does the conductivity, which in turn means the current density saturates at higher temperatures, and thus it affects this ratio. As explained in the previous subsection, the Joule heating effects are usually stronger at applied voltages larger than about 0.5 Volts, as can also be seen in Fig. 8.

![Graph showing the ratio of Joule heating over thermal energy vs. length and applied voltage.]

FIG. 8. Comparison of the ratio of Joule heating over thermal energy vs. length and applied voltage.

The results just described can also be reached by plotting the ratio of the heat losses at the boundaries over the Joule heating, as can be seen in Fig. 9. The ratio is defined as $|q''|/P_{Jh, max} L$, where $|q''| = k_L d T_L/dx$, and $k_L$ was evaluated at the mid point of the one sided second order finite difference. Also, it is observed that the total heat loss at the boundaries is $2|q''|$. It can be readily seen in Fig. 9 that only for values of applied voltage of 0.5 Volts and higher, the heat losses at the boundaries cannot dissipate the Joule heating generated, and the ratio decreases very sharply. In addition, the ratio is smaller for larger values of $L$, thus reaching the same conclusions as in the previous paragraph.

Lastly, applying scaling to Eq. (10d) the following expression can be obtained:

$$\frac{V_{app}}{L^2} \sim \frac{e}{\epsilon r_0 \epsilon_0} \max |n_h - n_e|, \quad (22)$$

thus, $V_{app} \propto \max |n_h - n_e| L^2$. Equation (22) also resembles the definition of $\alpha$ from the previous subsection. Using this relationship, the dependency of the maximum difference
between holes and electrons was plotted vs. the applied voltage shown in Fig. 10. As the applied voltage is increased, the maximum difference between holes and electrons increases in a non-linear fashion due to temperature-dependent properties, especially the mobility.

As a summary, these parametric studies, under the range of conditions analyzed here, show that there is a very small dependency of lattice temperature with respect to rod diameter, length and heat transfer coefficient. However, an increase in applied voltage
generates a significant rise in temperature. Therefore, in order to keep the rods at a certain
temperature, the voltage has to be carefully controlled to avoid overheating of the rods. On
the contrary, if one desires to heat up the rods quickly, relatively large values of applied
temperature should be used.

F. Application of Joule heating as an efficient method of thermochemical
decomposition of carbon-based materials

The effect of the carbonization temperature on the electrical conductivity of various wood
and wood-based materials was analyzed by Kwon et al.\cite{35}. Their results showed that the elec-
trical conductivity increases by several orders of magnitude with carbonization temperature.
For example, the conductivity of solid wood carbonized at 600 °C was nearly $10^{-3}$ (Ω m)$^{-1}$,
while the conductivity increased to $10^2$ (Ω m)$^{-1}$ after carbonization at 1100 °C. In compar-
ison, the electrical conductivity of pyrolytic graphite is roughly $10^6$ (Ω m)$^{-1}$ while that of
POCO graphites\cite{33} is in the order of $10^5$ (Ω m)$^{-1}$. Although the conductivity of heat treated
solid wood differs by three to four orders of magnitude from that of graphite, the electrical
conductivity of carbon-based materials tends to increase even more with higher carboniza-
tion temperatures. This trend is observed in graphite, whose electrical conductivity tends
to increase with graphitization temperature due to the development of larger single crystal
domains.\cite{8,9,33} Joule heating can be utilized as an enhancement of thermochemical conversion
processes such as pyrolysis and gasification, which tend to produce biochar, a carbon-rich
material, as byproduct. As discussed in the previous sections, the low heat losses, pro-
duce a sharp temperature rise in the rods due to Joule heating. Although in conventional
semiconductors and graphene-based semiconductors high temperatures are avoided, high
temperatures are desired in carbonization processes, and therefore, Joule heating may be
used as an efficient enhancement of these processes. It is shown in Fig. 11 that as the applied
voltage is increased, the total current density tends to reach a saturation value, while the
maximum lattice temperature increases sharply with higher applied voltages. For instance,
comparing the total power, $P = V_{app}J_{Ec}$, used in cases 2 and 3 described in section III C,
$P = 137$ Watts for Case 2, while $P = 190$ Watts, for Case 3. Therefore, an increase in
power of 39% raised the maximum lattice temperature from 173 °C to almost 800 °C. No
 thermochemical conversion occurs in Case 2 but with a modest increase in power, temper-
FIG. 11. \( J, T_{\text{L,max}} \) vs. \( V_{\text{app}} \). Parameter values: \( L = 10^{-2} \) m and \( D = 10^{-3} \) m.

... atures suitable for carbonization can be reached. The model developed in this work can be utilized as a first approximation of the behavior of graphitic materials in the presence of Joule heating.

IV. CONCLUSIONS

A one-dimensional steady-state drift-diffusion thermal-electric model has been developed to determine the difference between hole and electric densities, and to simulate charge and heat transport in rods of intrinsic graphite. Although the electron, hole, and thermal densities are nearly equal to each other for intrinsic graphite, there exists a small difference among them, i.e., \( n_h(x) \approx n_e(x) \approx n_{\text{th}}(T_L) \), and the maximum absolute difference, \( \max|n_h(x) - n_e(x)| \ll n_{\text{th}}(T_L) \). It was found that the difference between hole and electron densities, \( n_h(x) - n_e(x) \), varies along the length of the rod and that the maximum absolute difference occurs where the gradient of the electric field is largest, while the difference is zero at the midpoint along the rod. It was also determined that there are more holes than electrons on the half corresponding to the positive polarity of voltage, while there are more electrons on the half near the grounded end. Furthermore, when the applied voltage was negative, the variation of the difference between holes and electrons was inverted. The developed model was also used to analyze Joule heating effects for intrinsic graphite rods.
Dimensionless and scaling analysis showed that in the range of operating conditions, convective and radiation losses were negligible, with the lattice temperature increasing significantly when Joule heating was high. Although, total current density increases with applied voltage, the total current density tends to saturate at higher voltages, while the lattice temperature increases almost at an exponential rate. It was also discussed that the Joule effect can be utilized to enhance carbonization processes.

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Figure (a) shows the voltage $V$ as a function of position $x$ for different electric field magnitudes $E_0$. The plots are for $L/E_0 = 10^{12}$, $10^{13}$, and $10^{14}$.

Figure (b) illustrates the maximum density of $|n_n - n_e|$ as a function of $L/E_0$.

- For $L = 10^{-2}$ m, $|E_0| = 100$ V/m.
- For $L = 10^{-4}$ m, $|E_0| = 10^4$ V/m.
The graph shows the temperature $T_L$ (°C) as a function of $x$ (m) for two models: Present model (blue solid line) and Chandran (red dashed line). The temperature values range from 0 to 350 °C, and the $x$-axis is labeled in units of $10^{-4}$ meters.
Convection + radiation losses dominate

Conduction losses dominate

Low Joule heating

High Joule heating

$L^2/D = 1 \times 10^{-4}$ m

$1 \times 10^{-2}$ m

$1$ m
$\frac{|q''|}{|P'''|_{Jh,\text{max}}}$

- $L = 1e^{-2}$ m
- $1e^{-3}$ m
- $1e^{-4}$ m

$V_{\text{app}}$ (V)
The graph illustrates the relationship between the applied voltage \( V_{\text{app}} \) (V) and the maximum difference of electron density \( |n_h - n_e| \cdot L^2 \) (m\(^{-1}\)). The data points are categorized by different lengths \( L \) as follows:

- \( L = 1 \times 10^{-2} \) m (represented by red 'x' markers)
- \( 1 \times 10^{-3} \) m (represented by blue diamond markers)
- \( 1 \times 10^{-4} \) m (represented by red square markers)

The graph shows how the electron density difference changes with the applied voltage for different lengths.