Lecture 1, Intro and Section 12.1: 3D coordinates

Introducing myself: I have been a faculty here since 2006, I am Canadian, with an accent and I am allowed to say zed. I am (I think) known to be tough but fair. In this class, you will have to work to understand, and to understand to pass. I am not a stickler for details, but I don't like if you try to fool me and pretend you know more than you do. You all decided to pay more than $1000 to be here, so don't waste that money away.

Important points from the syllabus: Lots of homeworks for practice, but they don't count for much. You get quizzes a week in advance, they count for a lot, so **free** points there. There are 2 midterms, one final. Secret weapons are office hours and good discussion participation.

Homework 1 is online, on Webassign, due Wednesday. Later homeworks will be due on Wednesdays, and posted a week ahead of the due date. Also, please bring 4 green books to your TA within 3 weeks.

What is Math 23? Calculus in 3D.
That means we need to do geometry in 3D, differentiation, integration, in 3D.
This can be visualized, drawn, but not guessed.

The book is still Stewart, available on Webassign, via CatCourses. We cover chapters 12 to 16, most sections.

No laptop or tablets (or screens that others can see) in class please, unless you clear it with me first. Calculators are usually a bad idea, since they are not allowed on the tests,

Important point: We will build on early material in this class, so ask questions **early**. Don't let me go on if you don't understand a word I am saying.

OK, let's do math! We deal with space in this class, 3D space. One way to represent that is with good old cartesian coordinates, but 3 of them instead of 2.

2D cartesian coordinates figure

3D cartesian coordinates figure

What is the distance from the origin to the point $P = (1, 2, 4)$?
The *projection* of P onto the xy-plane is $Q = (1, 2, 0)$.
The distance from the origin O to Q is $|OQ|$ = $\sqrt{1^2 + 2^2}$.
The triangle OQP is a right-angle triangle. So the distance $|OP|$ is
$|OP|$ = $\sqrt{|OQ|^2 + 4^2} = \sqrt{1^2 + 2^2 + 4^2}$

In general, the distance from a point (x, y, z) to the origin is $d = \sqrt{x^2 + y^2 + z^2}$.
The distance between 2 points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
So where are the points such that $\sqrt{x^2 + y^2 + z^2} = 3$?

$$x^2 + y^2 + z^2 = 3^2$$

What about $(x-1)^2+(y+2)^2+(z-3)^2=4$?

$$(x-1)^2+(y+2)^2+(z-3)^2=4$$

Or $1 \leq (x-1)^2+(y+2)^2+(z-3)^2 \leq 4$?

$$1 \leq (x-1)^2+(y+2)^2+(z-3)^2 \leq 4$$

Or $x^2+y^2=9$?

$$x^2+y^2=9$$

What about $z = 2$?

$$z = 2$$

Or $x = 2$?

$$x = 2$$

Or $x = 2$ and $y = 1$?

$$x = 2 \text{ and } y = 1$$

Section 12.2: Vectors

If you connect two points in space, with a specific direction (from a starting point to a finishing point), you get a VECTOR. You can think of it as an arrow.

A vector

A vector has a given LENGTH and a DIRECTION (but it doesn't really have a single starting point or a single finishing point). A vector can be translated and remains the same vector, but if it is rotated, it becomes a new vector.

A vector will be denoted by its 3 Cartesian components, within $<,,>$,

$$\vec{v} =< x, y, z >$$

where each component stands for the displacement in the x, y, or z direction.

What is the vector connecting $P = (2, 2, 1)$ to $Q = (0, 4, 2)$?

$$\vec{v} = \vec{PQ} = Q - P = (0, 4, 2) - (2, 2, 1) =< -2, 2, 1 >$$

where we take the arrival point and subtract the starting point.

If we let $\vec{v} =< v_1, v_2, v_3 >$ and $\vec{w} =< w_1, w_2, w_3 >$ we can perform arithmetic component-by-component:
If $c \in \mathbb{R}$, $\quad c\vec{v} =< cv_1, cv_2, cv_3 >$
and $\quad \vec{v} + \vec{w} =< v_1 + w_1, v_2 + w_2, v_3 + w_3 >$

There are 3 special vectors that form the Cartesian basis:

$$\vec{i} =< 1, 0, 0 > \qquad \vec{j} =< 0, 1, 0 > \qquad \vec{k} =< 0, 0, 1 >$$

They are perpendicular (or ORTHOGONAL) to each other, and have length one. This is called an orthonormal basis.

$$\vec{i} =< 1, 0, 0 > \qquad \vec{j} =< 0, 1, 0 > \qquad \vec{k} =< 0, 0, 1 >$$

This gives us another way of writing vectors:

$$\vec{v} =< 3, -1, 4 >= 3\vec{i} - \vec{j} + 4\vec{k} = 3 < 1, 0, 0 > - < 0, 1, 0 > +4 < 0, 0, 1 >$$

Useful properties for vectors denoted $\vec{a}$, $\vec{b}$ and $\vec{c}$ are on in section 12.2 of Stewart. You can define a vector space from those properties, but that is done in Math 24.

We will look at the geometrical representation of some operators:

Addition subtraction multiplication by a scalar

What is the length, or magnitude, of a vector?
If you start at the origin, the length is the distance from O to the tip of the vector $\vec{v}$. So we write, for $\vec{v} =< v_1, v_2, v_3 >$,

$$|\vec{v}| = ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

What is the length of $c\vec{v}$? It is just the (absolute value) of the product: $|c||\vec{v}|$.

So how can you use this to get a vector of length 1, a so-called unit vector, going in the same direction as a given vector $\vec{v}$? You try with $\vec{v} = 2\vec{i} + 7\vec{j} - \vec{k}$.

$$\vec{u} = \text{unit vector parallel to } \vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 7\vec{j} - \vec{k}}{\sqrt{54}} =< 2/\sqrt{54}, 7/\sqrt{54}, -1/\sqrt{54} >$$

Finding and using unit vectors is a big deal.

You can also describe a vector by its length and the angle(s) it makes with a given direction: In 2D, you use polar coordinates.
Say $\vec{v}$ has length 5 and makes an angle of $3\pi/4$ with the x-axis, so $\vec{v} =< 5\cos 3\pi/3, 5\sin 3\pi/4 >$.
In general, $\vec{v} =< r\cos\theta, r\sin\theta >$

$$\vec{v} =< 5\cos 3\pi/4, \quad 5\sin 3\pi/4 >$$

In 3D, we would need 2 angles. One is the angle with the xz-plane, we still call it θ, and the other is the angle with the z-axis, we call is ϕ. For example $\vec{w} =< 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi >$. We will get back to this, the SPHERICAL COORDINATES.

$$\vec{w} =< 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi >.$$

Important point: When adding or multiplying, or later doing other operations, we have to make sure to compare comparable things. For example:
$\vec{v} + 3$ has no meaning
$\vec{v}\vec{w}$ has no meaning but $\vec{v} + \vec{w}$ does
$\vec{v} + |\vec{v}|$ has no meaning.

Section 12.3: Dot product of vectors (also called scalar or inner product)

We want to take 2 vectors, $\vec{v}$, $\vec{w}$, and combine them to get a scalar (number):

$$\vec{v} \cdot \vec{w} = \text{ a number}$$

If you know $\vec{v}$ and $\vec{w}$ in Cartesian coordinates, then we have a formula:
$\vec{v} =< v_x, v_y, v_z >$
$\vec{w} =< w_x, w_y, w_z >$
$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$
easy eh?

Example: $\vec{v} =< 1, e^t, e^{-t} >$ and $\vec{w} = e^t \vec{i} + 3\vec{j} + e^t \vec{k}$
$\vec{v} \cdot \vec{w} = e^t + 3e^t + 1 = 1 + 4e^t$.

The real question is what is the point of doing this? We need to know a bit more before we can answer that.

Basic properties:
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutativity)
$(s\vec{a}) \cdot \vec{b} = \vec{a} \cdot (s\vec{b}) = s(\vec{a} \cdot \vec{b})$ (associativity)
$\vec{a} \cdot \vec{a} = |\vec{a}|^2 =$ length of $\vec{a}$ squared.
$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ (distribituvity)

Geometric properties
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ with θ the angle between the two vectors.

Proof of scalar product formula

Proof: Consider the triangle with sides $\vec{a}$, $\vec{b}$ and $\vec{a} - \vec{b}$.
What is $|\vec{a} - \vec{b}|^2$?
First way:

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) && (1) \\ &= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} && (2) \\ &= |\vec{a}|^2 + |\vec{b}|^2 - 2\, \vec{a} \cdot \vec{b} && (3) \end{aligned}$$

Now let $\vec{b}_1$ be the projection of $\vec{a}$ onto $\vec{b}$. We then have $|\vec{b_1}| = |\cos\theta||\vec{a}|$. We also define $\vec{b_2} = \vec{b} - \vec{b_1}$ and $\vec{h}$ be the height vector of the triangle. then a second way to compute $|\vec{a} - \vec{b}|^2$ is

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= |\vec{h}|^2 + |\vec{b}_2|^2 \\ &= |\vec{a}|^2 \sin^2\theta + (|\vec{b}| - |\vec{b_1}|)^2 \\ &= |\vec{a}|^2 \sin^2\theta + |\vec{b}|^2 - 2|\vec{b}||\vec{a}|\cos\theta + |\vec{a}|^2\cos^2\theta \\ &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{b}||\vec{a}|\cos\theta \end{aligned}$$

So then we must have that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

So the dot product measures angles too: $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

Example: What is the angle between $\vec{a} =< 4, 1, 2 >$ and $\vec{b} =< 3, 0, -1 >$?

$$\cos\theta = \frac{10}{\sqrt{21}\sqrt{10}} = \sqrt{\frac{10}{21}}$$

so $\theta = \arccos(\sqrt{10/21})$.

Very important corollary: $\vec{v}$ and $\vec{w}$ are orthogonal ($\perp$) if and only if $\vec{v} \cdot \vec{w} = 0$.

If $\vec{a} =< 1, t, t, >$ and $\vec{b} =< 3, 1, -2 >$. When is $\vec{a} \perp \vec{b}$?
$\vec{a} \cdot \vec{b} = 3 + t - 2t = 3 - t$
so $\vec{a} \cdot \vec{b} = 0$ if $3 - t = 0$ so if $t = 3$ and $\vec{a} =< 1, 3, 3 >$.

Let's return to the idea of a projection:

Projection of $\vec{a}$ onto $\vec{b}$.

The vector projection of $\vec{a}$ onto $\vec{b}$ is the portion of $\vec{a}$ that is in the direction of $\vec{b}$. In other words, it is the "shadow" of $\vec{a}$ if the sun is perpendicular to $\vec{b}$. We denote it as: $\text{Proj}_{\vec{b}}\, \vec{a}$.

The length of the vector projection is called the scalar projection and denoted as: $\text{comp}_{\vec{b}}\, \vec{a} = |\, \text{Proj}_{\vec{b}}\, \vec{a}\, |$.

What is $|\vec{b}_1| = \text{comp}_{\vec{b}}\, \vec{a} = |\, \text{Proj}_{\vec{b}}\, \vec{a}\, |$? $\text{comp}_{\vec{b}}\, \vec{a} = |\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

So then , how do we compute $\text{Proj}_{\vec{b}}\, \vec{a}$?
First, how do we obtain a vector parallel to $\vec{b}$ of a prescribed length?
We use unit vectors: $\vec{u} = \frac{\vec{b}}{|\vec{b}|}$ has length 1, in the direction of $\vec{b}$.
So

$$\text{Proj}_{\vec{b}}\, \vec{a} = \vec{u}\, \text{comp}_{\vec{b}}\, \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}.$$

One example of projection you might have seen before is the work in physics, which is the force dotted with the displacement.

Example: $\vec{a} =< 1, 2 >$ and $\vec{b} =< 3, 1 >$

2D Projection example

$\text{Proj}_{\vec{b}}\, \vec{a} = \frac{3+2}{10} < 3, 1 >=< 3/2, 1/2 >$.

This works in 3D just as well, but it is harder to draw: Example: $\vec{a} =< 2, -1, 2 >$ and $\vec{b} =< 0, 3, 1 >$

3D Projection example

$\text{Proj}_{\vec{b}}\, \vec{a} = \frac{0-3+2}{10} < 0, 3, 1 >=< 0, -3/10, -1/10 >$.

What happens if you want to project a vector onto a vector that is perpendicular to the first one?

Lecture 4, Section 12.4: Cross product

Recall the dot product: scalar = $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos\theta$.

There is a second product between 2 vectors: the cross product. This one results in a vector: $\vec{v} \times \vec{w} = \vec{z}$.

Basic properties:
$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutativity, UNUSUAL)
$(s\vec{a}) \times \vec{b} = \vec{a} \times (s\vec{b}) = s(\vec{a} \times \vec{b})$ (associativity)
$\vec{a} \times \vec{a} = 0$
$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ (distribituvity)

Geometric properties:
1) $\vec{z} \perp \vec{v}$ and $\vec{z} \perp \vec{w}$ (orthogonality)
2) $\vec{z}$ points in the direction given by the right-hand rule: $\vec{v}$ = index, $\vec{w}$ = major, $\vec{z}$ = thumb
3) Length of $\vec{z}$ is the area of the parallelogram generated by $\vec{v}$ and $\vec{w}$.
4) $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$.

Property 3 is equivalent to property 4

How do you compute $\vec{v} \times \vec{w}$?
Formula: $\vec{v} \times \vec{w} = (v_y w_z - v_z w_y)\vec{i} + (v_z w_x - v_x w_z)\vec{j} + (v_x w_y - v_y w_x)\vec{k}$
To remember that, we introduce the determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = (v_y w_z - v_z w_y)\vec{i} + (v_z w_x - v_x w_z)\vec{j} + (v_x w_y - v_y w_x)\vec{k}$$

So why does this formula have properties 1, 2, 3 and 4?
We can check 1) directly: $\vec{v} \cdot (\vec{v} \times \vec{w}) = v_x(v_y w_z - v_z w_y) + v_y(v_z w_x - v_x w_z) + v_z(v_x w_y - v_y w_x) = 0$.
And the same goes for $\vec{w}$. For property 2), you can try one case, and then get all other cases via rotation (which doesn't change the cross product).

Properties 3) and 4) are the same really:

$$\begin{aligned}
|\vec{v} \times \vec{w}|^2 &= (v_y w_z - v_z w_y)^2 + (v_z w_x - v_x w_z)^2 + (v_x w_y - v_y w_x)^2 \\
&= (v_x^2 + v_y^2 + v_z^2)(w_x^2 + w_y^2 + w_z^2) - (v_x w_x + v_y w_y + v_z w_z)^2 \\
&= |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 \\
&= |\vec{v}|^2 |\vec{w}|^2 - |\vec{v}|^2 |\vec{w}|^2 \cos^2\theta \\
&= |\vec{v}|^2 |\vec{w}|^2 (1 - \cos^2\theta) \\
&= |\vec{v}|^2 |\vec{w}|^2 \sin^2\theta
\end{aligned}$$

Applications:
1) Area of a triangle or parallelogram
$\vec{a} =< 2, 1, 0 >$ and $\vec{b} =< 1, 3, 0 >$ Area of the parallelogram they span: $| < 0, 0, 5 > | = 5$

Computing Area with the cross-product

Area of the triangle formed by joining their ends: 5/2.

2) Finding a vector perpendicular to 2 given vectors (very important one!)
Take 3 points in space. They must lie within a plane. How can we find a vector that is normal (perpendicular) to that plane? Points are $P =< 1, 2, -1 >$, $Q =< 3, 5, 0 >$ and $R =< 0, 2, 1 >$

Cross-product gives a vector perpendicular to BOTH the vectors used to compute it.

Form vectors: $\vec{PQ} = Q - P =< 2, 3, 1 >$ and $\vec{PR} = R - P =< -1, 0, 2 >$
The normal is $\vec{n} = \vec{PQ} \times \vec{PR} =< 6, -5, 3 >$
Note that $\vec{n} \perp \vec{PQ}$ and $\vec{n} \perp \vec{PR}$. More generally, $\vec{n} \perp \vec{PX}$ for any point X in the plane.

We can now define the scalar triple product (not as critical as the dot or cross product, but still useful):

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Consider the box spanned by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ The height is $|\vec{a}|\cos\theta$ (θ is the angle to the vertical)

Squished box, of volume given by the triple product

The area of the bottom is $|\vec{b}\times\vec{c}|$
So the volume is $|\vec{b}\times\vec{c}||\vec{a}|\cos\theta = |\vec{a}\cdot(\vec{b}\times\vec{c})|$.

Important point: When is the triple product equal to 0?
If $\vec{a}\perp$ to $\vec{b}\times\vec{c}$
So if $\vec{a}$ is in the same plane as $\vec{b}$ and $\vec{c}$, in which case the volume of the box is 0.

With the same $\vec{PQ} =< 2, 3, 1 >$ and $\vec{PR} =< -1, 0, 2 >$ as before, is the point $S = (1, 1, 3)$ in the same plane as P, Q, and R?
Try $\vec{PS} =< -1, -2, 2 >$.
$\vec{PS}\cdot\vec{n} = -6 + 10 + 6 = 10$ so it is out of the plane.

Lecture 5, Section 12.5: Lines and Planes

Lines:

Say we pick a vector $\vec{v}$ and a point P. Let's make the length of $\vec{v}$ variable, by multiplying $\vec{v}$ by a parameter t, so we have $t\vec{v}, t \in \mathbb{R}$.

A point and a vector describe a line

Starting at P, I now add a multiple of $\vec{v}$: $\vec{r} = P + t\vec{v}$. If I allow t to take any value, what do I get? A line. Coordinate by coordinate, we get:

$$\begin{aligned} x(t) &= P_x + tv_x \\ y(t) &= P_y + tv_y \\ z(t) &= P_z + tv_z \end{aligned}$$

Example: Find the line going through $P = (-1, -2, -3)$ and $Q = (2, 0, 1)$.
The direction vector is $\vec{PQ} =< 3, 2, 4 >$. So we get

$$\begin{aligned} x_1(t) &= -1 + 3t \\ y_1(t) &= -2 + 2t \\ z_1(t) &= -3 + 4t \end{aligned}$$

If $t = 0$, we are at P. If $t = 1$, we are at Q, and by varying t we can cover the entire infinite line.

Does it intersect the following line?

$$\begin{aligned} x_2(s) &= 3 + s \\ y_2(s) &= 2s \\ z_2(s) &= -1 - s \end{aligned}$$

Or is it parallel to it?
The direction vector here is $< 1, 2, -1 >$, and that is not parallel (not a multiple) to $\vec{PQ}$. We look for an intersection:

$$x_1(t) = x_2(s) \text{ so } -1 + 3t = 3 + s \text{ so } s = -4 + 3t$$

Now we use the other coordinates to find s

$$y_1(t) = y_2(s) \text{ so } -2 + 2t = 2s = -8 + 6t \text{ so } t = 3/2, s = 1/2$$

In 2D we would be done, line are either parallel or they intersect. In 3D, we need to check the third coordinate

$$z_1(t) = -3 + 4t = 3 \text{ and } \quad z_2(s) = -1 - s = -3/2$$

The z coordinate is not the same, so they DON'T intersect.

Note that the lines may be given with the same parameter (both with t for example). YOU must change one of the parameters then.

One can also eliminate t to get the so-called symmetric equations

$$t = \frac{x - P_x}{v_x} = \frac{y - P_y}{v_y} = \frac{z - P_z}{v_z}$$

Planes:

We want an equation to describe all the points in a plane. To specify a unique plane, we need a point P and a vector NORMAL to the plane $\vec{n}$. How do we know if $\vec{R} = (x, y, z)$ is in the plane?

A plane and its normal vector.

Let $P = (P_x, P_y, P_z)$ and $\vec{n} = (n_x, n_y, n_z)$.
Then $\vec{PR} =< x - P_x, y - P_y, z - P_z >$
If R is in the plane, $\vec{PR} \perp \vec{n}$ so $\vec{PR} \cdot \vec{n} = 0$. So

$$n_x(x - P_x) + n_y(y - P_y) + n_z(z - P_z) = 0$$

or equivalently

$$xn_x + yn_y + zn_z - (n_xP_x) + n_yP_y + n_zP_z) = 0$$

which in general we write as

$$ax + by + cz + d$$

with x, y, z variables and a, b, c, and d constants.

Example: What plane goes through $P = (2, 0, 1)$ with a normal given by $\vec{n} =< 3, 2, -1 >$?
$3x + 2y - z - 5 = 0$ is the plane Π. What is the distance between $P_1 = (3, 1, 6)$ and Π? 0 Ha!
What is the distance between $P_2 = (4, 1, 6)$ and Π?

Calculating the distance between a point and a plane.

Find the length of the projection of $P\vec{P}_2$ onto $\vec{n}$:
$P\vec{P}_2 =< 2, 1, 5 >, \vec{n} =< 3, 2, -1 >$.
distance = $\cos\theta|P\vec{P}_2| = \frac{P\vec{P}_2 \cdot \vec{n}}{|\vec{n}|} = \frac{3}{\sqrt{14}}$ (less than 1).

Distance from a line to a plane? Either 0 or pick any point on the line and to what we just did, since the line must them be parallel to the plane.
Distance from a plane to a plane? Either 0 or pick any point in the plane. Distance from a line to a line? Either 0 or you need to make two planes with normal $\vec{n} = \vec{r_1} \times \vec{r_2}$ and each containing one line. They are now parallel, and we can use the method from the previous line.

How do you find the angle between two planes:
$\Pi = 3x + 2y - z - 5 = 0$ and
$\chi = 2x - y + 2z - 3 = 0$ Well it is the same as the angle between their normals so:

The angle between two planes is the same as that between their normals.

$\vec{n_\Pi} =< 3, 2, -1 >$ and $\vec{n_\chi} =< 2, -1, 2 >$.
$\cos\theta = \frac{\vec{n_\Pi} \cdot \vec{n_\chi}}{|\vec{n_\Pi}||\vec{n_\chi}|} = \frac{2}{\sqrt{14}\sqrt{9}}$

What is the equation of the line intersecting them? We need a point and a vector. A point P on both planes must satisfy both equations: $3x + 2y - z - 5 = 0$ and

$2x - y + 2z - 3 = 0$
Try $z = 0$, this leaves us with $3x + 2y = 5$ and $2x - y = 3$. So $y = 2x - 3$, $7x = 11$ and $x = 11/7$, $y = 1/7$.
So $P = (11/7, 1/7, 0)$.
The direction vector has to be perpendicular to both normals. So it is $\vec{r} = \vec{n_\Pi} \times \vec{n_\chi}$

$$\vec{r} = \vec{n_\Pi} \times \vec{n_\chi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 2 & -1 & c2 \end{vmatrix} = 3\vec{i} - 8\vec{j} - 7\vec{k} =< 3, -8, -7 >$$

So the line is $x(t) = 3t + 11/7, y(t) = -8t + 1/7, z(t) = -7t$.

Lecture 6, Section 12.6: Plotting surfaces in 3D: cylinders and quadrics

We want to plot functions in 3D: z = f(x,y) = height over the xy-plane.

Function of 2 variables over its domain.

More generally, we want to plot all points x, y, z satisfying some equation, not necessarily explicitly. We start here with first degree polynomials:

$$ax + by + cz + d = 0$$

This is a plane, with normal $\vec{n} =< a, b, c >$.
O.K. then on to second degree polynomials:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

That leaves lots of possibilities, so let's simplify things a little. By completing squares, and using a other axes, we can eliminate D, E and F. So for now, we set $D = E = F = 0$.
If $A \neq 0$, we can change $Ax^2 + Gx$ into $A(x + G/2A)^2 - G^2/4A^2$, which is a shift in x and a new value of J. So for now, if $A \neq 0$, we set $G = 0$. Similarly in y and z.

Simplest case: One variable doesn't appear at all → Cylinder. This is plotted as you would any 2D graph, and then extended, unchanged, in the direction of the missing variable. Example

$$z = y^2$$

For ANY x, we get the same parabola. Note that we draw a few curves and connect them.

Function of one variable seen in 3D, named a cylinder.

Second Example:

$$x^2 + \frac{z^2}{4} = 1$$

For any y, we get an ellipse. Fixing one variable yields a curve, which is called a trace, or a CONTOUR

Elliptic cylinder and its contours

when we fix $z = k =$constant.

A more complicated, but not so bad case, is one where A, B, and C all have the same sign.

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

This should look ”spherish”. If we look at some traces, we get:

Ellipsoid

If $x = 0$, we have an ellipse.
If $y = 0$, we have an ellipse.
If $z = 0$, we have an ellipse.
So overall, we get an ELLIPSOID.

Slightly harder is the case where A, B, C don't all have the same sign, and none of them is 0.

$$z^2 - x^2 - y^2 = 1$$

If $x = 0$, we have a hyperbola.
If $y = 0$, we have a hyperbola.
If $z = 0$, we have.. nothing! but if $z = 2$ we get circles
This is called a hyperboloid, of two sheets.

Two-sheet hyperboloid

$$z^2 - x^2 - y^2 = -1$$

gives a hyperboloid of one sheet, as all the contours are circles, and they are all connected.
The special case where $J = 0$ gives a CONE.

One sheet hyperboloid

Finally, say $C = 0$, but A and B are non-zero.
Case 1, they have the same sign

$$z = x^2 + 4y^2$$

If $x = k$, we have a parabola.
If $y = k$, we have a parabola.
If $z = k$, we get an ellipse
so this is a paraboloid (elliptic).

Paraboloid

Case 2, they don't have the same sign

$$z = x^2 - 4y^2$$

If $x = k$, we have a parabola, downward.
If $y = k$, we have a parabola, upward.
If $z = k$, we get a hyperbola
so this is a saddle.

Saddle or hyperbolic paraboloid

Lecture 7, Section 13.1, 13.2: Vector valued functions and space curves

We study functions that take one real number, say t,
and give a vector (like position $\vec{r}$, velocity $\vec{v}$, force $\vec{F}$, etc).

Such functions are called vector-valued functions: $\mathbb{R} \to \mathbb{R}^3$.
$t :\to (x(t), y(t), z(t))$
This can be thought of as a space curve, being traced over time t.

We already know some functions like this! Lines: $\vec{r}(t) =< 2, 3, 0 > +t < 1, -2, 1 >$ or
$x(t) = 2 + t$
$y(t) = 3 - 2t$
$z(t) = t$

These are parametric equations, like the ones you saw in 2D.
Remember this: $x(t) = 2t$, $y(t) = t^2$? This implies that $z = 0$ (and that $y = (x/2)^2$).

a 2D parametrized curve.

Another example: $x(t) = \cos t$, $y(t) = 3 \sin t$, is the same as $x^2 + (y/3)^2 = 1$, an ellipse.

A second example of a parametrized curve.

Note: the same curve may be represented by several parametrisations, through any legitimate change of variables: For example $u = 3t - 1$.

We use the same idea in 3D, although it is sometimes harder to see.
Example: $x(t) = \cos t$, $y(t) = t$, $z(t) = \sin t$ is a helix along the y axis.

A parametrized 3D curve: a helix

Example: $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$ is a cubish thingy.

A second parametrized 3D curve

One common occurrence is at the intersection of 2 surfaces: Intersection of $y^2 + z^2 = 16$ and $x + y = 1$.

A plane cutting through a cylinder

Try $y(t) = 4\cos t$, then $z(t) = 4\sin t$ from the first surface, and find $x(t) = 1 - 4\cos t$ from the second.

A plane cutting through an elliptical cone.

Intersection of $z^2 = 2x^2 + y^2$ and $z = x - 2$.
First, we eliminate a variable, in this case z:

$$\begin{aligned}
(x-2)^2 &= 2x^2 + y^2 \\
x^2 - 4x + 4 &= 2x^2 + y^2 \\
4 &= x^2 + 4x + y^2 \quad \text{now we complete the square} \\
4 &= (x+2)^2 - 4 + y^2 \\
8 &= (x+2)^2 + y^2
\end{aligned}$$

So in the variables x and y alone, we have a circle, which we parametrize as:
$x + 2 = \sqrt{8}\cos t$ and $y = \sqrt{8}\sin t$
and we can find z from $z = x - 2$, so that in the end we have:
$x = \sqrt{8}\cos t - 2$
$y = \sqrt{8}\sin t$
$z = \sqrt{8}\cos t - 4$.

We can now (finally!) do calculus on space curves. Really, it is easy: you do it component-by-component.

Say $\vec{r}(t) =< x(t), y(t), z(t) >$. Then we have

$$\begin{aligned}
\lim_{t\to 2} \vec{r}(t) &= < \lim_{t\to 2} x(t), \lim_{t\to 2} y(t), \lim_{t\to 2} z(t) > \\
\frac{d\vec{r}(t)}{dt} = \vec{r}'(t) &= < x'(t), y'(t), z'(t) > \\
\int \vec{r}(t)dt &= < \int x(t)dt, \int y(t)dt, \int z(t)dt >
\end{aligned}$$

There are differentiation rule, some old ones:

$$\begin{aligned}
(\vec{u}(t) \pm \vec{v}(t))' &= \vec{u}'(t) \pm \vec{v}'(t) \\
(c\vec{u}(t))' &= c\vec{u}'(t) \\
(\vec{u}(f(t)))' &= f'(t)\vec{u}'(f(t))
\end{aligned}$$

And some that are new, but they don't look very different from what you know

$$\begin{aligned}(f(t)\vec{u}(t))' &= f(t)\vec{u}'(t) + f'(t)\vec{u}(t)\\ (\vec{u}\times\vec{v})' &= \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}' \ \text{ still a vector}\\ (\vec{u}\cdot\vec{v})' &= \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}' \ \text{ still a scalar}\end{aligned}$$

We can easily check that last one, say in 2D:

$$(\vec{u}\cdot\vec{v})' = (u_x v_x + u_y v_y)' = u_x' v_x + u_x v_x' + u_y' v_y + u_y v_y' = u_x' v_x + u_y' v_y + u_x v_x' + u_y v_y' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$

That last one is useful when considering lengths. Consider a satellite in orbit around the earth. Say its distance to the center of the earth is constant: $|\vec{r}(t)| = R$. Then we have

$$\begin{aligned}\vec{r}(t)\cdot\vec{r}(t) &= R^2\\ \frac{d(\vec{r}(t)\cdot\vec{r}(t))}{dt} &= 0\\ \vec{r}'\cdot\vec{r} + \vec{r}\cdot\vec{r}' &= 0\\ 2\vec{r}'\cdot\vec{r} &= 0\\ \vec{r}'\cdot\vec{r} &= 0\end{aligned}$$

That means that the position (relative to the origin) is always perpendicular to the velocity $\vec{r}'$. This is true with higher derivatives too, so the velocity is perpendicular to the acceleration.

Position is perpendicular to velocity.

Lecture 8, Section 13.3: Arclength

Great, so we can do calculus component-by-component, but what does this mean?

For the limit, one can way to think of this is to think of t as time. Then the limit as $t \to t_0$ of $\vec{r}(t)$ is asking: Where were you as the time approached t_0? (You have the right to remain silent though)

The integral is mostly meaningful as an antiderivative really.

The derivative is the important one. Say $\vec{r}(t)$ represents a position changing in time. Then
$\frac{d\vec{r}}{dt}$ = rate of change of the position over time
which is the same as saying how fast is the position changing
which is the VELOCITY vector.
If you trace the curve $\vec{r}(t)$, then the velocity is TANGENT to the curve:

$$\frac{d\vec{r}(t)}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

tangent vector

We call $|\vec{r}'(t)|$ the speed (it is a scalar). We also introduce

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

the unit tangent vector, which has length one and points in the direction of motion.
Note that we have $\int \vec{r}'(t)dt = \vec{r}(t) + C$

Now, what is $\vec{r}''(t)$? It is the ACCELERATION. It describes how $\vec{r}'(t)$ changes over time, both in magnitude and in DIRECTION. It is also a vector.

Example: $\vec{r}(t) =< \cos t, t, \sin t >$
This is a helix moving along the y-axis. We have
$\vec{v}(t) = \vec{r}'(t) =< -\sin t, 1, \cos t >$ with speed $|\vec{r}'(t)| = \sqrt{2}$ and
$\vec{a}(t) = \vec{r}''(t) =< -\cos t, 0, -\sin t >$.

Position, velocity, and acceleration

How can we calculate the arclength of a curve?
Think of the curve as a string. There is an old folk tale where a God asked a warrior to measure the length of a sacred string, without straightening it out. The warrior's solution was to count how long it took for an ant to walk along it, and then multiply that by how fast the ant was walking! This is all true, and it works, except that there is no such old folk tale, I made it up.

So how fast does our "ant" go? $|\vec{r}'(t)|$ = speed. How far does it go if it walk for a time of Δt? $\Delta s = |\vec{r}'(t)|\Delta t$. But usually the speed is not constant, so we need to add up all those small contributions:

Adding up small displacements to compute the total arclength

Arclength is $s = \lim_{n\to\infty} \sum_{i=0}^{n} |\vec{r}'(t_i)|\Delta t_i = \int_0^t |\vec{r}'(\tau)| d\tau$
Or in more details:

$$s(t) = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + z'(\tau))^2} d\tau$$

What do you think is ds/dt? Well it is your speed again $|\vec{r}'(t)|$.

Often, we like to use s as a parameter instead of t, because it is a "natural" choice, that has more meaning than an arbitrarily chosen time. So we then have $\vec{r}(s)$ is a curve and for example $\vec{r}(1)$ is your position after you traveled one unit in distance.

What would be $\frac{d\vec{r}}{ds}$ then? and $|\frac{d\vec{r}}{ds}|$? That last one is 1, because your speed is then exactly 1, always. To calculate the derivative, we use the chain rule

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \vec{T}(t)$$

so it is the unit tangent vector again.

Note that a big deal if you study curves a bit more (like I do) is $\frac{d\vec{T}}{ds}$ and $\kappa = \left|\frac{d\vec{T}}{ds}\right|$. This last one is called the curvature, and we have $\frac{d\vec{T}}{ds} = \kappa\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the curve (and so is perpendicular to the unit tangent vector).

Lecture 9, Section 13.1,2,3,4: Examples of Parametrizations

Let us now look at a few examples of vector valued functions. We will look for parametrizations of curves. A parametrization is a specific equation, or equations describing a curve in terms of a parameter.

The cycloid

Example 1: Cycloid.
Consider a point fixed on a wheel of radius R rolling on a horizontal surface. What is the position of the point over time? (figure)
Assume the angle to the horizontal is changing at a constant rate $\theta = \omega t$. We then have the position $\vec{r}(t) =< x(t), y(t) >$

$$\begin{aligned} x(t) &= -R\theta + R\cos\theta = -R\omega t + R\cos\omega t \\ y(t) &= R\sin\theta = R\sin\omega t \end{aligned}$$

The velocity is then $\vec{r}'(t) =< x'(t), y'(t) >$

$$\begin{aligned} x'(t) &= -R\omega - \omega R\sin\omega t \\ y;(t) &= \omega R\cos\omega t \end{aligned}$$

with a speed of $|\vec{r}'(t)| = R\omega\sqrt{2 + 2\sin\omega t} = R\omega\sqrt{2}|\cos\omega t/2 + \sin\omega t/2|$.

We can calculate the arclength, if $\cos\omega t/2 + \sin\omega t/2 > 0$ or if $-\pi/4 < \omega t/2 < 3\pi/4$ or $-\pi/2 < \omega t < 3\pi/2$. Then we have

$$s = \int_0^t d\tau|\vec{r}'\tau| = \int_0^t d\tau R\omega\sqrt{2}|\cos\omega\tau/2 + \sin\omega\tau/2 = \sqrt{2}R\omega(2/\omega)(-\cos\omega t/2 + \sin\omega t/2)|_0^t = 2\sqrt{2}R(1 - \cos\omega t/2 + \sin\omega t/2)$$

The acceleration is $\vec{r}''(t) = R\omega^2 < -\cos\omega t, -\sin\omega t >$. This always points toward the center of the circle.

Note that $\vec{N} = |\frac{d\vec{T}}{ds}|^{-1}\frac{d\vec{T}}{ds}$ is the unit normal and
the acceleration has a normal component and a tangential component"

$$\vec{r}'' = r_N\vec{N} + r_T\vec{T}.$$

Second example: Projectile, a classic Say that North is the y-axis.
A missile is fired toward the NW, with an angle of $\pi/3$ from the vertical with an initial velocity of 240m/s.
1) Calculate its trajectory if air has been sucked out by the army (temporarily)
2) Determine where it will land.

A projectile in 2D and 3D

We want $\vec{r}t)$. What do we have?
$\vec{r}''(t) =< 0, 0, -g >\approx< 0, 0, -10 > m/s^2$
$\vec{r}'(0) = 240$m/s$< \sin \pi/3 \cos 3\pi/4, \sin \pi/3 \sin 3\pi/4, \cos \pi/3 >$
$\vec{r}'(0) = 240$m/s$< -\sqrt{6}/4, \sqrt{6}/4, 1/2 >$
We will assume for convenience that $\vec{r}(0) =< 0, 0, 0 >$.

Integrating the acceleration, we find
$\vec{r}'(t) = \vec{r}'(0)+ < 0, 0, -10t >=< -60\sqrt{6}, 60\sqrt{6}, 120 - 10t >$m/s
And integrating the velocity we found, we get
$\vec{r}(t) = \vec{r}(0)+ < -60\sqrt{6}t, 60\sqrt{6}t, 120t - 5t^2 >=< -60\sqrt{6}t, 60\sqrt{6}t, 120t - 5t^2 >$.
So we found an equation for the trajectory.

Where does it land? Wherever $z(t) = 0$. So 120t = $5t^2$ and $t = 24$s or $t = 0$.
The position is then $< -60\sqrt{6}24, 60\sqrt{6}24, 0 >$.
How far is that? $24 * 60 * \sqrt{12} \approx 5$km.

What is the speed? $v = (60^2 6 + 60^2 6 + (120 - 10t)^2)^{1/2} = 10(576 - 24t + t^2)^{1/2}$

The arclength? $s = \int_0^t 10(576 - 24\tau + \tau^2)^{1/2} d\tau$
Highest point? It is where $z'(t) = 0$ so when $120 = 10t$ so when $t = 12$. So the point is $\vec{r}(12) =< -720\sqrt{6}, 720\sqrt{6}, 720 >$m. So the maximum height is 720m.

Once again, we have

$$\vec{r}'' = r_N\vec{N} + r_T\vec{T}.$$

with part of the acceleration changing the speed of the projectile, and part changing its direction.

Lecture 10, Section 14.1: Multivariable functions

We now study more general two-variable scalar functions. They take 2 inputs and produce a single output: $\mathbb{R}^2 \to \mathbb{R}$ which is written as $(x, y) \to f(x, y)$.

The DOMAIN is the set of all acceptable inputs, living in the xy-plane.
The RANGE or the IMAGE is the set of all possible outputs, living in $\mathbb{R}$ (along the z-axis).

Example: $f(x, y) = \sqrt{x} + \sqrt{3 - y}$
So $f(1, 2) = 2$, for example. The domain is $\{x \geq 0, y \leq 3\}$
The range is $\{z \geq 0\}$.

Domain and range of $f(x, y) = \sqrt{x} + \sqrt{3 - y}$

Very often quantities depend on more than 2 variables. For example, your final grade depends on your quizzes, homeworks, and exams. Or the temperature depends of the location in 3D space and on time.

Consider $f(x, y) = \sqrt{y - x} \log(x + y)$.
The domain is given by $y \geq x$ and $x + y > 0$.
The range is $\mathbb{R}$.

Domain and range of $f(x, y) = \sqrt{y - x} \log(x + y)$

Consider $f(x, y) = e^{\sqrt{4 - x^2 - y^2}}$.
Domain is $\{(x, y) | x^2 + y^2 \leq 4\}$
Range is $\{z | e^0 \leq z \leq e^2\}$.

Let's graph it. Using traces, we find:
if $x = 0$, $z = e^{\sqrt{4 - y^2}}$.
if $x = \pm\sqrt{3}$, $z = e^{\sqrt{1 - y^2}}$. Note that these traces are 2D curves, like the ones we know well, and we could differentiate, them , find their tangent line, etc.
You get similar traces if you fix y, because we have x symmetric to y here.

Domain and range of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Looking at level curves $z = 0$ gives nothing.
$z = 1$ gives $\sqrt{4 - x^2 - y^2} = 0$ so $x^2 + y^2 = 4$.
$z = 2$ gives $\sqrt{4 - x^2 - y^2} = \log 2$ so $x^2 + y^2 = 4 - (\log 2)^2$.
so we get smaller and smaller circles.

Traces and level curves of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Putting is all together, we get a kind of ”cap” hanging in space.

Surface of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Note: What are the contours of $f(x, y) = z = -x - 2y + 4$?
If we fix $z = k$, we get $k = -x - 2y + 4$, which are straight lines (figure).
A good way to visualize a plane is to find where it intersects the x, y, and z planes and connect the triangle.
Here we have (figure)
$x = y = 0$ gives $z = 4$
$x = z = 0$ gives $y = 2$
$y = z = 0$ gives $x = 4$

Level curves and surface of $f(x, y) = z = -x - 2y + 4$

We can also try to represent 3-variable functions $w = g(x, y, z)$. The domain is then a portion of the 3D space, and the whole "hyper-surface" requires 4 dimensions to be represented. This is hard (but you could make a movie!)

Example: $w = f(x, y, z) = (x - 1)^2 + y^2 + z^2$. If we look at level surfaces, we would get:
$w = k = (x - 1)^2 + y^2 + z^2$.
so if $k = 0$, we get a point. If k increases, we get spheres of radius $\sqrt{k}$. This can be represented with a

Level surfaces of $w = f(x, y, z) = (x - 1)^2 + y^2 + z^2$

movie. More importantly, we can still do calculus on such an object.

Lecture 13, Section 14.3: Derivatives of functions of several variables

Recall the definition of a derivative:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{ slope of the tangent}$$

Derivative of a function of one variable.

But surfaces don't have one slope. Rather, their slope depends on which way you look at them. Think of a mountain, where you can stay level, or go up, or down.

The slope of a surface depends on the direction in which you look.

Strategy: Consider only one direction at a time, then compute the slope.
How can we easily do that? Recall our traces, where one variable was **fixed**.

So to look in the x-direction, we fix $y = b$.
Then $f(x, y)$ becomes $f(x, b)$, a function of only one variable. We can also denote it $f_b(x)$.

Slope of the tangent in the x-direction

We can now take its derivative. We differentiate only part of $f(x, y)$, so we call it the PARTIAL DERIVATIVE of $f(x, y)$ with respect to x:

$$\left.\frac{\partial f}{\partial x}\right|_{y=b} = f_x(x, b) = f_b'(x) = \lim_{h\to 0} \frac{f(x+h, b) - f(x, b)}{h}$$

Note that the other variable (here y) has to be fixed. This is the slope of the tangent of the trace when $y = b$, or the slope of the surface when looking in the x-direction.

Similarly

$$\left.\frac{\partial f}{\partial y}\right|_{x=a} = f_y(a, y) = f_a'(y) = \lim_{h\to 0} \frac{f(a, y+h) - f(a, y)}{h}$$

You may think of these 2 traces (one with fixed x and one with fixed y) as intersecting roads, with $f(x, y)$ representing the height of the ground.

Standing in front of the library, we'll call y the gym-bound direction and x the library-bound direction. Then $f_y \approx 0$, as this is pretty flat, but $f_x < 0$ as it is going down toward the gym.

A contour diagram allows to ESTIMATE partial derivatives

Using contours to estimate partial derivatives

$$f_x(0, 1) = \frac{(f(3, 1) - f(0, 1)}{3} = \frac{2-1}{3} = 2/3$$

$$f_y(0, 1) = \frac{(f(0, 1.8) - f(0, 1)}{0.8} = \frac{2-1}{0.8} = 1.25$$

In practice, we find $f_x(x,y)$ and $f_y(x,y)$ by treating the other variable as a constant. For example, consider $f(x,y) = 3x^2y^4$.
Then $\frac{\partial f}{\partial x} = f_x(3y^4)(2x) = 6xy^4$
and $\frac{\partial f}{\partial y} = f_y(3x^2)(4y^3) = 12x^2y^3$.

Consider $z = f(x,y) = \sqrt{4-x^2-y^2}$, the top half of a sphere, and look at the point $(0,1)$.

$$f_x(1,0) = f_x(x,y)\|_{x=1,y=0} = \left.\frac{-2x}{2\sqrt{4-x^2-y^2}}\right|_{(1,0)} = \frac{-1}{\sqrt{3}} \text{ (down)}$$

$$f_y(1,0) = f_y(x,y)\|_{x=1,y=0} = \left.\frac{-2y}{2\sqrt{4-x^2-y^2}}\right|_{(1,0)} = 0 \text{ (flat)}$$

Slopes on a half-sphere

Could we get the equation of a tangent plane out of those 2 slopes?

Lecture 12, Section 14.4: Tangent planes & Linear Approximations

We know how to calculate the slope in x ($\partial f/\partial x$) and in y ($\partial f/\partial y$). We want to use this to approximate the surface.

Recall the tangent line approximation in 2D
Starting with $f(x)$, we find a linear approximation near the point $(x_0, y_0 = f(x_0))$. Our linear approximation is $l(x) = y_0 + f'(x_0)(x - x_0)$.

Tangent line to a curve, and illustration of a tangent plane to a surface

We would like to use a TANGENT PLANE, $\Pi(x, y)$ to approximate $z = f(x, y)$. We require a few things of this plane:
1) Plane has to agree with the surface at a point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$.
So we will have $\Pi(x, y) - z_0 = m(x - x_0) + n(y - y_0)$ or $z = m(x - x_0) + n(y - y_0) + f(x_0, y_0)$.
2) The x-slope of the plane and of the surface should be the same at (x_0, y_0).

$$\frac{\partial \Pi}{\partial x} = m = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$$

3) The y-slope of the plane and of the surface should be the same at (x_0, y_0) too.

$$\frac{\partial \Pi}{\partial y} = n = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0)$$

So we have $\Pi(x, y) = z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$.

For example: find the tangent plane to $z = f(x, y) = x^2 + y^2/4$ above (1/2,1)
Plane is $z = (x - 1/2) + 1/2(y - 1) + 1/2$.

A function is said to be differentiable if a tangent plane can be found and if it gives a "good" approximation to $f(x, y)$. Then $\Pi(x, y)$ is a LINEARIZATION of $f(x, y)$.

Example: We had $z = 1/2 + (x - 1/2) + 1/2(y - 1)$ as an approximation to $f(x, y) = x^2 + y^2/4$ near $(1/2, 1)$.

What is $f(0.47, 1.02)$? I don't know exactly, it is too hard.
Our approximation will be: $f(0.47, 1.02) \approx \Pi(0.47, 1.02) = 1/2 - 0.03 + 0.01 = 0.48$.

More precisely, a function is differentiable if the following condition holds

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \Pi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

Some example of surfaces that are not differentiable include the tip of a cone and a step.

Theorem: If $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and are continuous, then $f(x, y)$ is differentiable at (x_0, y_0).

We can express our linearisation in terms of differentials (amount of change).
Define $\Delta z = z - z_0$ and $\Delta x = x - x_0$ and $\Delta y = y - y_0$.
Our tangent plane formula is then

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

This formula is exact on the plane, but only approximative for the original function.

If we take the limit of $\Delta x, \Delta y, \Delta z \to 0$, we get the total differential

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

This is an easy way to calculate how much a function changes. In fact it is the same method as using the tangent plane, but we refer to both methods indiscriminately.

Why does it matter? Why bother?
Because real functions, real life, is too hard to handle exactly. So we use linear functions, which we can work with, instead of realistic ones we can't work with.

Lecture 13, Section 14.5: Chain Rule for grown-ups

Very often, functions depend on variables which themselves depend on other variables.
For example: The price of burritos depends on the price of rice (r) and of salsa (s).

$$B(r, s) = 20r + s^2$$

Both the prices of rice and salsa depend on time: $r(t) = 1 + t/2$ and $s(t) = 2 + 3t^3$.
So $B(r(t), s(t))$ is really a function of time (only).
Given t, we can find $r(t)$, $s(t)$ and then B:

$$\begin{aligned} B(t) &= = B(r(t), s(t)) = 20r(t) + (s(t))^2 \\ &= 20(1 + t/2) + (2 + 3t^3)^2 \\ &= 20 + 10t + 4 + 12t^3 + 9t^6 \end{aligned}$$

So what is $\frac{dB}{dt}$?
We can calculate it easily from the last formula obtained:

$$\frac{dB}{dt} = 10 + 36t^2 + 54t^5$$

or from the previous one:

$$\frac{dB}{dt} = 20(1/2) + 2(2 + 3t^3) * 9t^2 = 10 + 36t^2 + 54t^5$$

But we can also get it directly from the first line:

$$\frac{dB}{dt} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} = 20(1/2) + 2(2s) * 9t^2 = 10 + 36t^2 + 54t^5$$

This is what I call the chain rule for grown-ups.

What if r and s depend on more than one variable? Say time t and distance to Mexico d.

$$\begin{aligned} r(t, d) &= (1 + t/2)(1 + d^2 - 4d) \\ s(t, d) &= (2 + 3t^3)e^{dt} \end{aligned}$$

So do we plug in again? Yuck, that sounds like a recipe for mistakes. But the chain rule is still the same:

$$\frac{\partial B}{\partial t} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} = 20(1/2)(1 + d^2 - 4d) + 2(2s)e^{dt}(9t^2 + d)$$

and similarly

$$\frac{\partial B}{\partial d} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial d} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial d} = 20(1 + t/2)(2d - 4) + 2(2s)e^{dt}(t)$$

So how do we get these formulas? With a diagram

We need to find all the ways (paths) in which B depends on t and add them up. Here $\frac{dB_1}{dt} = (1) + (2)$.
As you go down, you multiply derivatives:

$$\frac{\partial B}{\partial t} = (1) + (2) + (3) = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial B}{\partial u}\frac{\partial u}{\partial t}$$

Chain rule diagram

Try $f(x, y, z)$ with $x = r\cos\theta$, $y = r\sin\theta$ and $z = 1 - r^2$. Note that extra connections in your diagram are harmless (they will just give a 0 derivative).

$$\frac{\partial f}{\partial r} = f_x x_r + f_y y_r + f_z z_r = f_x(\cos\theta) + f_y(\sin\theta) + f_z(-2r)$$

$$\frac{\partial f}{\partial \theta} = f_x x_\theta + f_y y_\theta + f_z z_\theta = f_x(-r\sin\theta) + f_y(r\cos\theta) + f_z(0)$$

Does that work with implicit formulas? Yes. Say $f(x, y, z) = g(x, y, z)$ with $x(r, \theta)$ and $y(r, \theta)$.
So implicitly, we must have $z(r, \theta)$. What is $\frac{\partial z}{\partial r}$? We have

$$f_x x_r + f_y y_r + f_z z_r = g_x x_r + g_y y_r + g_z z_r$$

So

$$\frac{\partial z}{\partial r} = \frac{g_x x_r + g_y y_r - f_x x_r + f_y y_r}{f_z - g_z}$$

In general, if we have $F(x, y, z) = 0$, then this formula becomes

$$\frac{\partial z}{\partial r} = -\frac{F_x x_r + F_y y_r}{F_z}$$

Lecture 14, Section 14.6: Directional derivatives

We can now take a derivative in the x-direction (f_x) or y-direction (f_y). How about other directions?

Derivative in a general direction (seen on a surface and on a contour plot.

Consider a UNIT vector $\vec{u} = u_x\vec{i} + u_y\vec{j} =< u_x, u_y >$, $||\vec{u}|| = 1$.
We define the DIRECTIONAL DERIVATIVE as:

$$D_{\vec{u}}f(a,b) = f_{\vec{u}}(a,b) = \lim_{h\to 0}\frac{f(a+hu_x, b+hu_y) - f(a,b)}{h} = \lim_{h\to 0}\frac{f((a,b)+\vec{u}h) - f(a,b)}{h}$$

Meaning: How fast does f change as you look in the direction $\vec{u}$?

Let us use our linear approximation on $f(a+hu_x, b+hu_y)$:

$$f(a+hu_x, b+hu_y) \approx f(a,b) + hu_xf_x(a,b) + hu_yf_y(a,b) + h\epsilon$$

with ϵ going to 0 as $h \to 0$.

So we get

$$D_{\vec{u}}f(a,b) = \lim_{h\to 0}\frac{hu_xf_x(a,b) + hu_yf_y(a,b) + h\epsilon}{h} = u_xf_x(a,b) + u_yf_y(a,b)$$

Note that this only works if $||\vec{u}|| = 1$.
If the given direction $\vec{v}$ is not unitary, just use $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$ in the formula.

Example: $f(x,y) = x\cos y$, $\vec{v} =< -3, 4 >$, find $f_{\vec{v}}(2,\pi/4) = D_{\vec{v}}(2,\pi/4)$
We get first $\vec{u} =< -3/5, 4/5 >$
and $f_x = \cos y$ so $f_x(2,\pi/4) = \frac{\sqrt{2}}{2}$
and $f_y = -x\sin y$ so $f_y(2,\pi/4) = -\sqrt{2}$
so $D_{\vec{v}}(2,\pi/4) = -3/5\sqrt{2}2 - 4/5\sqrt{2} = \sqrt{2}(-11/10)$

We now introduce an important vector:
The GRADIENT of a function $f(x,y)$ is the VECTOR

$$\text{grad} f(a,b) = \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j} =< f_x(a,b), f_y(a,b) >$$

First use: Rewrite the directional derivative formula:

$$D_{\vec{u}}f(a,b) = f_{\vec{u}}(a,b) = \nabla f(a,b)\cdot\vec{u} = ||\nabla f(a,b)||||\vec{u}||\cos\theta$$

Properties of the gradient:

1. ∇f is a vector living in the domain of $f(x, y)$.
2. ∇f points in the direction of maximum increase of $f(x, y)$
   Why? $D_{\vec{u}} f$ is maximum if $\cos\theta = 1$ so $\theta = 0$ so $\vec{u}||\nabla f$
3. The direction of maximum decrease (minimum increase) is $-\nabla f$
   because there $\theta = \pi$, $\cos\theta = -1$.
4. The length of ∇f is the maximum rate of increase of f
   because if $\theta = 0$, $D_{\vec{u}} = ||\nabla f||$.
5. ∇f is perpendicular to CONTOURS in the domain because if $\vec{u}$ points to a contour then $D_{\vec{u}} f = 0 = \nabla f \cdot \vec{u}$.

We can draw gradients from contours.

Gradient from contour plot.

Example: $f(x, y) = \sqrt{4 - x^2 - y^2}$ a half-sphere.

$$\nabla f = f_x \vec{i} + f_y \vec{j} = \frac{-x}{\sqrt{4 - x^2 - y^2}} \vec{i} - \frac{y}{\sqrt{4 - x^2 - y^2}} \vec{j} = \frac{1}{\sqrt{4 - x^2 - y^2}} < -x, -y >$$

Gradient of $f(x, y) = \sqrt{4 - x^2 - y^2}$

The gradient is easy to define in higher dimension: $g(x, y, z)$ has gradient $\nabla g = < g_x, g_y, g_z >$
it has all the properties mentioned above.

In particular, if $g(x, y, z) = K$, then $\nabla g \perp$ the level surface $g(x, y, z) = K$.

So we have a vector perpendicular to a surface.
Like a normal to a tangent plane? Yes, exactly!

Say $z = f(x,y)$. Let $g(x,y,z) = z - f(x,y) = 0$

then $\nabla g = < -f_x, -f_y, 1 >$ is the normal to the surface, which is the normal to the tangent plane. So $(a, b, f(a,b))$, the equation of the plane is:

$$-(x-a)f_x(a,b) - (y-b)f_y(a,b) + (z - f(a,b)) = 0$$

Better yet, this works for implicit functions: $2x^2 + y^2 + z^2 = 4 = g(x,y,z)$, an ellipsoid.
We have $\nabla g = < 4x, 2y, 2z >$
The tangent plane at $(1,1,1)$ (which is on the surface) is:
Normal is $< 4, 2, 2 >= \hat{n}$.
The plane is $4(x-1) + 2(y-1) + 2(z-1) = 0$ or $4x + 2y + 2z = 8$

so even if $g(x,y,z) = x^{xy^2z^3} + xy - z^2 = 0$, the tangent plane is really easy to find!

Finally, the directional derivative in 3D works the same way

$$g_{\vec{u}}(a,b,c) = D_{\vec{u}}g(a,b,c) = \nabla g(a,b,c) \cdot \vec{u}$$

with $\vec{u}$ a unit vector.

Lecture 15, Section 15.7: Optimization

Remember local min/max in 2D: requires $f'(x) = 0$

2D optimization: Max and min occur at points where the curve is horizontal.

Similarly in 3D, at a local min/max, the surface is flat. That is to say, the tangent plane is horizontal. So at

3D optimization: Max and min occur at points where the tangent plane is horizontal.

a local min/max (a,b), we have
$f_x(a, b,) = 0$ and $f_y(a, b) = 0$ SIMULTANEOUSLY.
But that isn't quite enough: Remember the pringles?

So how do we tell? And how do we know if we have a min or a max?
Use Taylor Series expansion, to the SECOND order. Recall:

$$f(a + \Delta x) = f(a) + f'(a)\Delta x + f''(a)(\Delta x)^2/2 + O((\Delta x)^3)$$

So in 3D, we get:

$$f(a+\Delta x, b+\Delta y) = f(a, b)+f_x(a, b)\Delta x+f_y(a, b)\Delta y+f_{xx}(a, b)(\Delta x)^2/2+f_{xy}(a, b)(\Delta x\Delta y)+f_{yy}(a, b)(\Delta y)^2/2$$

Why does that help? Because I know:
$f(u, v) = u^2 + v^2 + c$ has a minimum at $(0, 0)$.
$f(u, v) = -u^2 - v^2 + c$ has a maximum at $(0, 0)$.
$f(u, v) = u^2 - v^2 + c$ has a saddle point at $(0, 0)$.

So at a critical point (where $f_x = f_y = 0$), we have:

$$f(a + \Delta x, b + \Delta y) = f(a, b) + f_{xx}(a, b)(\Delta x)^2/2 + f_{xy}(a, b)(\Delta x\Delta y) + f_{yy}(a, b)(\Delta y)^2/2$$

If we complete the square (assuming that $f_{xx} \neq 0$) we get

$$f(a + \Delta x, b + \Delta y) - f(a, b) = 1/2(f_{xx}(\Delta x + f_{xy}/f_{xx}\Delta y)^2 + (\Delta y)^2(f_{yy} - (f_{xy})^2/f_{xx}))$$

and if we let $u = \Delta x + f_{xy}/f_{xx}\Delta y$, $v = \Delta y$ and $D = f_{xx}f_{yy} - f_{xy}^2$, we get

$$f(a+u, b+v) - f(a,b) = 1/2(f_{xx}u^2 + v^2(D/f_{xx}))$$

So if:
$f_{xx} > 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then we have a minimum.
$f_{xx} < 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then we have a maximum.
$D < 0$ we have a saddle point.

If $D = 0$, this is inconclusive. We would then need to look either at higher derivatives, or use other means to figure it out. For example, $f(x,y) = x^6 + y^8$ has a minimum at $(0,0)$ because everywhere else it is positive.

Example:$f(x,y) = x^3/3 - 5x^2/2 + 4x + 100 + (y-2)^2$
Then $f_x = x^2 - 5x + 4 = (x-4)(x-1)$
and $f_y = 2(y-2)$. So we have TWO critical points: (1,2) and (4,2). Let us classify them:
$f_{xx} = 2x - 5$
$f_{xy} = 0$
$f_{yy} = 2$
So $D = 4x - 10$. If $x = 1$, $y = 2$, we have $D = -6 < 0$ so it is a saddle-point at (1,2).
If $x = 4$, $y = 2$, $D = 6 > 0$ and $f_{xx} = 3$ so we have a local minimum at (4,2)

Constrained Optimization: A global max or min is a point (x_0, y_0) such that: $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) in the domain under consideration.

A Global max/min may occur:
1) At a local max/min
2) On a boundary of the domain, including at corners if applicable. 3) Nowhere, if $f(x,y) \to \infty$ or $-\infty$ within the domain (including as x or y approaches infinity.

Example: Let x be the time spent studying math in a day, and y be the time spent studying anything in a day.
Let the accumulated knowledge be $f(x,y) = x^3/3 - 5x^2/2 + 4x + 100 + (y-2)^2$.
We want to optimize this over the region $x \geq 0$, $y \geq x$ and $y \leq 24$.

Domain for constrained optimization.

Is there a local min or max? Well we found critical points (1,2) and (4,2) before. Only (1,2) is within our domain, and it was a saddle-point.

So we need to check along the boundaries.
Along $x = 0$, we have
$f(0,y) = (y-2)^2 + 100$
which has a minimum at $y = 2$, where $f(0,2) = 100$.

Along $y = 24$, we have

$f(x, 24) = g(x) = x^3/3 - 5x^2/2 + 4x + 100 + 484$
. So we find $g'(x) = 0$ if an only if $x = 1$ and $x = 4$. Both are within our domain.
We have $g''(x) = 2x - 5$ so at $x = 1$, we have a local maximum, and at $x = 4$ a local minimum.
We have $f(1, 24) = 1/3 - 5/2 + 4 + 584 = 584 + 11/6$ and
$f(4, 24) = 64/3 - 40 + 16 + 584 = 584 - 8/3$

Along $y = x$, we have
$f(x, x) = g(x) = x^3/3 - 5x^2/2 + 4x + 100 + (x - 2)^2$
. $g'(x) = x^2 - 5x + 4 + 2(x - 2) = x^2 - 3x$
So $g'(x) = 0$ at $x = 0$ and $x = 3$. We also have
$g''(x) = 2x - 3$ so at $x = 0$ we have a maximum and $f(0, 0) = 104$ and at $x = 3$ we have a local minimum and $f(3, 3) = 9 - 45/2 + 12 + 100 + 1 = 99 + 1/2$.

Finally, we need to check the corner of our domain:
$f(0, 0) = 104$ as we know already
$f(0, 24) = 100 + 484 = 584$ and
$f(24, 24) = 24^3/2 - 524^2/2 + 4 * 24 + 100 + 484 = 6152$.

So among all our candidates for global min and max, we see that the maximum knowledge is achieved at $(24, 24)$, and the minimum at $(3, 3)$.

So the best is to study math all the time, but if you don't do that enough you get the worst, by thinking that you know when actually you don't.

Lecture 16, Section 15.1: Double integrals

Recall the Riemann sums:

Riemann sum for a function of one variable

To calculate the area under a curve, use rectangles (because you know their area).

The idea is that, as you take more and more rectangles (smaller and smaller ones), you get a better approximation of the area under the curve.

$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x^*)(x_i - x_{i-1}) \text{ with } x^* \in [x_{i-1}, x_i]$$

A few remarkable facts:
1) The limit exists for any continuous functions (and more really)
2) This works for any $x^* \in [x_{i-1}, x_i]$.
3) $\int_0^x f(x)dx = F(x)$ is the ANTIDERIVATIVE of $f(x)$. So $F'(x) = f(x)$.

The goal of this class is for you to be able to integrate over any domain:
1) A part of the xy-plane
2) A part of $\mathbb{R}^n$
3) A curve
4) A general surface.

The idea is always the same: Add up all the values of $f(x^*)$, multiplied by the size of the region within which x^* is taken.

So consider $f(x,y): \mathbb{R}^2 \to \mathbb{R}$
Its domain is in $\mathbb{R}^2$ so we will integrate over a region of $\mathbb{R}^2$, a rectangle for now.

We will approximate the VOLUME under the surface $f(x,y)$ using rectangular prisms (boxes).

2D Domain of integration and Riemann sum for a function of 2 variables.

$$V \approx \sum_{j=1}^{n}\sum_{i=1}^{n} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

with $(x^*, y^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

We will get progressively better estimates as the rectangles get smaller, so that

More boxes yield a more accurate approximation of the integral.

$$V = \int_a^b dx \int_c^d dy f(x,y) = \lim_{m\to\infty, n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{n} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

Remarkably:
1) This still works for any continuous function $f(x,y)$ (and more really).
2) You can still use any (x^*, y^*) in each small rectangle
3) Using antiderivatives, you can evaluate it again (we'll see how).

The best point to use to approximate these integrals is the one in the middle of the rectangle:

$$(x^*, y^*) = \left(\frac{x_i + x_{i-1}}{2}, \frac{y_j + y_{j-1}}{2}\right)$$

So we have a first way to evaluate integrals: add up volumes of all the boxes (computerized is better). We will see other ways next time.

There are other useful interpretations of the integrals:
Average of a function. In 2D, we had

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^n (x_i - x_{i-1})} \sum_{i=1}^n f(x^*)(x_i - x_{i-1}) = \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

Similary, in 3D:

$$\bar{f} = \frac{\int_a^b dx \int_c^d dy f(x,y)}{\int_a^b dx \int_c^d dy} = \frac{\int_a^b dx \int_c^d dy f(x,y)}{(b-a)(d-c)}$$

where the bottom represents the area of the rectangle over which we are integrating.

Note that this will be true even if the area over which we integrate is not a rectangle.

Note that we still have:

$$\int\int_D f(x,y)dxdy + \int\int_D g(x,y)dxdy = \int\int_D (f(x,y) + g(x,y))dxdy$$

$$\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA$$

$$\int\int_{D_1} f(x,y)dxdy + \int\int_{D_2} f(x,y)dxdy = \int\int_{D_1+D_2} f(x,y)dxdy$$

And finally

$$\int\int_D dA = \text{ Area of D}$$

Lecture 17, Section 15.2: Iterated integrals

Recall: Signed Volume between $z = 0$ and $z = f(x, y)$ is

$$V = \int_a^b dx \int_c^d dy\, f(x,y) = \lim_{m\to\infty, n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1}) = \int_a^b dx \int_c^d dy f(x,y)$$

Idea: Add up $f(x, y)$ multiplied by the size of the region over which it is applied.

How do you compute such a signed volume?

Idea: Add up slices. What is the volume of a slice? It is the area under the curve, times Δx.

Slicing a volume with fixed x.

$V = \lim_{n\to\infty} \sum_{i=1}^{n}$ area under the curve $(x_i - x_{i-1})$.
This is the same as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\int_c^d f(x^*, y) dy \right) (x_i - x_{i-1})$$

where each integral is taken for some fixed x^*. As a result, we get an area for each x^*.

The limit of the sum that is left is actually an integral over x, so we can complete the computation by taking the integral of the result we just obtained:

$$\int_a^b dx[\int_c^d f(x,y)dy]$$

Similarly, we can start with x $V = \lim_{m\to\infty} \sum_{j=1}^{m}$ area under the curve $(y_j - y_{j-1})$.
This is the same as

$$\lim_{m\to\infty} \sum_{j=1}^{m} \left(\int_a^b f(x, y^*) dx \right) (y_j - y_{j-1})$$

where each integral is taken for some fixed y^*. And again, we can finish the problem by taking the integral of the result

$$\int_c^d dy[\int_a^b f(x,y)dx]$$

Either order gives the same answer if the original function is continuous.
We evaluate the integrals one at a time, treating the other variable as constant:

Slicing a volume with fixed y.

A rectangular domain of integration.

Example $D = [0, 2] \times [1, 3]$

$$\int\int_D (x + 2y + 3)dA = \int_0^2 dx \int_1^3 dy(x + 2y + 3)dA = \int_0^2 dx \left(\int_1^3 (x + 2y + 3)dy\right)$$

or alternatively

$$\int_1^3 dy \left(\int_0^2 (x + 2y + 3)dx\right) = \int_1^3 dyx^2/2{+}2xy{+}3x\ |_0^2 = \int_1^3 (2{+}4y{+}6{-}0)dy = 8y{+}2y^2\ |_1^3 = 24{+}18{-}8{-}2 = 32$$

Fubini's theorem: If $f(x, y)$ is continuous, then iterated integrals can be evaluated in either order.

Sometimes one order is easier than the other (see examples later).

Example

$$\int_1^2 \int_0^1 \frac{xe^x}{y} dxdy = \int_1^2 \frac{dy}{y} \int_0^1 xe^x dx = \log y \,\big|_1^2 (xe^x - e^x\big|_0^1) = \log 2$$

What if the domain of integration is not a rectangle? Use arrows Here $0 \leq y \leq x/2$ for $0 \leq x \leq 3$ so we can

Using arrows to determine the bounds of integration.

integrate

$$\int_0^3 dx \int_0^{x/2} dy f(x, y) = \int_0^{3/2} dy \int_{2y}^3 dx f(x, y)$$

Important: These integrals compute a signed volume, so they should give a NUMBER. That means
1) The bounds of the inner integral can depend on the other variable ONLY
2) The bounds of the outer integral HAVE TO be numbers.

Lecture 18, Section 15.3: Integrating over general domains

In this section, we need to be able to:
1) Integrate over a given domain.
2) Find a domain given bounds of integration.

Given an integral, we can find the corresponding domain of integration

$$\int_{x=a}^{x=b} dx \int_{y=f(x)}^{y=g(x)} dy\ h(x,y)$$

Recall that the outer bound must be constant.
The inner bound can depend on the OTHER variable.
Here, the domain of integration is $D = \{a \leq x \leq b, f(x) \leq y \leq g(x)\}$. This is called a Type I domain (by the

A domain where you should integrate y first.

book anyway).

Similarly, we can have:

$$\int_{y=c}^{y=d} dy \int_{x=m(y)}^{x=n(y)} dx\ h(x,y)$$

with $D = \{c \leq y \leq d, m(y) \leq x \leq n(y)\}$, a type II domain.

A domain where you should integrate x first.

Note that we can integrate over a mixture of type I and type II domains using

$$\int\int_{D_1} h(x,y)dA + \int\int_{D_2} h(x,y)dA$$

Coming up with a figure is ESSENTIAL to understanding the domain of integration.

A domain that needs to be broken up.

How can we go from the bounds of integration to the picture of the domain? Given a domain, find the correct bounds of integration. We call on Cupid and his arrows.

First, we shoot an vertical arrow, to find the inner integral: $\int_{in}^{out}$.
For the outer integral, we look for bounds on where an arrow can be shot from: $\int_{left/bottom}^{right/top}$.
Here $\int_1^4 dx \int_{x/2+12}^{3x-2} dy h(x,y)$.

Domain corresponding to $\int_1^4 dx \int_{x/2+12}^{3x-2} dy h(x,y)$.

How do we find the area of that triangle? Just use $h(x,y) = 1$, so that all we integrate is the area element. We find

$$\int_1^4 (3x - 2 - x/2 - 1/2)dx = 5x^2/4 - 5x/2\big|_1^4 = 20 - 10 - 5/4 + 5/2 = 11\ 1/4$$

Example: Half-circle We have 2 choices:

A half-circle.

Vertically first:

$$\int_0^3 dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\ h(x,y)$$

or horizontally first

$$\int_{-3}^3 dy \int_0^{\sqrt{9-y^2}} dx\ h(x,y)$$

So if $h(x,y) = x$, we find

$$\int_{-3}^3 dy \int_0^{\sqrt{9-x^2}} dyx = \int_{-3}^3 x^2/2\Big|_0^{\sqrt{9-y^2}} dy = \int_{-3}^3 \frac{9-y^2}{2} dy = 9/2y - y^3/6\Big|_{-3}^3 = 27 - 9 = 18$$

or similarly in the other direction

$$\int_0^3 dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dyx = \int_0^3 dx\ xy|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} = \int_0^3 dx 2x(\sqrt{9-x^2}) = -(2/3)(9-x^2)^{3/2}\Big|_0^3 = (2/3)27 = 18$$

We can now deal with more complicated domains

A more complicated domain.

What to do? Break it!

$$\int\int_D h(x,y)dA = \int\int_{D_1} h(x,y)dA + \int\int_{D_2} h(x,y)dA + \int\int_{D_3} h(x,y)dA + \int\int_{D_4} h(x,y)dA$$

$$= \int_{-2}^0 dx \int_{-2x-4}^0 dyh(x,y) + \int_{-4}^0 dy \int_0^{(y+4)/2} dxh(x,y) + \int_{-2}^0 dx \int_0^{\sqrt{1-(x+1)^2}} dyh(x,y) + \int_0^2 dx \int_0^{\sqrt{1-(x-1)^2}} dyh(x,y)$$

Lecture 21, Section 15.4: Integration in polar coordinates

Recall double integrals:
The volume between $z = 0$ and $z = f(x, y)$ over D is $V = \int\int_D f(x, y)dA$, a number
The area of a region in the xy-plane is $A = \int\int_D dA$
The average of a function over a domain D is

$$\bar{f} = \frac{\int\int_D f(x, y)dA}{\int\int_D dA}$$

with the following properties:

$$\int\int_D (af(x, y) + bg(x, y))dA = a\int\int_D f(x, y)dA + b\int\int_D g(x, y)dA$$

$$\int\int_{D_1} f(x, y)dA + \int\int_{D_2} f(x, y)dA = \int\int_{D_1+D_2} f(x, y)dA$$

and with definition

$$\int\int_D f(x, y)dA = \lim_{n\to\infty, m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x\Delta y$$

with $\Delta x\Delta y = dA$.

This definition is equivalent of splitting the domain D into rectangles, and adding them up.

Splitting a domain in cartesian pieces

But there are other ways to split a domain into small pieces

Two more ways to split a domain.

You can even use polar coordinates!
Every point (x, y) may be written as $(r\cos\theta, r\sin\theta)$, or in polar form (r, θ).
In the other direction, we have $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

In polar form, $f(x, y)$ may be rewritten $f(r\cos\theta, r\sin\theta)$. So we can rewrite the integral as

$$\int\int_D f(x,y)dA = \int\int_D f(r\cos\theta, r\sin\theta)dA = \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i, \theta_j)dA$$

We can now try to describe D and dA in terms of r and θ.

But what is dA?

The area element in polar coordinates.

$$dA = \frac{\pi(r+\Delta r)^2 - \pi r^2}{2\pi}\Delta\theta = r\Delta r\Delta\theta + (\Delta r)^2\Delta\theta/2$$

This last term is much smaller than the others and so it is negligible.

So we will use $dA = rdrd\theta$ in our integrals. This has the units of area!
You may think of this area as a twisted rectangle of size Δr by $\Delta\theta r$.

For the bounds of integration, you still shoot ”arrows” (automatic weapon/boomerang)

Arrows (cartesian) vs boomerang (polar)

Example 1: a half-circle of radius 2

What is the average of $f(x,y) = \sin(x^2+y^2)$ over D?

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^2 \sin(r^2) r dr d\theta = \frac{1}{2\pi}\pi(-\cos(r^2)1/2 \mid_0^2 = 1/4(1-\cos 2)$$

Example 2: Let $f(x,y) = xy$. Domain is $(x-1)^2+y^2 \leq 1$. We rewrite $(x-1)^2+y^2 = 1$ as $x^2+y^2 = 2x$ so

Example 2: a translated circle of radius 1

$r^2 = 2r\cos\theta$ so $r = 2\cos\theta$.

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 \sin\theta\cos\theta dr d\theta = \int_{-\pi/2}^{\pi/2} 16\cos^5\theta\sin\theta d\theta$$

And the rest is (supposed to be by now) easy.

Example 3

What is the area of the region bounded by $r = \cos(2\theta)$, for $-\pi/4 \leq \theta \leq \pi/4$.

$$A = \int_{-\pi/4}^{\pi/4} \int_{0}^{\cos(2\theta)} dA = \int_{-\pi/4}^{\pi/4} \int_{0}^{\cos(2\theta)} r dr d\theta$$

$$\int_{-\pi/4}^{\pi/4} \frac{\cos^2\theta}{2} d\theta = \frac{2\theta + \sin 4\theta}{4} \ |_{-pi/4}^{\pi/4} = \pi/4$$

Their are 2 reasons to use polar coordinates rather than cartesian coordinates:
1) Either the integrand is suitable to polar coordinates
2) Or the domain is suitable to polar coordinates.

Lecture 22, Section 15.7: Triple Integrals

Going from double to triple integral is a "natural" extension.

We still have Riemann sum definition:

$$\int\int\int_V f(x,y,z)dV = \lim_{m,n,p\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i,y_j,z_k)\Delta V$$

where ΔV is the volume element.

Break space up into small rectangular prism (boxes). Our domain in now in 3D.

Breaking up 3D space into small boxes.

Our integrand is now a function of 3 variables.

What does this all mean?
2 common uses:
1) $\int\int\int_V dV =$ Volume of V.
all double integrals give : $\int\int_D f(x,y)dA = \int\int_D \left(\int_0^{f(x,y)} dz\right) dA$

Volume computed by adding areas.

2) If $f(x,y,z)$ is a density, $\int\int\int_V f(x,y,z)dV$ is a mass.

Recall: density = mass / volume.

This can be made into a local statement: density changes in space.

What is the mass of water over Merced County?
Let $\rho(x,y,z)$ be the density of water in the air, in kilograms per meter cubed.

Mass computed via a triple integral.

$$M = \int\int_A \int_0^\infty \rho(x,y,z) dz dA$$

We still have to shoot arrows, but now in one more direction. Example calories in a brownie.

$$\int\int\int_V (3 + xz - y^3) dV$$

with $V = \{(x,y,z)| - 1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1\}$.
First, you must sketch V. This is important! For a box, we can start in any direction: You try.

A first triple integral.

$$\int_{-1}^1 dx \int_0^2 dy \int_0^1 dz(xz - y^3) = \int_{-1}^1 dx \int_0^2 dy(x/2 - y^3) = \int_{-1}^1 dx(x-4) = -8 + 12 = 4$$

More challenging is the following domain. Consider

A more complex domain.

$$I = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx$$

What is V? (figure) Same as $\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)dzdxdy$

Or start it with y (figure)

$$I = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydxdz$$

or

$$I = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydzdx$$

Or start it with x (figure)

$$I = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)dxdydz$$

or

$$I = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)dxdzdy$$

Lecture 21, Section 15.8: Triple Integrals in cylindrical coordinates

We still want to integrate over a Volume in space, but now we want to describe space using Cylindrical coordinates: r, θ, z.

Recall cylindrical coordinates

Cylindrical coordinates.

r and θ are the same as in polar coordinates, and z is the same as in Cartesian coordinates.
r is the distance to the z-plane, $r^2 = x^2 + y^2$
$0 \leq \theta \leq 2\pi$: Angle to the xz-plane, so $x = r\cos\theta, y = r\sin\theta$
z is the vertical height.

Surfaces easily described in cylindrical coordinates:
$z = K$ is a horizontal plane of height K.
$r = R$ is a vertical cylinder
$\theta = \pi/4$ is a half-plane.

Fixing r, θ, and z in cylindrical coordinates.

To use this in triple integrals $\int\int\int_V f(x, y, z)dV$, we need to:
1) Convert the integrand $f(x, y, z)$ to $f(r, \theta, z)$.
2) Express V in cylindrical coordinates.
3) Express dV in cylindrical coordinates.

So what is dV? $dV =$ length X width X height, so

Volume element in cylindrical coordinates.

$dV = rd\theta drdz$

Example: What is the volume of the cone $z = 2\sqrt{x^2+y^2}$ below the height $z = H$?

$$\begin{aligned} V &= \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{H/2}\int_{2r}^{H} dz\ r\ dr\ d\theta \\ &= (2\pi)\int_0^{H/2}(H-2r)rdrd\theta = (2\pi)(H(r^2/2)-2r^3/3) = 2\pi(H^3/8-2H^3/24) = \pi H^3/12 \end{aligned}$$

Note that this is the same as $V = \pi R^2 H/3$, because here $R = H/2$.

Cone $z = 2\sqrt{x^2+y^2} = 2r$ and paraboloid $z = 2(x^2+y^2) = 2r^2$

What changes if we want the volume inside the paraboloid $z = 2(x^2+y^2)$? Our integral is then:

$$V = \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{\sqrt{H/2}}\int_{2r^2}^{H} dz\ r\ dr\ d\theta$$

Example: what is the average height of points inside the upper half-sphere of radius R? (This is the vertical coordinate of the center of mass).
Here the surface bounding the volume above is $x^2+y^2+z^2 = R^2$ or $z = \sqrt{R^2-(x^2+y^2)}$ or $z = \sqrt{R^2-r^2}$.

So we want:

$$\begin{aligned}
\bar{z} &= \int\int\int_V z dV = \int_0^{2\pi}\int_0^R\int_0^{(R^2-r^2)^{1/2}} z\; dz\; r\; dr\; d\theta \\
&= (2\pi)\int_0^R (R^2-r^2) r dr = 2\pi(R^2r^2/2 - r^4/4)\,\big|_{r=0}^R = \pi R^4/2
\end{aligned}$$

upper half-sphere of radius R and sphere of radius R

Finally, what is the volume of a sphere of radius R?

$$\begin{aligned}
V &= \int\int\int_V dV = \int_0^{2\pi}\int_0^R\int_{-(R^2-r^2)^{1/2}}^{(R^2-r^2)^{1/2}} dz\; r\; dr\; d\theta \\
&= (2\pi)\int_0^R 2(R^2-r^2)^{1/2} r\; dr = 2\pi(2/3)(1/2)(R^2-r^2)^{3/2}\,\big|_{r=0}^R = 4\pi R^3/3
\end{aligned}$$

Fantastic, no?

Lecture 22, Section 15.9: Triple Integrals in spherical coordinates

We still want to integrate over a Volume in space, but now we want to describe space using Spherical coordinates: ρ, θ, ϕ.

Recall spherical coordinates $0 \leq \theta \leq 2\pi$: Angle to the xz-plane

Spherical coordinates.

$0 \leq \phi \leq 2\pi$: Angle to the z-axis
$0 \leq \rho \leq 2\infty$: Distance to the origin

Surfaces easily described in spherical coordinates:
$\rho = R$ is a sphere of radius R, centered at the origin. $\theta = \pi/4$ is a half-plane. $\phi = \pi/4$ is a cone, the same as $z = r$ in cylindrical coordinates.

Fixing ρ, θ, and ϕ in spherical coordinates.

The relation between Cartesian and spherical coordinates.
$\rho^2 = x^2 + y^2 + z^2$, $\cos\phi = z/\rho$, $\tan\theta = y/x$
$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$.

To use this in triple integrals $\int\int\int_V f(x, y, z)dV$, we need to:
1) Convert the integrand $f(x, y, z)$ to $f(\rho, \theta, \phi)$.
2) Express V in spherical coordinates.
3) Express dV in spherical coordinates.

So what is dV? $dV =$ length X width X height, so
$dV = (\rho\sin\phi)(\rho d\phi)d\rho$
$dV = \rho^2 \sin\phi d\theta d\phi d\rho$.

Volume element in spherical coordinates.

Just the spherical volume element.

Example: What is the mass of the Earth if the density is

$$d(x,y,z) = \log(1 - \frac{1}{5}\left(\frac{x^2+y^2+z^2}{R^2}\right)^{3/2})$$

with $R = 6500$ km.

$$\begin{aligned} M &= \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R \log(1 - \frac{1}{5}\frac{\rho^3}{R^3}\rho^2 \sin\phi d\rho d\phi d\theta & (1) \\ &= (2\pi)(-\cos\phi\Big|_0^{\pi})(-5/3R^3(\log(1-\frac{1}{5}\frac{\rho^3}{R^3} - 1)(1 - \frac{1}{5}\frac{\rho^3}{R^3})\Big|_0^R & (2) \end{aligned}$$

where we used $u = 1 - 1/5\rho^3/R^3$ and $du = -3/5\rho^2/R^3 d\rho$.
This simplifies to $4\pi R^3(5/3)(1 + (4/5)(1 - \log(4/5))$.

One other example: Mass of an ice cream cone: $\phi = \pi/6$, so the cone is $z = \sqrt{3}(x^2+y^2)^{1/2}$
The ice cream is a spherical cap of radius $R = 2/\sqrt{3}$ centered at $(0,0,R)$.
So the sphere is $x^2 + y^2 + (z-R)^2 = R^2$ or $\rho^2 = 2\rho R\cos\phi$. And finally the density is $d(x,y,z) = \frac{4zy^2}{x^2+y^2}$
So our integral becomes:

$$\int_0^{2\pi}\int_0^{pi/6}\int_0^{2R\cos\phi} \frac{4\rho\cos\phi\rho^2\sin^2\phi\sin^2\theta}{\rho^2\sin^2\phi}\rho^2\sin\phi d\rho d\phi d\theta$$

Ice cream cone.

Lecture 23, Section 15.10: General change of variables

We have seen several coordinate systems with which to integrate:
Cartesian: (x, y, z) with $dA = dxdy$ and $dV = dxdydz$
Cylindrical: (r, θ, z) with $dA = rdrd\theta$ and $dV = rdrd\theta dz$
Spherical: (ρ, θ, ϕ) with $dA = \rho^2 \sin\phi d\phi d\theta$ and $dV = \rho^2 \sin\phi d\phi d\theta d\phi$

But sometimes none of these work, and you need a General change of coordinates:
Going from (x, y) to (u, v) through some known functions $u(x, y)$ and $v(x, y)$.

You might also want to know their inverse $x(u, v)$ and $y(u, v)$.

Example $u = y - x$ and $v = y + x/2$

A general change of variables.

$x = (2v - 2u)/3$ and $y = (u + 2v)/3$

Example

A second general change of variables.

$x = \bar{r}\cos\bar{\theta}$ and $y = 2\bar{r}\sin\bar{\theta}$
$\bar{r} = \sqrt{x^2 + y^2/4}$ and $\tan\bar{\theta} = \frac{y}{2x}$.

Example $u = xy$ and $v = y/x$
$x = \sqrt{u/v}$ and $y = \sqrt{uv}$.

How do you integrate using any of these? $\int\int_D f(x, y)dA$.
1) Find the proper change of variables $u(x, y), v(x, y)$ and their inverse $x(u, v), y(u, v)$.
2) Describe the domain D in terms of u and v (should be easy).
3) Rewrite the integrand in terms of u and v (only!).
4) Express dA in terms of u and v only.

We will focus on 4) now.

A third general change of variables.

The Math world (u,v) (figure) is related to the Real world (x,y)

From Math world to the Real world, and back.

What is the real area of that rectangle in the uv-plane?

In the xy-plane, we have $\vec{v}_1 =< (x(u+\Delta u, v), y(u+\Delta u, v)) - (x(u,v), y(u,v)) >$
$\vec{v}_1 =< (x(u+\Delta u, v) - x(u,v), y(u+\Delta u, v) - y(u,v)) >$
$\vec{v}_1 =< \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u >$

And similarly $\vec{v}_2 =< \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v >$

So the area is $||\vec{v}_1 \times \vec{v}_2||$ which gives $|\Delta u \Delta v (x_u y_v - x_v y_u)|$.

So finally, $dA = |x_u y_v - x_v y_u| dudv$ The part before $dudv$ is called the Jacobian and is denoted by $\frac{\partial(x,y)}{\partial(u,v)}$.

Note: sometimes, we only have $u(x,y)$ and $v(x,y)$. Then it is good to know that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

For example, in polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. So

$$\frac{\partial(x,y)}{\partial(r,\theta)} = (\cos\theta r\cos\theta - r(-\sin\theta)\sin\theta) = r$$

Like we derived geometrically!

Lecture 24, Section 16.1: Vector Fields

So far, we have dealt with scalar function of several variables, $f(x, y, z)$. To each point in space, these functions associate a number.
And, we also saw vector-valued functions of one variable: $\vec{r}(t) =< x(t), y(t), z(t) >$. To a single number (time) associate a point in space.

Now we want to associate a VECTOR to each point in space. In 2D, that gives:

$$\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j} =< F_1(x, y), F_2(x, y) >$$

with $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and both F_1 and F_2 are scalar functions.
For example: $\vec{F} =< xy, e^y \sin x >$.

To visualize vector fields, we draw vectors $\vec{F}$ starting at point (x, y). So for $\vec{F}(x, y) = x\vec{i} + y\vec{j}$

A 2D vector field.

We get a kind of explosion, or a source.

Similarly, we can go to 3D $\vec{F}(x, y, z) =< F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) >$.
Those are harder to see though. Try

$$\vec{F}(x, y, z) =< \frac{-x}{(x^2 + y^2)^{1/2}}, \frac{-y}{(x^2 + y^2)^{1/2}}, z >=< -\cos\theta, -\sin\theta, z >$$

which points to the z-axis, upward in the upper plane and downward in the lower plane.

A 3D vector field.

In 3D, straight hair is a good example. Other examples come from science:
1) Force fields (like gravity)

Gravity force field.

2) Velocity fields (wind)

A velocity field.

3) Gradient fields $\nabla f =< f_x, f_y, f_z >$
Say $f(x,y) = \frac{M}{\sqrt{x^2+y^2}}$, $\nabla f = M < \frac{-x}{(x^2+y^2)^{1/2}}, \frac{-y}{(x^2+y^2)^{1/2}} >$

A special kind of vector field is a CONSERVATIVE vector field. A vector field $\vec{F}(x,y)$ is conservative if there exists a scalar function $\phi(x,y)$ such that $\vec{F}(x,y) = \nabla\phi(x,y)$. We call this scalar function the POTENTIAL ϕ (from physics).

How do you check if $\vec{F}$ is conservative? Look for $f(x,y)$. If is exists, you should have that $f_{xy} = f_{yx}$ (if they are both continuous). That would mean that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

Lecture 25, Section 16.2: Line Integrals

We are back in 1D domains:
Idea: Your domain is a curve (2D or 3D)

The domain of integration of a line integral is a curve.

$C = \vec{r}(t) =< x(t), y(t) >$. Over this curve and maybe more, a function is defined: $f(x, y)$.

Riemann sum version: $\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$

Riemann sum using a curve as a domain.

where Δs_i is arclength between consecutive points.

$\int_{\vec{r}(t)} f(x, y)ds = \int_C f(x, y)ds = \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$

Example: 1) Say $f(x, y)$ is a density of raised money and C is the trajectory of a political candidate. Then $\int_C f(x, y)ds$ = total money amassed.
2) $f(x, y)$ is the density of "star" in a video game, in star per length, and C is the trajectory of your "player". $\int_C f(x, y)ds$ = total numer of stars amassed.

Right. Given a trajectory, and a function to integrate, how do we calculate $\int_C f(x, y)ds$?
There are 3 steps:
1) Parametrize C with $\vec{r}(t)$.
2) Rewrite $f(x, y)$ as $f(x(t), y(t))$, a function of t only.
3) Recall $ds = \sqrt{x'(t)^2 + y'(t)^2}dt$.

Example:
$\vec{r}(t) =< t, t^2 >$, for $0 \le t \le 2$, so $ds = \sqrt{1+4t^2}dt$

Example 1: a parabola

with $f(x,y) = y/x$, so $f(x(t), y(t)) = t^2/t = t$.

$$\int_C \frac{y}{x} ds = \int_0^2 t\sqrt{1+4t^2}dt = \frac{1}{8}\left.\frac{(1+4t^2)^{3/2}}{3/2}\right|_0^2 = \frac{1}{12}((1+16)^{3/2}-1) = \frac{1}{12}(17^{3/2}-1)$$

Second example: The domain is a circle of radius 3 centered at the origin $\vec{r}(t) =< 3\cos t, 3\sin t >$ so $ds =$

Example 2: a circle

$\sqrt{9\cos^2 t + 9\sin^2 t}dt = 3dt$
$f(x,y) = 15 + y^2$ is the fuel consumption rate. Then the total consumption is

$$\int_C f(x,y)ds = \int_0^{2\pi} (15 + 9\sin^2 t)(3dt) = (15 \cdot 2 \cdot 2\pi + 9\pi)3 = 117\pi$$

Third example: This is an old one: a horizontal line segment, along the x-axis: $\vec{r}(t) =< t, 0 > ds = dt$

Example 3: a horizontal line!

$f(x,y) = g(x)$.

$$\int_C f(x,y)ds = \int_a^b g(t)dt$$

Our good old integral!

Lecture 26, Section 16.2: Line Integrals, continued

IMPORTANT SPECIAL CASE: We want to integrate

$$\vec{F}(x,y) \cdot \frac{d\vec{r}(t)/dt}{|d\vec{r}(t)/dt|}$$

What is that?
Well $\vec{F}(x,y)$ is a vector field.
$d\vec{r}/dt$ is a tangent vector to the curve $\vec{r}(t)$.
$\vec{r}'/|\vec{r}'|$ is a UNIT tangent vector, which we also write as $\vec{T}(t)$.

So we have

$$\vec{F}(x,y) \cdot \vec{T} = \text{Proj}_{\vec{T}}\vec{F}$$

which is the portion of $\vec{F}$ in the direction of $\vec{T}$.

Integrating the componenf of $\vec{F}$ tangent to the curve $\vec{r}(t)$.

What does it mean? If $\vec{F}$ is a velocity field, then
$\vec{F} \cdot \vec{T}$ is the velocity of an object moving on C and
$\oint_C \vec{F} \cdot \vec{T} ds$ is called the circulation if C is closed

The Circulation is the integral of the tangent portion of $\vec{F}$ over a closed curve.

It measures how fast things are going around the contour C.

If $\vec{F}$ is a force, $\vec{F} \cdot \vec{T}$ is the portion of the force in the direction of $\vec{T}$, the part that can be used to move things along C.
$\int_C \vec{F} \cdot \vec{T} ds$ = Work done to move an object along C.

Now, given C and $\vec{F}(x, y)$, how do we compute $\int_C \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} ds$? This is also written as $\int_C \vec{F} \cdot d\vec{r}$.

1) Parametrize C using $\vec{r}(t) =< x(t), y(t) >$
2) Note that

$$\vec{T} ds = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| dt = \frac{d\vec{r}}{dt} dt$$

3) Compute $\vec{F}(x, y)$ on C as $\vec{F}(x(t), y(t)) =< F_1(x(t), y(t)), F_2(x(t), y(t)) >$.
4)Take $\vec{F} \cdot \vec{T} ds$ as $F_1(x(t), y(t)) \frac{dx}{dt} dt + F_2(x(t), y(t)) \frac{dy}{dt} dt$
note that $\frac{dx}{dt} dt$ is sometimes written as dx and $\frac{dy}{dt} dt$ is sometimes written as dy.
5) Integrate the result

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b (F_1(x(t), y(t)) \frac{dx}{dt} + F_2(x(t), y(t)) \frac{dy}{dt}) dt$$

Example: Gravity force $\vec{F}(x, y) =< \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} >$ Let C be a circle centered at $(0, 0)$ of radius 2:

Gravitational force, integrated over a half-circle.

$\vec{r}(t) =< 2\cos t, 2\sin t >$. We will take the upper half only, so $0 \leq t \leq \pi$. Then we have

$$\vec{F}(x(t), y(t)) =< \frac{-2\cos t}{8}, \frac{-2\sin t}{8} >= -1/4 < \cos t, \sin t >$$

$$\frac{d\vec{r}}{dt} = 2 < -\sin t, \cos t >$$

So the Work is $\int_0^\pi -1/2(-\sin t \cos t + \sin t \cos t) dt = 0$.
Here the force is perpendicular to the displacement.

Example: C is the line from (0,-1) to (2,0), $\vec{r}(t) =< t, -1+t/2 >$ with $0 \leq t \leq 2$.

A line segment as a domain over which to find the work.

$$\vec{F}(x(t), y(t)) =< -\frac{t}{(5t^2/4 - t + 1)^{3/2}}, \frac{1 - t/2}{(5t^2/4 - t + 1)^{3/2}} >$$

and $\vec{r}'(t) =< 1, 1/2 >$. Then work is then

$$W = \int_0^2 \frac{-t + 1/2 - t/4}{(5t^2/4 - t + 1)^{3/2}} dt = \int_0^2 \frac{1/2 - 5t/4}{(5t^2/4 - t + 1)^{3/2}} dt$$

Let $u = 5t^2/4 - t + 1$ and $du = (5t/2 - 1)dt$.

$$\int_1^4 -1/2 \frac{du}{u^{3/2}} du = \frac{-1}{2} \frac{u^{-1/2}}{-1/2} |_1^4 = -1/2$$

Example $\vec{F}(x, y) =< -y, x >$, solid body rotation.
C is the circle of radius 1 centered at 0 $\vec{r}(t) =< \cos t, \sin t >$.

$$\int_0^{2\pi} < -\sin t, \cos t > \cdot < -\sin t, \cos t > dt = \int_0^{2\pi} 1dt = 2\pi = \text{ Circulation}$$

If $\vec{F}$ is the velocity of vehicles, $\oint_C \vec{F} \cdot d\vec{r}$ counts cars that go by.

Lecture 27, Section 16.3: Fundamental theorem of Line Integrals

Exercise: Consider $\phi(x, y)$ a scalar function and $\vec{r}(t) =< x(t), y(t) >$ a path.
Along that path, what is ϕ? It is $\phi(x(t), y(t))$.
What is $\frac{d\phi}{dt}$?

$$\frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt} + \frac{d\phi}{dy}\frac{dy}{dt} = \nabla\phi \cdot \frac{d\vec{r}}{dt}$$

So what is $\int_a^b \nabla\phi \cdot \frac{d\vec{r}}{dt} dt$?

$$\int_a^b \nabla\phi \cdot d\vec{r} = \int_a^b \frac{d}{dt}\left(\phi(x(t), y(t))\right) dt = \phi(x(b), y(b)) - \phi(x(a), y(a))$$

So if we call $\vec{F} = \nabla\phi$, $\int_C \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P)$.

The work along ANY path going from P to Q, for a conservative vector field.)

Important points:
1) This only works if $\vec{F} = \nabla\phi$ for some potential $\phi(x, y)$. That is it only works if $\vec{F}$ is CONSERVATIVE.
2) If C is a closed curve, $\oint \nabla\phi \cdot d\vec{r} = \phi(P) - \phi(P) = 0$
3) $\int_C \nabla\phi \cdot d\vec{r}$ does NOT depend on the path taken to go from P to Q. It is called path-independent.
4) $\phi(x, y) = \int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$ is a potential to any conservative field $\vec{F}$.

Example: Consider $\phi(x, y) = x \sin y$, then $\nabla\phi = \vec{F} =< \sin y, x \cos y >$
What is the work done by $\vec{F}$ on a particle traveling along a spiral starting at $(0, 0)$ and ending at $(1, \pi/2)$?

A spiral (or other complicated path from P to Q.)

$$W = \int_C \vec{F} \cdot d\vec{r} = \phi(1, \pi/2) - \phi(0, 0) = \sin(\pi/3) - 0 = \frac{\sqrt{3}}{2}$$

Given $\vec{F} =< F_1, F_2 >$, it would be really good to know:
1) Is $\vec{F}$ conservative?
2) If it is conservative, what is the potential $\phi(x, y)$?

Start with 1). Suppose $\vec{F}$ is conservative and continuous, then there exists $\phi(x, y)$ such that $\nabla\phi = \vec{F}$ and $\frac{\partial\phi}{\partial x} = F_1$ and $\frac{\partial\phi}{\partial y} = F_2$.

Then $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_1}{\partial y}$ and $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_2}{\partial x}$. If both partial derivatives are continuous, we must have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

It turns out that this works the other way too: if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, then $\vec{F} =< F_1, F_2 >$ is conservative.

We call $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}$ the VORTICITY of $\vec{F}$. If the vorticity is 0, $\vec{F}$ is conservative.

Now for 2), how would we find $\phi(x, y)$ to have $\nabla\phi = \vec{F}$?
We want $\frac{\partial\phi}{\partial x} = F_1$ and $\frac{\partial\phi}{\partial y} = F_2$
So $\phi(x, y) = \int F_1(x, y)dx + C_1(y)$ and
$\phi(x, y) = \int F_2(x, y)dy + C_2(x)$
This is enough to find $\phi(x, y)$, up to a constant c.

Example $\phi(x, y) = x^2e^y + x^3 + 3y + 6$, but we don't know that (supposedly).
Then we have $\vec{F} =< 2xe^y + 3x^2, x^2e^y + 3 >$. The vorticity is $2xe^y - 2xe^y = 0$.

$$\phi(x, y) = \int F_1 dx = \int (2xe^y + 3x^2)dx = x^2e^y + x^3 + C_1(y)$$

and

$$\phi(x, y) = \int F_2 dy = \int (x^2e^y + 3)dy = x^2e^y + 3y + C_2(x)$$

So $C_1(y) = 3y$ and $C_2(x) = x^3$ and $\phi(x, y) = x^2e^y + x^3 + 3y$

What happens in 3D?

Take a potential, $\phi(x, y, z)$ a vector field $\vec{F} =< F_1, F_2, F_3 >= \nabla\phi =< \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} >$. So

$$\frac{\partial\phi}{\partial x} = F_1, \ \ \frac{\partial\phi}{\partial y} = F_2, \ \ \frac{\partial\phi}{\partial z} = F_3,$$

So to have a conservative field, there will now be 3 conditions:

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_2}{\partial x} = \frac{\partial F_2}{\partial y},$$

and

$$\frac{\partial^2\phi}{\partial x\partial z} = \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z},$$

and

$$\frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y},$$

Try $\vec{F}(x, y, z) =< xy, z + x^2/2 + z^2 + xy >$ But here $(F_1)_z = 0$ and $(F_3)_x = y$, so the field is not conservative.

Now try $\phi(x, y, z) = e^{xz} + x \log y + y^2 + xyz$. Then

$$\vec{F} =< ze^{xz} + \log y + yz, x/y + 2y + xz, xe^{xz} + xy >$$

And $(F_1)_y = 1/y + z$ and so is $(F_2)_x$.
$(F_1)_z = e^{xz}(1 + xz) + y$ and so is $(F_3)_x$.
and $(F_2)_z = x$ and so is $(F_3)_y = x$.
So $\vec{F}$ is conservative, and we can find its potential:

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial x} dx = \int F_1 dx = \int (ze^{xz} + \log y + yz) dx = e^{xz} + x \log y + xyz + C_1(y, z)$$

And also

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial y} dy = \int F_2 dy = \int (x/y + 2y + xz) dy = x \log y + y^2 + xyz + C_2(x, z)$$

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial z} dz = \int F_3 dy = \int (xe^{xz} + xy) dz = e^{xz} + xyz + C_3(x, z)$$

So $\phi(x, y, z) = e^{xz} + x \log y + xyz + y^2 + C$

And $\int_P^Q \vec{F} \cdot d\vec{r} = \int_P^Q F_1 dx + F_2 dy + F_3 dz = \phi(Q) - \phi(P)$ and $\oint \vec{F} \cdot d\vec{r} = 0$.

Lecture 28, Section 16.4: Green's theorem

Back to 2D: We study now non-conservative fields but with closed C.

Start simple Let $\vec{F} =< F_1, F_2 >$

A simple contour, to integrate in 2 ways.

We will approximate $\vec{F}$ on this square, taking $\Delta x, \Delta y \to 0$.
Then we have $F_1(x, y) \approx F_1(x_0, y_0) + (x - x_0)\frac{\partial F_1}{\partial x} + (y - y_0)\frac{\partial F_1}{\partial y}$.
Similarly $F_2(x, y) \approx F_2(x_0, y_0) + (x - x_0)\frac{\partial F_2}{\partial x} + (y - y_0)\frac{\partial F_2}{\partial y}$. Using this, we have a fairly simple integrals to take.
Along C_1, we have $\vec{r}(t) =< x_0 + (\Delta x)t, y_0 >$ for $0 \leq t \leq 1$.
So $\frac{d\vec{r}}{dt} =< \Delta x, 0 >$, and we get

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot \frac{d\vec{r}}{dt} dt &= \int_0^1 F_1(x_0 + (\Delta x)t,\ y_0)\ \Delta x\ dt \approx && (1)\\
& \int_0^1 \left(F_1(x_0, y_0) + t\Delta x\ \frac{\partial F_1}{\partial x} \right) \Delta x dt = \Delta x F_1(x_0, y_0) + \frac{(\Delta x)^2}{2}\frac{\partial F_1}{\partial x} && (2)
\end{aligned}$$

Along C_3, $\vec{r}(t) =< x_0 + \Delta x - (\Delta x)t, y_0 + \Delta y >$ with t going from 0 to 1. so we have

$$\begin{aligned}
\int_{C_3} \vec{F} \cdot \frac{d\vec{r}}{dt} dt &= \int_0^1 F_1(x_0 + \Delta x(1 - t), y_0 + \Delta y) dt \approx && (3)\\
& \int_0^1 \left(F_1(x_0, y_0) + \Delta x(1 - t)\frac{\partial F_1}{\partial x} + \Delta y \frac{\partial F_1}{\partial y} \right) (-\Delta x) dt && (4)\\
&= (-\Delta x) F_1(x_0, y_0) + \frac{(-\Delta x)^2}{2}\frac{\partial F_1}{\partial x} + (-\Delta x \Delta y)\frac{\partial F_1}{\partial y} && (5)
\end{aligned}$$

So $\int_{C_1 + C_3} = -\Delta x \Delta y \frac{\partial F_1}{\partial y}$

Similarly, for the other 2 sides:
Along C_2, we have $\vec{r}(t) =< x_0 + \Delta x, y_0 + t\Delta y >$ for $0 \leq t \leq 1$.
So $\frac{d\vec{r}}{dt} =< 0, \Delta y >$, and we get

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot \frac{d\vec{r}}{dt} dt &= \int_0^1 F_2(x_0 + \Delta x, y_0 + t\Delta y)\ \Delta y dt \approx && (6)\\
& \int_0^1 \left(F_2(x_0, y_0) + (\Delta x)\frac{\partial F_2}{\partial x} + \Delta y\ t\frac{\partial F_2}{\partial y}) \right) \Delta y dt && (7)\\
&= \Delta y F_2(x_0, y_0) + \frac{(\Delta y)^2}{2}\frac{\partial F_2}{\partial y} + \Delta x \Delta y \frac{\partial F_2}{\partial x} && (8)
\end{aligned}$$

Along C_4, we get $-\Delta y F_2(x_0, y_0) - \frac{(\Delta y)^2}{2}\frac{\partial F_2}{\partial y}$ so we have
$\int_{C_2+C_4} = \Delta x \Delta y \frac{\partial F_2}{\partial x}$
So finally over the entire square, we get

$$\oint \vec{F} \cdot d\vec{r} = \Delta x \Delta y \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = dA \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

So the vorticity, $(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$ is a circulation/area = circulation density!

What happens if I put small squares side-by-side?

Building a larger contour from a single square.

The common side cancels out, and we are left with only the outer boundary.

$$\oint_{S_1} \vec{F} \cdot d\vec{r} + \oint_{S_2} \vec{F} \cdot d\vec{r} + ... = \oint_C \vec{F} \cdot d\vec{r}$$

By taking the limit $\Delta x, \Delta y \to 0$, we can match any closed curve that way:

Building a general contour from squares.

and find

$$\oint_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n}\sum_{j=1}^{m} \oint_{S_{i,j}} \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta x \Delta y \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)|_{x_i, y_j} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

where D is the domain inside C. This is Green's theorem.

A few notes:
1) Here C is positively oriented (otherwise multiply by -1).

2)C is made of smooth pieces and CLOSED.
3) C doesn't cross itself.
4) Both $\frac{\partial F_2}{\partial x}$ and $\frac{\partial F_1}{\partial y}$ have to be continuous.

Important: Green's theorem works both ways: The two integrals are EQUAL, so we can compute whichever is the simplest.

Example: Compute $\oint \vec{F} \cdot d\vec{r}$, with C the unit circle $x^2 + y^2 = 1$ and $\vec{F} = -y\vec{i} + x\vec{j}$.

An example of usage of Green's theorem.

The curl is: $\text{curl}\vec{F} = \partial F_2/\partial x - \partial F_1/\partial y = 1 - (-1) = 2$. So we have

$$\oint \vec{F} \cdot d\vec{r} = \int\int_D 2dA = 2\pi(1)^2 = 2\pi$$

Example: find the area inside $x^2 + y^{2/3} = 1$. The area is $A = \int\int_D dA$. But here that is hard to do.

Using Green's theorem to compute surface area.

If we had a vector field $\vec{F}$ with a curl of 1, we could do

$$\int\int_D dA = \int\int_D \text{curl}\vec{F} dA = \oint \vec{F} \cdot d\vec{r}$$

Several vector fields are like that. Some simple ones are $\vec{F} = x\vec{j}$ and $\vec{F} = -y\vec{i}$.
Here parametrizing C is not so hard: $x(t) = \cos t, y(t) = \sin^3 t$ with $0 \leq t \leq 2\pi$. So we have

$$\frac{d\vec{r}}{dt} =< -\sin t, 3\sin^2 t \cos t >$$

And we can have $\vec{F} \cdot d\vec{r} = 2\sin^2 t \cos^2 t$ or $\sin^4 t$. Neither is so bad to integrate:

$$\int_0^{2\pi} 3\sin^2 t \cos^2 t dt = \int_0^{2\pi} \frac{3}{4}(\sin 2t)^2 dt = \frac{3\pi}{4} = \frac{3}{4}\left(\frac{t - \sin(4t)/2}{2}\right)\Big|_0^{2\pi}$$

One more example. Say $\vec{F} =< -yx^4 - x^2y^3/2, y^2x + x^3y^2/3 >$ Then the curl is $(y^2 + x^2)^2$.
What is $\oint \vec{F} \cdot d\vec{r}$ where C_1 is the upper half circle of radius 1.

How to deal with an open contour? Close it!

We could parametrize , but the resulting integral is messy $\int_0^\pi (4/3)(\sin^2\theta\cos^4\theta + \sin^4\theta\cos^2\theta)d\theta$.

On the other hand, we could also close the contour with a line on the x-axis (figure). We would then have

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int\int_D \text{curl}\vec{F} dA$$

So

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int\int_D \text{curl}\vec{F} dA - \int_{C_2} \vec{F} \cdot d\vec{r}$$

But on C_2, $\vec{F} =< 0, 0 >$. so all we need is to integrate over the interior.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^\pi \int_0^1 r^5(\cos^2\theta + \sin^2\theta) dr d\theta = \pi/6$$

Lecture 29, Section 16.5: Curl and divergence

We have met the curl before, in 2D. and briefly in 3D it is

$$\text{curl}\vec{F} =< \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} >$$

and we saw that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ was a circulation density.

Here we give a more systematic way to calculate the curl and we interpret its meaning.

Introduce the differential operator ∇ or $\vec{\nabla} =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$.

We have met it before: grad $f = \nabla f =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$.

Now we can use it again:

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The physical meaning of the curl is that it measures how much $\vec{F}$ pushes a marker to rotate. Recall it is a circulation density.

Meaning of the curl: local rotation.

The direction of curl$\vec{F}$ is the rotation axis , using the right-hand rule, and $||\text{curl}\vec{F}||$ is the angular velocity.

Two special cases to keep in mind: Green's theorem

$$\oint \vec{F} \cdot d\vec{r} = \int\int_D \text{curl}\vec{F} \cdot \vec{k} dA$$

And if $\vec{F}$ is conservative, that is $\vec{F} = \nabla f$, then

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Well that is what we checked before (we called it 3 conditions, here it is one).

Second new operator: Divergence, div $\vec{F} = \nabla \cdot \vec{F}$
The definition is: $< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > \cdot < F_1, F_2, F_3 >= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. It is a scalar, not a vector.

What does this mean? It measures how much of $\vec{F}$ is coming out of a point.
1) If div $\vec{F} > 0$, then more is coming out than going in.

Divergence as Flow out - Flow in.

2) If div $\vec{F} = 0$, then as much as coming in as in going out.
3) If div $\vec{F} < 0$, more is coming in than going out.

We will see why in §16.7 and §16.8.

So at a point (x_0, y_0, z_0) if $\vec{F}$ is a velocity, then:
1) $\vec{F}(x_0, y_0, z_0)$ is how fast it moves.
2) curl $\vec{F}/||$curl $\vec{F}||$ is the axis about which it rotates.
3) $||$curl $\vec{F}||$ is how fast it rotates.
4) div $\vec{F}$ = going out - coming in.

One application: $\oint \vec{F} \cdot \vec{n} ds$ = how much is coming out of C. We have $\vec{F} =< F_1, F_2 >$.

Using Green's theorem to compute a flux.

$C = \vec{r}(t) =< x(t), y(t) >$
$\vec{n}(t) = \frac{<y'(t), -x'(t)>}{|d\vec{r}/dt|}$ so $\vec{n} ds =< y'(t), -x'(t) >$.
So we have $\vec{F} \cdot \vec{n} ds = (F_1 y' - F_2 x')dt =< -F_2, F_1 > \cdot < x', y' > dt$. So we get

$$\oint \vec{F} \cdot \vec{n}\, ds = \oint < -F_2, F_1 > \cdot \frac{d\vec{r}}{dt} dt = \int\int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA = \int\int_D \text{div}\vec{F}\, dA$$

cool huh?

Lecture 30, Section 16.6: Surface parametrization

We want to do calculus on general surfaces, not just the xy-plane. So far, we used $z = f(x, y)$ to describe a surface. What it really mean is given 2 parameters, x and y, your position in space, and on the surface is: $x = x, y = y, z = f(x, y)$.
This is a surface parametrization. It needs TWO parameters.

In general, we can use the parameters u and v, and define
$x(u, v)$
$y(u, v)$
$z(u, v)$
These are the parametric equations of a surface. Given u and v, you are at a point in space, on the surface. All the values of u and v describe the entire surface.

If we fix $u = u_0$, and let v vary. You get a curve (like $\vec{r}(t)$), a trace.
If we fix $v = v_0$, and let u vary. You get a curve (like $\vec{r}(t)$), a trace.
Varying both u, v, you get the whole surface.

Example, the simplest kind: $x = u, y = v, z = f(x, y) = f(u, v)$.

We can do that based on other coordinates: (r, θ, z)
$x = r\cos\pi/3$
$y = r\sin\pi/3$
$z = z$.
This is a half plane.

Other example
$x = 2\cos\theta = 2\cos u$
$y = 2\sin\theta = 2\sin u$
$z = v$ describe a cylinder of radius 2, along the z-axis.

Based on spherical coordinates
$x = 2\cos\theta\sin\phi$
$y = 2\sin\theta\sin\phi$
$z = 2\cos\phi$.
is a sphere of radius 2.

$x = \rho\cos\theta\sin\pi/4$
$y = \rho\sin\theta\sin\pi/4$
$z = \rho\cos\pi/4$.
is a cone.

In general, we can do surfaces of revolution in a systematic manner: If you rotate a curve $y = f(x)$ around

Parametrization of a surface of revolution.

the x-axis, you are rotating in the xz-plane, and y acts like a radius. So $y^2 + z^2 = r^2 = (f(x))^2$.
So our parametrization is
$x = x$
$y = f(x)\cos\theta$
$z = f(x)\sin\theta$
So when $\theta = 0$ you are in the xy-plane, and if $\theta = \pi/2$, you are in the xz-plane.

One more example: a plane, if you know that $\vec{v}_1$ and $\vec{v}_2$ are in the plane, and P_0 is a point on the plane. Then we have $\vec{n} = \vec{v}_1 \times \vec{v}_2$ and $\vec{n} \cdot (< x, y, z > -P_0) = 0$. If $\vec{v}_1$ and $\vec{v}_2$ are not parallel, you can describe the

Parametrization of a plane.

whole plane through

$$\vec{r}(u, v) = P_0 + u\vec{v}_1 + v\vec{v}_2$$

And one more rotation example Consider the curve $x^2 - z^2 = 1$, rotated about the z axis. This can be written

Surface of revolution, rotated about the z-axis.

as $x = f(z) = \sqrt{1 + z^2}$.
In the xy-plane, we have $x^2 + y^2 = (f(z))^2 = 1 + z^2$ with $f(z)$ the radius. So we find
$x = (1 + z^2)^{1/2}\cos\theta$
$y = (1 + z^2)^{1/2}\sin\theta$
$z = z$

Lecture 31, Section 16.6: Surface parametrization continued

Let's say you are given a parametrization $\vec{r}(u, v)$. What can you say about the vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$?

If you fix $v = v_0$, as you do when taking a derivative with respect to u, you get a trace, a curve in space which is part of the surface $\vec{r}(u, v)$. The vector $\frac{\partial \vec{r}}{\partial u}$ will be tangent to that curve, and so tangent to the surface. Similarly, $\frac{\partial \vec{r}}{\partial v}$ is also tangent to the surface.

Tangent vectors to a parametrized surface.

So if $< x_u, y_u, z_u >$ and $< x_v, y_v, z_v >$ are tangent to the surface, how do you find a vector normal to the surface? By taking the cross product
$\vec{n} = \vec{r}_u \times \vec{r}_v$.
and a unit normal would be $\hat{n} = \vec{n}/||\vec{n}|| = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$.

Example: Given $\vec{r}(u, v) =< \cos u \sin v, 3 \sin u \sin v, 3 \cos v >$, with $0 \le u \le 2\pi$ and $0 \le v \le \pi$, an ellipsoid, what is its normal?
Well $\vec{r}_u =< -\sin u \sin v, 3 \cos u \sin v, 0 >$
and $\vec{r}_v =< \cos u \cos v, 3 \sin u \cos v, -3 \sin v >$
and their cross-product is $\vec{n} =< -9 \cos u \sin^2 v, -3 \sin u \sin^2 v, -3 \sin v \cos v >$ So given any u, v, $\vec{r}(u, v)$ gives a point on the surface and $\vec{n}$ gives a normal vector at that point.

Question: What would be the element of surface area along this surface?

Area element of a parametrized surface.

The area of the tangent plane at a point would be

$$\Delta A = \Delta u \Delta v ||\vec{r}_u \times \vec{r}_v||$$

This is the same approach we used to find the area element of a general change of coordinates, except that then $z = 0$ and here z can also depend on u and v.

So to calculate a surface area, we do:

$$A = \int\int_S dA = \int_{u=u_0}^{u=u_1} \int_{v=v_0}^{v=v_1} ||\vec{r}_u \times \vec{r}_v|| dv du$$

Example with a sphere, same as the ellipsoid from before, with coefficients of 3 for all coordinates. Then $\vec{r}_u \times \vec{r}_v =< -9 \cos u \sin^2 v, -9 \sin u \sin^2 v, -9 \sin v \cos v >$.
The length of that is $9 \sin v$, so the surface area of a sphere is

$$A = \int_0^{\pi} \int_0^{2\pi} 9 \sin v du dv = 9 \cdot 2\pi(-\cos v|_0^{\pi}) = 9 \cdot 4\pi = 4\pi R^2$$

One last example of a parametrisation:

$$\begin{aligned} x(u,v) &= (3 + \cos v) \cos u \\ y(u,v) &= (3 + \cos v) \sin u \\ z(u,v) &= \sin v \end{aligned}$$

Parametrizing a donut/bagel/ring.

Note that if $u = 0$, we have $(x-3)^2 + z^2 = 1$ and $y = 0$. If $u = \pi/2$ we have $(y-3)^2 + z^2 = 1$ with $x = 0$. If we fix v, we get circles or varying radii.

The whole thing is a doughnut.

Lecture 32, Section 16.7: Surface Integrals

We have been integrating over a single surface so far, the xy-plane.

Integrating over a domain in the xy-plane.

Now we want our domain to be curved.

Integrating using a general surface as a domain.

For example, we might want to integrate a quantity (density of pollutants) over a sphere (the Earth).

To do this, we need to:
1) parametrize the surface with $\vec{r}(u, v)$
2) Write your integrand with u, v: $f(u, v) = f(x(u, v), y(u, v), z(u, v))$
3) write dA in terms of u and v.

Integrating over a domain given by a half-sphere.

We then have

$$\lim_{n\to\infty, m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta A f(x_{i,j}, y_{i,j}, z_{i,j}) = \int\int_S f(u,v) dA = \int\int_S f(u,v) ||\vec{r}_u \times \vec{r}_v|| dudv$$

For example, on Earth, we would use spherical coordinates, with $\rho = R$ fixed. Say we wanted to integrated $f(x,y,z) = z^2$. The area element is $R^2 \sin\phi d\phi d\theta$ and we have

$$< x, y, z >=< R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi >$$

So we get the integral

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta R^2 \cos^2\phi (R^2 \sin\phi)$$

which give $4\pi R^4/3$.

Why was the surface area element $R^2 \sin\phi$? We can get this from the volume element in spherical coordinates, or by taking $||\vec{r}_\theta \times \vec{r}_\phi||$.

If we consider the simplest parametrization, $x = u$, $y = v$, $z = f(u,v)$, which represents the surface $z = f(x,y)$, we find:
$\vec{r}_u =< 1, 0, f_x >$ and
$\vec{r}_v =< 0, 1, f_y >$
So $dA = ||\vec{r}_u \times \vec{r}_v|| dudv = \sqrt{1 + f_x^2 + f_y^2} dudv$.
To integrate the function $g(x,y,z)$ over the surface $z = f(x,y)$, we then have

$$\int_{u=u_0}^{u=u_1} \int_{v=v_0}^{v=v_1} g(u,v,f(u,v)) \sqrt{1 + f_x^2 + f_y^2} dudv$$

Later, we will integrate $\vec{F} \cdot \vec{n}$, to compute the flux through a surface.

Lecture 33, Section 16.7: Surface Integrals: Flux

We now consider flux integrals, which measure how much of a certain vector field, $\vec{F}$, is flowing through a certain surface S.

Recall that we had:
$\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$
$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$ and
$dS = ||\vec{r}_u \times \vec{r}_v|| dudv$.

A special type of integrand is the FLUX through a surface. If we let $\vec{F}(x,y,z) =< F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$ be a velocity field, and $\hat{n}$ be a unit normal to the surface S.

The flux is the amount of $\vec{F}$ that crosses a surface.

The product $\vec{F} \cdot \hat{n}$ is the component of $\vec{F}$ that is parallel to $\hat{n}$, which is to say the part of $\vec{F}$ that goes THROUGH the surface S. In general we have at a point on the surface that

$$\vec{F} = (\vec{F} \cdot \hat{n})\hat{n} + \text{ a vector tangent to } S$$

Example: $\vec{F} =< -x+y, -x-y, -z >$ is the amount of solar wind in space, per unit area, with $(0,0,0)$ being at the center of the Earth. How much solar wind enters the atmosphere? Let's say that the atmosphere has radius 7 (thousand km).

Example: solar wind into the atmosphere.

Solar wind in = $\int\int_S \vec{F} \cdot \hat{n} dS$, with $\hat{n}$ a normal pointing inward. This becomes

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta < -7\cos\theta\sin\phi + 7\sin\phi\sin\theta, -7\cos\theta\sin\phi - 7\sin\theta\sin\phi, -7cos\phi > \cdot < -\cos\theta\sin\phi, -\sin\theta\sin\phi, -\cos\phi > 49\sin$$

because $\vec{r}(\theta,\phi) =< 7\cos\theta\sin\phi, 7\sin\theta\sin\phi, 7\cos phi >$ and
$\vec{r}_\theta \times \vec{r}_\phi = 49\sin\phi < \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi > (-1)$

This integral simplifies to

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta 7^3 \sin(\phi)(\sin^2\phi + \cos^2\phi) = 7^3 2\pi 2 = 4\pi 7^3$$

You can also think of the flux as the amount of water going through a net, with $\vec{F}$ is the velocity field, and $\hat{n}$ the unit normal to the net. The product $\vec{F} \cdot \hat{n}$ is how much crosses the net.

There are a few common surfaces you are likely to encounter:
Sphere:
$\vec{r}(\theta, \phi) =< R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos phi >$
$\hat{n} =< \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi >$
$dS = R^2 \sin\phi d\phi d\theta.$

Cylinder (vertical)
$\vec{r}(\theta, z) =< R\cos\theta, R\sin\theta, z >$
$\hat{n} =< \cos\theta, \sin\theta, 0$
$dS = Rd\theta dz.$

Surface $z = f(x, y)$.
$\vec{r}(x, y) =< x, y, f(x, y) >$
$\hat{n} =< f_x, f_y, -1 >$
$dS = \sqrt{1 + f_x^2 + f_y^2} dxdy.$

Lecture 34, Section 16.9: Divergence Theorem

Recall the definition of the Divergence:

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

I claimed at the time that this was measuring (Flux out - Flux in) at a point. Let's see why.

Flux coming out of a small cube centered at the origin.

Consider a box with sides of length Δx, Δy, and Δz. Let's compute the flux out of faces 1 and 3, with normals given by
$\vec{n}_1 =< -1, 0, 0 >$ and $\vec{n}_3 =< 1, 0, 0 >$, respectively.

On surface 1, we have the following parametrization: $x = -\Delta x/2, y = y, z = z$.
So $\vec{F} \cdot \vec{n}$ = -$F_1(\Delta x/2, y, z)$ and dA = dy dz, and the flux is

$$\text{Flux}_1 = \int_{-\Delta z/2}^{\Delta z/2} \int_{-\Delta y/2}^{\Delta y/2} -F_1(-\Delta x/2, y, z) \text{dy dz} \approx \ \Delta z \Delta y \ - F_1(-\Delta x/2, 0, 0)$$

Similarly, on surface 3, we have the following parametrization: $x = \Delta x/2, y = y, z = z$.
So $\vec{F} \cdot \vec{n}$ = $F_1(\Delta x/2, y, z)$ and dA = dy dz, and the flux is

$$\text{Flux}_3 = \int_{-\Delta z/2}^{\Delta z/2} \int_{-\Delta y/2}^{\Delta y/2} F_1(\Delta x/2, y, z) \text{dy dz} \approx \ \Delta z \Delta y \ F_1(\Delta x/2, 0, 0)$$

So that together, we have:

$$\begin{aligned} \text{Flux}_1 + \text{Flux}_3 &= \Delta z \Delta y \ (F_1(\Delta x/2, 0, 0) - F_1(-\Delta x/2, 0, 0)) = \Delta x \Delta z \Delta y \ \frac{F_1(\Delta x/2, 0, 0) - F_1(-\Delta x/2, 0, 0)}{\Delta x} \\ &\approx \Delta V \frac{\partial F_1}{\partial x}(0, 0, 0) \end{aligned}$$

On the other faces of the cube, we find something similar, so over the whole cube, the flux outward is:

$$Flux = \Delta V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = \Delta V \text{ div} \cdot \vec{F}$$

So the divergence of $\vec{F}$ is a local FLUX DENSITY.

This leads us to the divergence theorem:
If $\vec{F}$ is differentiable and S is a smooth, CLOSED surface, then

$$\int\int_S \vec{F}\cdot\hat{n}dA = \int\int\int_V \text{div}\vec{F}dV$$

where V is the INTERIOR of S and $\hat{n}$ points outward.

Illustration of the divergence theorem.

Example: $\vec{F}(x,y,z) = (x + x^2 + 1 + y)\vec{i} + (3y - 2xy + 4z)\vec{j} + (z^2 + e^{xy})\vec{k}$. What is the flux out of the sphere of radius 2 centered at the origin?

Flux coming out of a sphere of radius 2 centered at the origin.

We could parametrize S, plug in, integrate. It would be long. OR

$$\int\int_S \vec{F}\cdot\hat{n}dA = \int\int\int_V \text{div}\vec{F}dV$$

div$\vec{F}$ = 1 + 2x+3-2x+ 2z = 4 + 2z. So the flux is:

$$\int\int\int_{Sphere}(4+2z)dV = 4\left(\frac{4\pi 2^3}{3}\right) + 0 \text{ by symmetry}$$

So the flux is $128\ \pi/3$

Last example: Let S be the surface of a half-sphere $x^2 + y^2 + z^2 = 9$, $y \leq 0$ and consider a normal pointing away from the sphere. If $\vec{F} = (4x - z^2, (y-1)^2, \sin y + z)$, what is $\int\int_S \vec{F} \cdot \hat{n} dA$?

Using the divergence theorem for a half-sphere

We note that div $\vec{F} = 2 + 2(y-1) + 1 = 3 + 2y$, which is nice and simple.

S is not closed, so we can't use the divergence theorem, at least not directly.
We can close the half-sphere though. Let D be the disk $x^2 + z^2 \leq 9$, $y = 0$. Then $S + D$ is a closed half-sphere. The divergence theorem then says

$$\begin{aligned} \int\int_{S+D} \vec{F} \cdot \hat{n} dA &= \int\int\int_V \text{div } \vec{F} dV \\ &= \int_0^\pi \int_\pi^{2\pi} \int_0^3 (3 + 2\rho \sin\theta \sin\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= 3V + \int_0^\pi \int_\pi^{2\pi} \int_0^3 (2\rho^3 \sin\theta \sin^2\phi) \, d\rho \, d\theta \, d\phi \\ &= \frac{3}{2}\frac{4\pi 3^3}{3} + 2\frac{3^4}{4}(-\cos\theta)\Big|_\pi^{2\pi} \left(\frac{\pi}{2}\right) \\ &= 54\pi - 81\pi/2 = (27/2)\pi \end{aligned}$$

We we are really after is

$$\int\int_S \vec{F} \cdot \hat{n} dA = \int\int_{S+D} \vec{F} \cdot \hat{n} dA - \int\int_D \vec{F} \cdot \hat{n} dA$$

On D we have $\hat{n} =< 0, 1, 0 >$, and $y = 0$, so we find

$$\int\int_S \vec{F} \cdot \hat{n} dA = (27/2)\pi - \int\int_D (y-1)^2 dA = (27/2)\pi - \int\int_D (-1)^2 dA = (27/2)\pi - 9\pi = (9/2)\pi$$

And that is our FINAL answer.