Today's plan

1. Go over how the class will work
2. Introduce what is a Partial Differential Equation (PDE).
3. What are examples of PDEs?
4. Where are PDEs from?
5. What else do we need to get a solution?
6. We will begin with some classification of PDEs

What are PDEs?

PDEs are a very powerful mathematical tool to describe physical, biological, and sociological systems. They can accurately model VERY complicated situations.

On the flip side, PDEs are usually hard to solve. You need a mixture of: 1) experience, 2) theory, 3) numerical simulations.

Here we will focus on the first two points. By the end of this class, for the types of equations we consider I would like you to: 1) Know what the solution is expected to look like. 2) Know how to get a solution in simple cases.

We begin by recalling a definition, that of **partial derivatives**.

Given a function of several variables $u(x,t)$, we define the *partial derivative with respect to* x as:

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,t)-u(x,t)}{h} \quad \text{WITH} \quad t \quad \text{FIXED}.$$

A Partial Differential Equation (PDE) is an equation (what is an equation?) where a partial derivative appears.

Here is a simple PDE, maybe your first one!

$$u_t = 0.$$

What is the general solution? Any function whose partial time derivative is zero, so:

$$u(x,t) = \int u_t \, dt = \int 0 dt = f(x)$$

Where do they come up? Who cares about PDEs?

Where do they come up? Who cares about PDEs?

Most common example: x is a spatial coordinate, t stands for time.
The unknown function u can describe:

1. Temperature
2. Price
3. Velocity
4. Population
5. Infection level

In short, they are everywhere!

In our simple example, we did not get a unique solution. For that we need some IMPORTANT additional information.

1. If time is involved, we usually need some INITIAL conditions (we call those IC).
2. We need some BOUNDARY conditions on the edge of our domain (we call those BC).

Only when we have the PDE and the proper conditions (initial, boundary) can we get a unique solution.

Classifying PDEs

We can solve only a relatively small number of PDEs. To know if we have a chance, we first need to classify them. There are a few criteria we use:

1. Number of (input) variables.
   If there is only 1, it is an Ordinary Differential Equation, or Differential Equation.
   If there are $n > 1$, it is a PDE on n variables.

2. Order of the highest derivative:
   $u - x + 2u_t = 0$ is a first order PDE.
   $Du_{xx} - u_t = 0$ is a second order PDE.
   $uu_{xx} + 4u_{xxt} + u^5 = 0$ is a third order PDE.

3. Linearity, or non-linearity *in the unknown function and its derivatives*.
   So if our function is u, for the equation to be linear u, u_x, u_t, etc. can only appear to the first power. Moreover, they may NOT multiply each other.

In general, we can write PDEs using an *operator*, denoted by L, applied to a function u:

$$u_{xx} - u_t = Q(x) \text{ becomes } \quad L[u] = Q(x).$$

So here L is a differential operator that says: take two derivatives with respect to x and subtract from that a derivative with respect to t:

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \text{ so } \quad L[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}.$$

We can now give a more precise definition of linearity. A PDE is linear if $L[u]$ is a linear operator in u, which is to say if, for any real numbers α and β and for any functions u_1 and u_2 in its domain, we have

$$L[\alpha u_1 + \beta u_2] = \alpha L[u_1] + \beta L[u_2].$$

Using this definition, you can verify if $L = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$ is a linear operator.

Note: In general, an *operator* is a mathematical object that operates on other mathematical objects. Differential operators involve derivatives. Here, you can think of it as a function of function (even though that is not formally correct because functions act on numbers only).

Second order PDEs classification

In this class, we will focus on linear, second order PDEs, of two variables. In general, they look like

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Note that all the letters $A, B, C...$ can be functions of x and y, but not of u or its derivatives. Here we will assume that $A \neq 0$.

There is a strong analogy between PDEs and scalar equations.
Replace $\partial/\partial x \to x$ and $\partial/\partial y \to y$. We then get a conic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = G$$

This is a curve in space, but what does it look like? We complete the square to figure it out:

$$\begin{aligned}
&A\left(x^2 + \frac{B}{A}xy + \frac{C}{A}y^2\right) + Dx + Ey + F = G \\
&A\left(x + \frac{B}{2A}y\right)^2 + A\left(\frac{C}{A} - \frac{B^2}{4A^2}\right)y^2 + Dx + Ey + F = G \\
&A\,r^2 + \left(\frac{4AC - B^2}{4A}\right)y^2 + D\left(r - \frac{B}{2A}y\right) + Ey + F = G
\end{aligned}$$

for a new variable $r = x + \frac{B}{2A}y$. So what does that look like? It depends on the sign of A and of $det = 4AC - B^2$.

If $det = 4AC - B^2 > 0$, we have an ellipse. The corresponding PDE is said to be elliptic.

If $det = 4AC - B^2 < 0$, we have a hyperbola. The corresponding PDE is said to be hyperbolic.

If $det = 4AC - B^2 = 0$, we have a parabola. The corresponding PDE is said to be parabolic.

There are several features of the behavior of solutions to PDEs that are either typical of elliptic, hyperbolic, or parabolic systems. We will learn them so we can know what to expect.

Solving PDEs is so hard that there are many opportunities to make mistakes. It is therefore critical to know what to expect from our solution.

Today's plan

1. Introduce the advection equation. What is it?
2. Describe circumstances where it occurs
3. Present a way to solve a simple advection equation

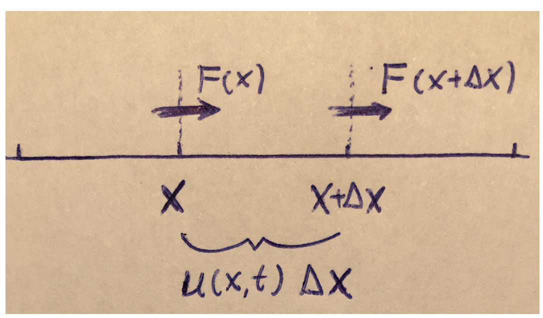
We want to study what is arguably the simplest "true" PDE: the advection equation. Even though it is fairly simple, the solutions can be quite varied, at least if we allow for non-linearities.

The simplest PDE is a first order PDE, linear, of two variables. Consider the variables x for position in space and t for time and the function $u(x,t)$. Our PDE must involve u_x and u_t, both to the first power, with coefficients that do not depend on u. A general form of such an equation is

$$u_t + c(x,t)u_x = Q(x,t,u)$$

This is the advection equation (or one-dimensional wave equation, see chapter 12.2.2).
Note that if the coefficient of u_t is not one, we can divide the equation by that coefficient to recover the form above (so long as that coefficient is not zero).
Also, it is possible that the coefficient $c(x,t)$ also depends on u. This would make the equation non-linear (and more complicated).

When can this occur? The most common occurrence is if our function u represents an **amount or concentration** that is being moved by a flux F.



Schematics of a concentration modified by a flux F.

Consider the segment between x and $x + \Delta x$, and suppose that $u(x,t)$ represents the concentration of something (like a chemical) at that location. Because we are looking at a 1D system, we will have a concentration in mass per unit length.

The question now is how does this concentration change in time when the flux of u toward the right is $F(x,t)$. Recall that the flux is the rate at which the quantity of interest (like a chemical) moves across a location. When a concentration is *advected* (transported at a speed that doesn't depend on its derivative) the flux is a velocity c multiplied by the local concentration:

$$F = c(x,t)u(x,t)$$

and we have, in terms of units

$$\begin{aligned}\text{units of } u &= \frac{M}{L}\\ \text{units of } F &= \frac{M}{T}\\ \text{units of } c &= \frac{L}{T}\end{aligned}$$

In our segment between x and $x+\Delta x$, the total amount of chemical changes because some flows in from x and some flows out at $x+\Delta x$ (note that a negative flux to the right corresponds to a positive flux to the left). So we have that the total amount in the segment at time $t+\Delta t$ is

$$\Delta x\, u(x,t+\Delta t) = \Delta x\, u(x,t) + \Delta t\,(F(x,t) - F(x+\Delta x,t))$$

which is: what we had before, plus what came in on the left, minus what came out on the right.

We can now rearrange terms to try to get a PDE when we take the limits of $\Delta t \to 0$ and $\Delta x \to 0$. We find:

$$\Delta x\, u(x,t+\Delta t) - \Delta x\, u(x,t) + \Delta t\,(F(x+\Delta x,t) - F(x,t)) = 0$$

and if we divide by $\Delta t \Delta x$

$$\frac{u(x,t+\Delta t) -\ u(x,t)}{\Delta t} + \frac{F(x+\Delta x,t) - F(x,t)}{\Delta x} = 0$$

and taking the limits $\Delta t \to 0$ and $\Delta x \to 0$, we find

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{u(x,t+\Delta t) -\ u(x,t)}{\Delta t} + \lim_{\Delta x\to x}\frac{F(x+\Delta x,t) - F(x,t)}{\Delta x} = 0\\ &\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0\end{aligned}$$

Finally, if we use our definition that $F = cu$, we find the advection equation

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = -u\frac{\partial c}{\partial x}$$

In the simplest case, which we will study first, the velocity is constant, $c(x,t) = c$ and $c_x = 0$. We then get the constant coefficient advection equation

$$u_t + cu_x = 0$$

Boundary and initial conditions

To have a chance of finding a solution to any PDE, including the advection equation, we must provide initial and boundary conditions. These are often as determinant as the equation itself.

Because the advection equation involves a first order time derivative (and no higher), we need to provide a single initial condition. This is almost always given in the form:

$$u(x, t = 0) = f(x)$$

for a given function $f(x)$ which is differentiable over the domain of interest. Note that things can also be done if $f(x)$ is not differentiable, but more theory is required.

Similarly, the advection equation involves a single spatial derivative, so a single boundary condition can be provided. This typically takes the form

$$u(x = x_0, t) = g(t).$$

However, we will see soon that this boundary cannot be just anywhere in relation to the domain.

Lastly, it is possible to have the entire real axis as a domain, in which case, no boundary condition is needed.

Solution to the constant coefficient advection equation

Consider the advection equation for a constant velocity $c > 0$

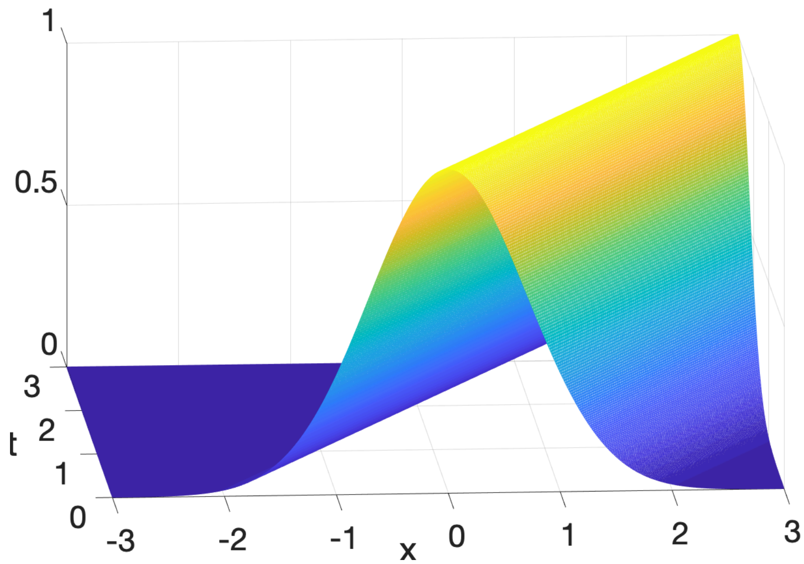
$$u_t + cu_x = 0, \quad \text{subject to } u(x, 0) = e^{-x^2} \quad \text{over the entire real line}$$

What do we expect the solution to look like?

Recall how we got this equation: it describes the concentration of a chemical subject to a flux given by $F = uc$, which is moving the concentration to the right with velocity c. What do you expect the solution to look like?

The initial condition here is a bell curve, a blob of higher concentration centered at $x = 0$. As it gets pushed by the flux, it should move to the right, with speed c. Moreover, given this flux, it should keep its shape.

Mathematically, this is described by $u(x,t) = e^{-(x-ct)^2}$, which is equivalent to moving the location where the argument of the exponential is zero to be ct.



A Gaussian curve advected to the right at a constant speed.

More generally, we can think that any initial condition $u(x, t = 0) = f(x)$ would have a corresponding solution

$$u(x,t) = f(x - ct).$$

We can verify this by plugging it in our equation. We have $u_x = f'(x - ct)$ and $u_t = (-c)\, f'(x - ct)$. So we get:

$$u_t + cu_x = (-c)\, f'(x - ct) + c\, f'(x - ct) = 0$$

so indeed it works! We have solved our first non-trivial PDE.

Next time we will see what happens for a finite domain, and how we can find similar solutions for more complicated speeds.

Today's plan

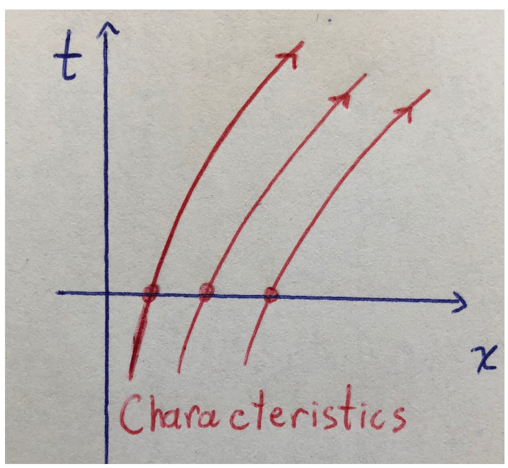
1. When do you use the methods of characteristics?
2. How do you use it?
3. Conceptually, what is going on?
4. What problems can it encounter?

We return now to a more general version of the advection equation, one which may have a source term (RHS) and may be non-linear (c may depend on u).

$$u_t + c(x,t,u)u_x = Q(x,t,u)$$

Recall that we saw that if $Q = 0$ and c is constant, $u(x,t) = f(x - ct)$ is a solution. Can we use this result to get a more general one?

The idea of many methods to solve PDEs is to find a way to reduce them to ODEs. The method of characteristics does this as well, by selecting *curves in the domain* over which the solution is easier to describe. **These curves are called characteristics**.



Schematics of the method of characteristics

We want to introduce some curves in the xt-plane, our domain. We will write these curves as functions $x(t)$. The good news is that along these curves, our solution can be viewed as a function of a single variable, t:

$$u(x(t),t) = U(t).$$

Can we get a differential equation for $U(t)$? Let's take its (time) derivative:

$$\frac{dU}{dt} = \frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

The key here is to make this derivative look like our advection equation. To do so, we pick our characteristics carefully. Here, to have dU/dt match the LHS of the advection equation, we select

$$\frac{dx}{dt} = c(x,t,u) \; = \; c(x(t),t,u(x(t),t)). \tag{1}$$

With this choice, we get that

$$\frac{dU}{dt} = \frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + c(x,t,u)\frac{\partial u}{\partial x} = Q(x,t,u)$$

so more simply that

$$\frac{dU}{dt} = Q(x,t,u) \tag{2}$$

We therefore have a system of two ODEs (equations 1 and 2) to solve. In general, this can be hard, but we will focus on easier cases.

We start by the case we have seen before: $Q = 0$ and c constant. Our system of ODEs is:

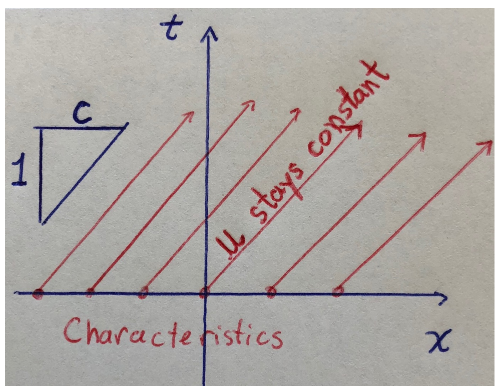
$$\frac{dx}{dt} = c \qquad\qquad \frac{dU}{dt} = 0$$

These are decoupled, and indeed quite easy to integrate. From the first one we have

$$x = ct + x_0 \quad \text{or} \quad x_0 = x - ct$$

for a constant x_0. This means that the special curves, the characteristics, are straight lines in the xt-plane. x_0 indicates the location on the x-axis at time $t = 0$.

From the second one, we have that $U(t) = u(x(t), t) = U_0$, a constant. This means that along each characteristic, the value of u does not change (note that here $u(x(t), t) = U(t)$ is the value of the function u ALONG a characteristic). However, u can change from one characteristic to another. So the initial value of u does not have to be constant, but once it is given, it stays the same along a characteristic.



Schematics of characteristic for a constant c and $Q = 0$.

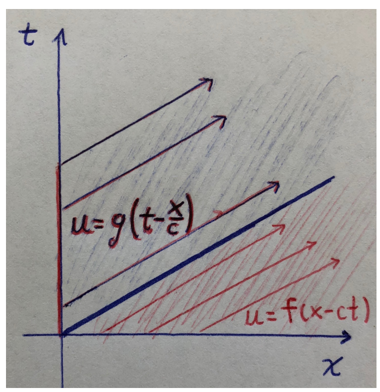
When c is positive, we can see that the characteristics travel to the right. You can think of the characteristics as carrying information. In this case, the information is that u does not change. When $c > 0$, this information goes to the right only. This means that if we have a spatial boundary at $x = 0$, it will only have an effect on what happens for $x > 0$, $t > 0$.

Consider the boundary condition $u(x=0,t) = g(t)$.
If we trace some characteristics, we can see that one goes through $x = 0$, $t = 0$. It is: $x = ct$. To the right and below it, the value of u is given by the initial value at time $t = 0$. However, to the left and above it, the value of u depends on the boundary value at $x = 0$.

More precisely, how can we express $u(x,t)$? It will depend on $x - ct$. When $x_0 = x - ct > 0$, the value comes from the initial conditions, $f(x_0)$.
When $x_0 = x - ct < 0$, the value comes from the boundary conditions at the time for which $x = 0$ and $x - ct = x_0$, that is $t_0 = \frac{-x_0}{c}$ and the value of u is , $g(t_0)$. Putting it all in a single formula, we have

$$u(x,t) = \begin{cases} f(x-ct) & \text{if } x - ct > 0 \\ g(t - \frac{x}{c}) & \text{if } x - ct < 0 \end{cases}$$

Note that for $x - ct = 0$, if the initial condition is different than the boundary condition, the solution will not be continuous, and will not have a well-defined value along the line $x = ct$.



Schematics of the solution to the advection equation with a boundary condition at $x = 0$.

Today's plan

1. What does a non-linear equation look like?
2. Why does it matter if the equation is non-linear?
3. Can you still use the method of characteristics?
4. What can go wrong with this method?

Recall the method of characteristics, for the advection equation:

$$u_t + c(x,t,u)u_x = Q(x,t,u)$$

We obtained the couple ODEs:

$$\frac{dx}{dt} = c(x,t,u) \; = \; c(x(t),t,u(x(t),t)). \tag{1}$$

$$\frac{dU}{dt} = Q(x,t,u) \quad \text{for} \quad U(t) = u(x(t),t) \tag{2}$$

We saw that when $Q = 0$ and c is constant, the characteristics are straight lines in the xt-plane, and u remains constant along them.

Let's see a more complicated example. Note that sometimes the system of ODEs cannot be solved explicitly (or can be VERY hard to solve explicitly). The method of characteristics can be used numerically in general, but only certain simple setups can be solved analytically. Let's try a case where c is a function of time, like a time dependent transport velocity, and there is a source term proportional to the concentration. So imagine a population walking and getting tired in which we want to keep track of the number of people who have a virus. We will use $c(x,t,u) = \frac{1}{(t+1)^{1/2}}$ and $Q(x,t,u) = 0.1u$. Our equation is therefore

$$u_t + \frac{u_x}{(t+1)^{1/2}} = 0.1u$$

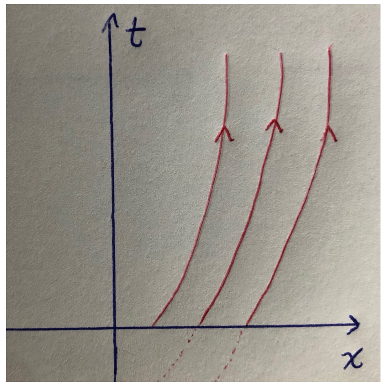
This example is chosen so that the system of ODEs is actually decoupled. We get for the characteristics:

$$\frac{dx}{dt} = \frac{1}{(t+1)^{1/2}}$$

This can be integrated with respect to t and yields

$$x(t) = x_0 - 2 + 2(t+1)^{1/2} \quad \text{or} \quad t = \left(\frac{x - x_0}{2} + 1\right)^2 - 1 \quad \text{or} \quad x_0 = x + 2 - 2(t+1)^{1/2}$$

Note that here the characteristics never cross and cover the entire domain. **That is not always the case**.

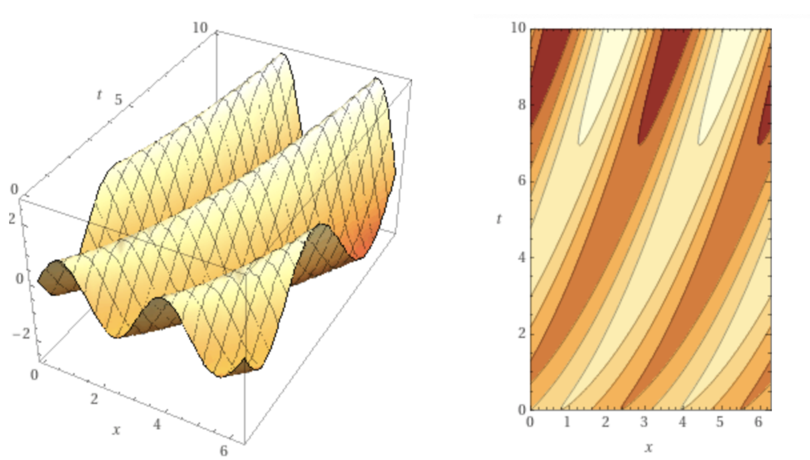


Characteristic curves following a parabola in the xt-plane.

Along the characteristics, the function u is subject to $\frac{dU}{dt} = 0.1U$. This is easily solved to give $U(t) = u_0 e^{0.1t}$. So along each characteristic, there is an exponential growth. The initial value u_0 is given by the initial condition, at $t = 0$ so $u_0 = f(x_0)$. Note that along a characteristic this completely determines the function. There is actually no dependence on neighboring values of u.

We may therefore write an expression for our full solution. Suppose the initial condition is $u(x, t = 0) = f(x)$. For a given point (x, t), we need to know what characteristic we are on. So what is the value of x_0 in the expressions given above? We find $x + 2 - 2(t + 1)^{1/2} = x_0$. Our solution is therefore

$$u(x, t) = u_0 e^{0.1t} = f(x_0)e^{0.1t} = f(x + 2 - 2(t + 1)^{1/2})e^{0.1t}$$



Solution for initial condition $f(x) = \sin(2x)$ (left) and contour plot of this solution (right)

Issues with the method of characteristics and traffic flow

As was hinted at earlier, some issues may occur with the method of characteristics. For a given point (x,t), it is possible that

- One and only one characteristic goes through that point. This is good (and what we saw so far).
- No characteristics go through this point. We have no information about u.
- More than one characteristics goes through that point. We have conflicting information about u.

We will present brief descriptions of how to deal with the last two points. This is a complex topic and the goal here is simply to give a sense of what might happen. To illustrate those setups, we will consider a model of traffic flow, following 12.6.2 (loosely).

Consider a model in which we keep track of the *density* of cars, ρ as a function of one spatial dimension x and of time t. Our conservation law is applicable

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0$$

but we need to define a flux F. For cars on a single road, it makes sense to assume that $F = v\rho$, where v is the speed of car. However, this speed with depend on the density ρ. For simplicity, we will assume here that the speed decreases linearly with density $v = 1 - \rho$. If the density is ever greater than 1, the speed remains zero. Note here that we are effectively looking at a rescaled density, where the maximum density has been set to one. This could be defined as $\rho = \frac{\text{density}}{\text{max density}}$. We have also rescaled time so that the maximum velocity is one.

Our flux is then $F = (1-\rho)\rho = \rho - \rho^2$. Our PDE is therefore

$$\frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0$$

Note that this is a non-linear equation. It is simple enough that we can solve it exactly with the method of characteristics, but it can give rise to more complex solutions. Our characteristic equations are

$$\frac{dx}{dt} = 1 - 2\rho$$

and

$$\frac{d\rho}{dt} = 0.$$

Thankfully, this last one is easy and states that $\rho = \rho_0$ is constant along a characteristic.

We can now solve for the shape of the characteristics, assuming a constant ρ, and find

$$x(t) = x_0 + (1 - 2\rho)t.$$

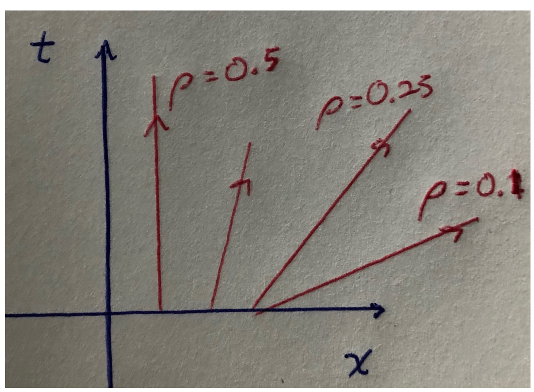
These are straight lines again. **However**, their slope depends on the initial value of ρ.

Subtle but important point: Here, characteristics have a slope which is different than the speed at which cars flow. This is analogous to the difference between phase velocity and group velocity in physics. It is basically saying that the speed of cars is different that the speed at which *information* about the car density travels, which is the slope of the characteristics.

For example, imagine you are at a protest or rally with a lot of people (all wearing masks of course). Suppose I want to track the density of people who had heard a rumor. If the people are standing still, the rumor may travel by word of mouth, at a certain speed. If the people are walking, the information will be moving because of both the speed of people AND the speed of the spread by word of mouth. But information is moving separately from people. The same idea applies to the density of cars, which travels at a different speed than any individual car.

Red light turns green: Expansion Fan

Because the slope of the characteristics depends on the value of ρ, the initial conditions play a determinant role. We consider a few examples:



Constant initial density remains constant.

First, suppose the density is constant initially: $\rho = \rho_0$. In that case, all characteristics have the same slope and nothing changes in time. In fact, $\rho = \rho_0$ satisfies our PDE.

Suppose now that a right light at $x = 0$ turns green at time $t = 0$. We then have an initial density of cars given by

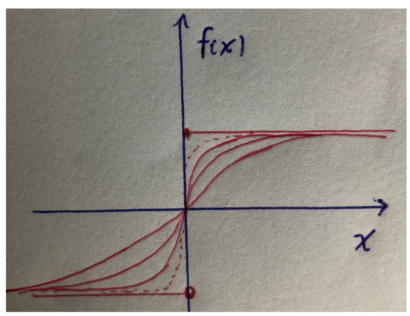
$$\rho(x, t=0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

The corresponding characteristics in each regions have slope -1 for $x < 0$ and 1 for $x > 0$. This describes well what happens if $x > t$ and if $x < -t$, no changes are seen in those regions. But what about in between? Because the initial condition was discontinuous, the characteristics yield no information about that region.

We therefore need to introduce a mathematical trick. We are going to think of the initial condition as the limit of a continuous function that approaches a discontinuous function. We will use a tanh function that gets sharper and sharper as it transitions from 0 to 1.

$$\rho(x, t) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases} = \lim_{\epsilon \to 0} \frac{1 + \tanh(-x/\epsilon)}{2}$$

As a result, we can think of ρ as taking ALL the values between 0 and 1 at the point



Continuous functions approaching a step function

$x = 0$. We therefore need to draw all the corresponding characteristics. This is convenient as those characteristics will fan out and cover the entire region of the xt-plane between the two regions where we already know the solution. This is called an **expansion fan**.

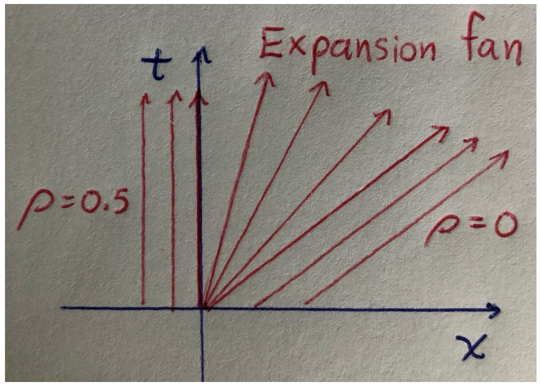
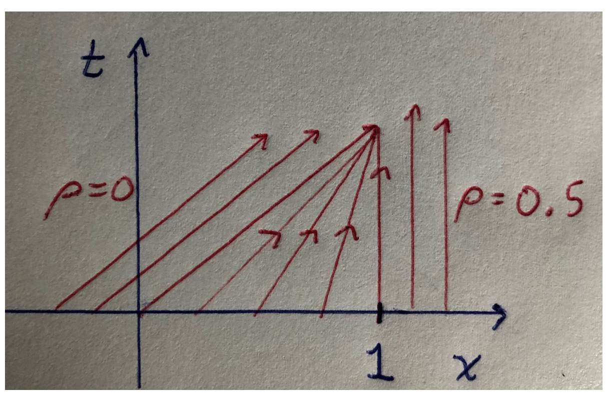


Figure 1: Expansion fan at a red light turning green. Note that here we assumed a more realistic density of $\rho = 0.5$ for $x < 0$ (the discussion above used $\rho = 1$ in that region).

Traffic jam formation: Shocks

We now consider the opposite situation, where the density of cars increases with x. Suppose we start with a density of $\rho = 0.5$ for $x \geq 1$ and a density of $\rho = 0$ for $x \leq 0$, and a linear interpolation between the two

$$\rho(x, t = 0) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } x \geq 1 \\ \frac{x}{2} & \text{otherwise} \end{cases}$$



Initial condition of increasing density with x.

Once again, there are regions well described by characteristics. If $x \geq 1$, the characteristics are vertical and the density stays at $\rho = 0.5$. Note that in the case the cars are still moving to the right (with speed 0.5). If $x < t$, the characteristics have slope one, and the density stays $\rho = 0$.

So what is the problem? The issue is that those two regions overlap. The characteristics cross! So who are we to listen to?

In fact, with this initial condition, all the characteristics that start from $0 \leq x \leq 1$ meet at $x = 1, t = 1$. This is like an expansion fan in reverse: we are creating a point where ρ appears to take on many values. This is called a **shock**, and it corresponds to a discontinuity in the function ρ. This is how traffic jams can form from nothing in particular. See an cool illustration of this at:

`https://www.youtube.com/watch?v=7wm-pZp_mi0`

Here the drivers were tasked with driving in a circle at a constant speed (maybe about 20mph), and shocks naturally formed!

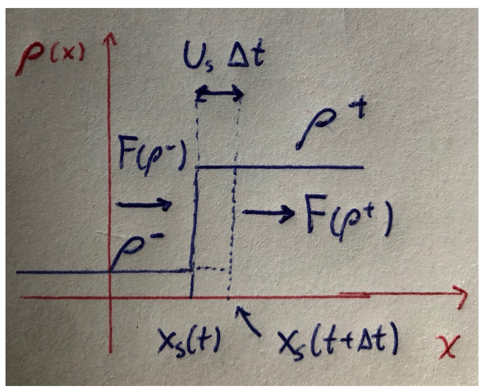
Shock speed

What happens on either side of the shock is clear: the density keeps its value, and there is a jump at the shock. However, the position of the shock is generally going to change in time too. To compute how it changes, we need to return to our derivation of the equation, using the flux, and assume that a shock is present. We will look for the speed v_s at which a shock is propagating.

To compute this speed, we again consider how much flows into and out of a segment over a short time Δt. Suppose the density on the right of the shock is ρ^+ and that on the left is ρ^-. We denote the size of the jump as

$$[\rho] = \rho^+ - \rho^-.$$

Similarly, we can speak of the flux to right of the shock as F^+ and the flux coming in from the left as F^-. We have $[F] = F^+ - F^-$.



Derivation of the speed of a shock

Because the shock moves, the amount of cars that moved to the right over Δt is $u_s[\rho]$. This has to be the same as the difference in the fluxes are either side of the shocks. We get:

$$\text{Shock related flux} = u_s[\rho] = [F]$$

and solving for the shock velocity, we find the so-called Rankine-Hugoniot velocity

$$u_s = \frac{[F]}{[\rho]} = \frac{F^+ - F^-}{\rho^+ - \rho^-}. \tag{3}$$

In our example, the flux was the density multiplied by a velocity $v = 1 - \rho$, so we have

$$u_s = \frac{\rho^+ v^+ - \rho^- v^-}{\rho^+ - \rho^-}.$$

Using the numerical values in our example, we have $\rho^+ = 1/2$, $v^+ = 1/2$, $\rho^- = 0$, $v^- = 0$, we find for our example that

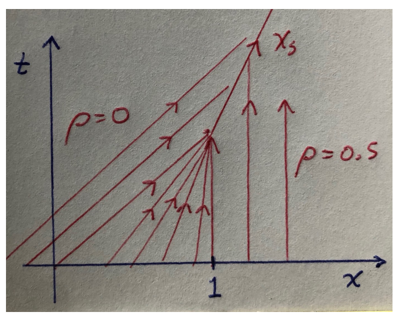
$$u_s = \frac{1/2 \cdot 1/2 - 0 \cdot 0}{1/2 - 0} = \frac{1}{2}.$$

The position of the shock starting at $(1,1)$ and with $\frac{dx_s}{dt} = 1/2$ is therefore $x_s = (t+1)/2$. After $t = 1$, when the shock first forms, we therefore have

$$\text{for } t > 1 \quad \rho(x,t) = \begin{cases} 0 & \text{if } x < \frac{t+1}{2} \\ 0.5 & \text{if } x > \frac{t+1}{2} \end{cases}$$

and prior to that

$$\text{for } 0 < t < 1 \quad \rho(x,t) = \begin{cases} 0 & \text{if } x \leq t \\ 0.5\frac{x-t}{1-t} & \text{if } t < x < 1 \\ 0.5 & \text{if } x \geq 1 \end{cases}$$



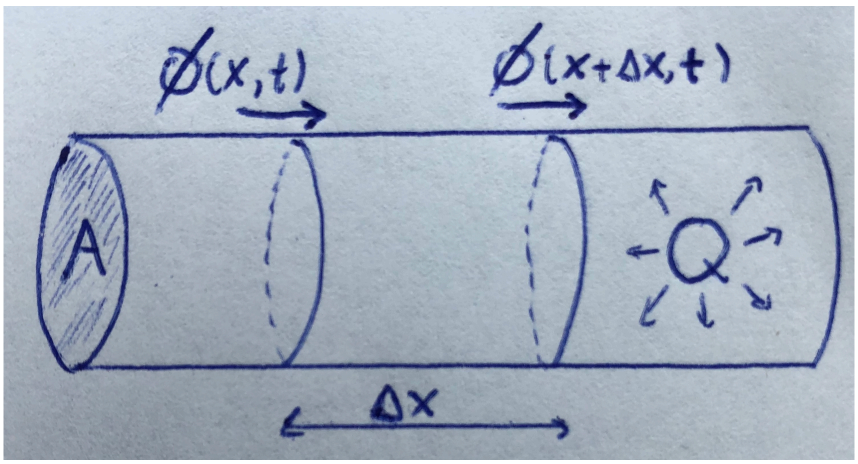
Density in the presence of a shock.

Actually, because the density left of the shock is 0, the speed of the shock is the same as the speed of the cars to the right of the shock. However, in general that will not be the case. The shock may even go to the left (with a negative speed). For example, if $\rho^+ = 0.8$ and $\rho^- = 0.3$, the shock speed would be $u_s = -0.1$. This is the case of a traffic jam that is spreading: even as cars keep moving to the right, so many cars arrive that the jam gets bigger and bigger.

Today's plan

1. Come up with a simple system.
2. Think of all that can cause heat to change.
3. Do a heat budget for a finite time step.
4. Take the limit of time step going to zero.

To derive the Heat equation, we track the amount of heat in a thin slice of a rod.



Heat in a thin slice of a rod.

We introduce the heat density $e(x,t)$, which is the amount of heat by volume. The heat in our little slice is

$$H = e(x,t)A\Delta x.$$

How does it change in time?

It depends on the FLUX: $\phi(x,t) =$ time rate of change of heat per area moving to the right.

There might also be some heat creation: $Q(x,t) =$ rate at which heat per volume is created.

What will be the heat in our slide of rod at time $t + \Delta t$ in terms of how things were at time t?

In words, we expect:

New total heat = Old heat + Flux in - Flux out + created heat

Mathematically, this becomes

$$A\Delta x e(x, t + \Delta t) = A\Delta x e(x, t) + A\phi(x, t)\Delta t - A\phi(x + \Delta x, t)\Delta t + Q(x, t)A\Delta x\Delta t$$

Note that here ϕ is associated with right-going heat. If the heat flows left, ϕ becomes negative.

Now, we will rearrange terms to try to obtain terms that look like derivatives. First, we can divide by A. Then we put all the e terms on the left and divide by $\Delta t\Delta x$.

$$\frac{e(x, t + \Delta t) - e(x, t)}{\Delta t} = \frac{\phi(x, t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x, t)$$

Now we can think of taking limits. Assume that as $\Delta t \to 0$, we also have $\Delta x \to 0$. Taking those limits gives us a *continuous* approximation (otherwise we can try to solve the previous equation numerically):

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x, t)$$

So this is great, we now have a PDE! The only problem is that it has too many unknowns.

Usually, $Q(x, t)$ is given, and does not have to be solved for. But what is ϕ? This is easier to explain in terms of temperature.

Consider the relation between heat and temperature: $e(x, t) = \rho(x)C_p(x)u(x, t)$ where
$\rho(x)$ is the density of the rod, in mass per volume
$C_p(x)$ is the specific heat in heat per mass per degree C
$u(x, t)$ is the temperature in degree C.

Now we can try to relate the flux ϕ to the observable property that is the temperature $u(x, t)$.

Following the derivation of Fourier, we will want a heat flux that satisfies:

1. If the temperature is constant, $\phi = 0$.
2. The heat should flow from hot to cold.
3. The flux should be greater when the temperature difference is greater.
4. The magnitude of the flux should depend on the material of the rod.

The simplest such heat flux is

$$\phi = -K_0 \frac{\partial u}{\partial x}, \quad \text{with } K_0 > 0$$

We can check properties 1-4.
Here K_0 is a material dependent constant. We usually write $K_0 = C_p(x)\rho(x)\, k$, where k is the thermal diffusivity, with units $[k] = L^2/T$.

So our PDE becomes

$$C_p\rho \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-C_p\rho k \frac{\partial u}{\partial x}\right) + Q$$

In most cases, we can assume that $c, \rho,$ and k are constant and this simplifies to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q}{C_p\rho} \quad \text{the Heat Equation!!}$$

Note that chemical concentrations diffuse the same way as heat, and satisfy the same equation. It is also called the diffusion equation.

In many cases, there is no source term ($Q = 0$) and we only have to solve $u_t = ku_{xx}$.

So what do solutions look like? How does the temperature spread in time? Our intuition from everyday life can help, thinking of a hot spoon and a hot pot of soup:

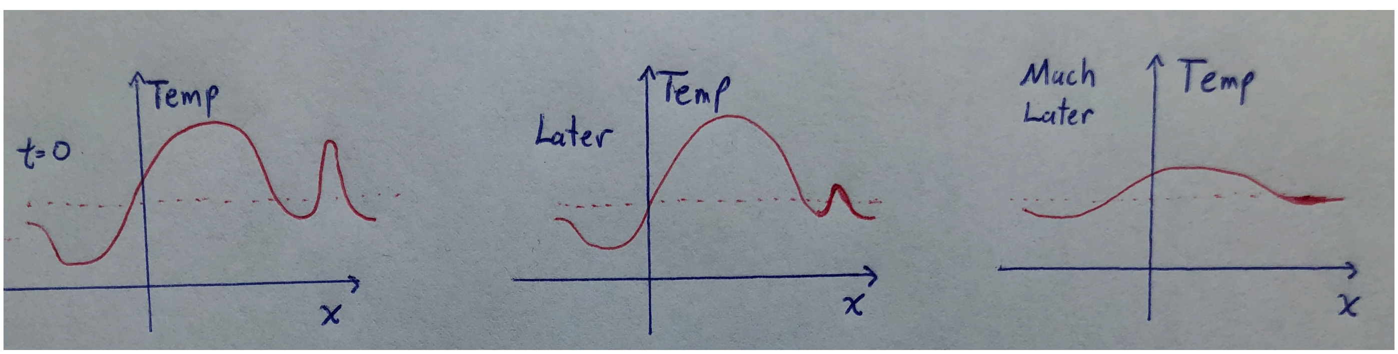


Illustration of the time-evolution of temperature.

Today's plan

1. Define boundary and initial conditions for the heat equation.
2. Provide example and physical interpretation of common BCs.
3. Define what an equilibrium solution is.
4. Find some equilibrium solutions.

It is IMPOSSIBLE to find a unique solution to $u_t = ku_{xx}$ without more information. First we need a domain where we want our solution to hold. Typically the domain is something like:

$$D = \{\ (x,t) \in \mathbb{R}^2 \ \mid \ t > t_0, 0 \leq x \leq L\}$$

The domain may be infinite in space, but we must have a starting point in time. Most often, it is $t_0 = 0$.

Initial Condition (IC)

To know the temperature distribution over time, we need a starting point. We need to know over our whole spatial domain what the temperature is at some given time:

$$\text{IC:} \quad u(t_0, x) = f(x) \quad \text{for } 0 \leq x \leq L$$

Then we can proceed to keep track of how it changes in time. In fact, this is how we derive our PDE in the first place.

Boundary Condition (BC)

In our derivation, we did not talk about what happens at the ends of our rod. However, this is determinant. If you heat both ends, you will get a very different temperature profile than if you insulate both ends. This information makes up the BC and it is AS IMPORTANT as the equation itself.

The most common setups are:

1. The temperature u is given at the ends

   $$u(x=0,t) = u_0(T) \quad u(x=L,t) = u_L(t)$$

   These are also called Dirichlet boundary conditions.
   Very often, u_0 and u_L are constant in time.

2. The heat flux is given:
   At the left boundary, this corresponds to heat flowing INTO the rod:

$$\phi(x=0,t) = \phi_0(t) = -K_0 \left.\frac{\partial u}{\partial x}\right|_{x=L} = -kc\rho \left.\frac{\partial u}{\partial x}\right|_{x=0}$$

   and at the right most end this corresponds to heat flowing OUT of the rod

$$\phi(x=L,t) = \phi_L(t) = -K_0 \left.\frac{\partial u}{\partial x}\right|_{x=L} = -kc\rho \left.\frac{\partial u}{\partial x}\right|_{x=L}$$

   Most commonly, the flux is $\phi_0 = \phi_L = 0$ (insulated ends). This is then called a Neumann condition, which simplifies to $u_x(0,t) = u_x(L,t) = 0$.

3. You can mix 1. and 2., most commonly in Newton's cooling law

$$\phi_0(t) = -kc\rho \left.\frac{\partial u}{\partial x}\right|_{x=0} = (u_0(t) - u(x=0,t))H$$

   where H is a heat transfer constant.
   In general, if both u and u_x appear in the boundary condition, it is called a Robin boundary condition.

4. If the domain is infinite, we usually require that the temperature does NOT go to infinity anywhere in the domain ($u < \infty$). Sometimes, we may also require that it approaches a given value ($\lim_{x\to\infty} u = u_\infty$).

Equilibrium temperature distribution

Before tackling the harder question of how temperature changes in time, we ask a simpler question: What will happen after a very long time?

There are two main possibilities:
1) If heat is constantly inputed or removed from the system, then no equilibrium will be reached, and we will have

$$\lim_{t\to\infty} u(x,t) = \text{Does not exist.}$$

This will happen when the TOTAL heat flux is non-zero. The total flux into the system is

$$\text{Total heat flux} = \phi(x=0,t) - \phi(x=L,t) + \int_0^L Q(x)\,dx =$$

So if the Total heat flux is non-zero, no equilibrium is possible.

2) If the total heat flux is 0 (a more common case for physically relevant setups), then an equilibrium WILL be reached.

At equilibrium, nothing changes in time, so we have for the equilibrium distribution $u_e(x)$

$$\frac{\partial u_e}{\partial t} = 0$$

This makes the remaining equation easier to solve because it becomes an ordinary differential equation (in x only). We then have to solve:

$$k\frac{\partial^2 u_e}{\partial x^2} + Q(x) = 0.$$

Depending on the boundary conditions, we can get various solutions. For example:

Ex. 1: If $Q = 0$, with $u_0 = A$, $u_L = B$, we have:
$u_e(x) = C_1 + C_2 x$ which can be found to yield: $u_e(x) = B\frac{x}{L} + A\frac{L-x}{L} = A + (B - A)\frac{x}{L}$.

Ex. 2: The system is insulated ($\frac{\partial u}{\partial x} = 0$ at both ends) and $Q = 0$.
Then $u_e(x) = C$ is a solution. But what is the value of the constant? It depends on the initial condition. Since no heat escapes our system, we must have, for a given initial condition $u(x, t = 0) = f(x)$.

$$\int_0^L C dx = CL = \int_0^L f(x)\, dx \quad \text{so } C = \frac{\int_0^L f(x)\, dx}{L}$$

So really the equilibrium temperature is the *average* of the initial temperature, which makes intuitive sense.

Now let's see concrete examples:
A) $u_t - u_{xx} = 0$, BC: $u(x = 0, t) = 1$ and $u(x = 2, t) = 7$ and IC: $u(x, t = 0) = \sin x$.
Here the total flux is zero, so there will be an equilibrium.
$u_e(x) = 1 + \frac{6}{2}x = 1 + 3x$

B) $u_t - ku_{xx} = \sin x$, BC: $u(x = 0, t) = 0$ and $u(x = \pi, t) = 1$ and IC: $u(x, t = 0) = x(\pi - x)$.
Here there is a source, but we have $\int_0^\pi Q(x)\, dx = \int_0^\pi \sin x\, dx = 0$. So again, there will be an equilibrium.
We find it by integrating:

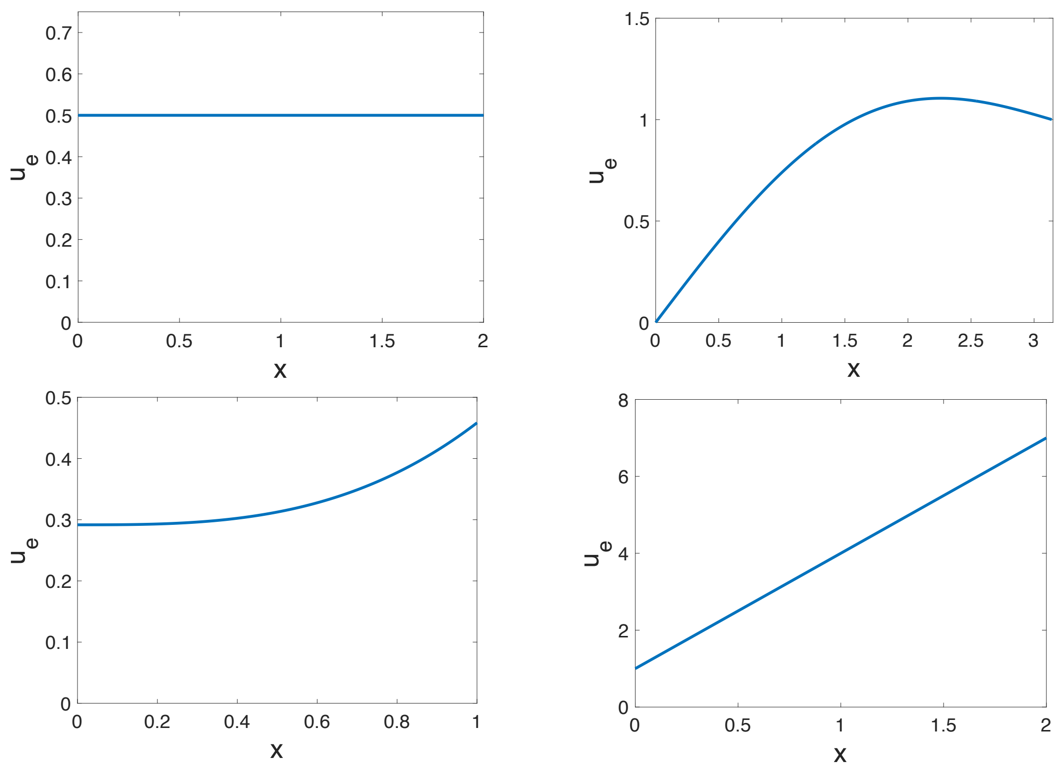
$$\begin{aligned}
\frac{\partial^2 u_e}{\partial x^2} &= -\frac{\sin x}{k} \\
\frac{\partial u_e}{\partial x} &= \frac{\cos x}{k} + C_1 \\
u_e &= \frac{\sin x}{k} + C_1 x + C_2 \\
u_e(0) &= 0 = C_2 \\
u_e(\pi) &= 1 = C_1\pi, \quad \text{so finally} \\
u_e(x) &= \frac{\sin x}{k} + \frac{x}{\pi}
\end{aligned}$$

C) $u_t - u_{xx} = 0$ BC: $u_x(x=0,t) = 1$ (influx) and $u_x(x=2,t) = 0$ (insulated) and IC: $u(x,t=0) = x^3$.
If we try to solve for the equilibrium, we get: $u_e(x) = C_1 x + C_2$
The BC at $x=0$ yields that $C_1 = 1$, but the BC at $x=1$ yields that $C_1 = 0$. A contradiction. So there is no equilibrium possible. This is confirmed by the fact that there is a net influx of heat in the system.

D) $u_t - 4u_{xx} = 0$, BC: $u_x(x=0,t) = 0$ and $u_x(x=2,t) = 0$ (insulated) and IC: $u(x,t=0) = (\sin x)^2$.
Here there is no source and the system is insulated, so there is an equilibrium.
Integrating yields $u_e(x) = C_1 x + C_2$.
Using the BC gets us that $C_1 = 0$. But what about C_2?
Conservation of heat in the system shows that the total heat is (ρc times)

$$C_2 \pi \int_0^{\pi} (\sin x)^2 \, dx = \frac{\pi}{2}$$

So $C_2 = 1/2$ and out equilibrium is $u_e(x) = \frac{1}{2}$.



Equilibrium temperatures

E) $u_t - u_{xx} = -x$ with BC: $u_x(x=0,t) = 0$ and $u_x(x=1,t) = 1/2$ and IC: $u(x,t=0) = x^2$.
Note that in class in 2022, I mistakenly used Q = x
I should have used Q/K_0 = -x

Then the total heat flux is 0 and the solution given here is correct.

Here there is a source, and the total creation of heat is

$$\int_0^L Q(x)\, dx = \int_0^1 -x\, dx = -\frac{1}{2}$$

However, there is also an inflow of heat at the right boundary of 1/2 (be careful with the signs: ϕ has the opposite sign of u_x) which cancels the heat production.

So we expect to be able to find an equilibrium. If we integrate:

$$\begin{aligned}
\frac{\partial^2 u_e}{\partial x^2} &= x \\
\frac{\partial u_e}{\partial x} &= \frac{x^2}{2} + C_1 \\
\left.\frac{\partial u_e}{\partial x}\right|_{x=0} &= 0 = C_1 \quad \text{using one BC} \\
\left.\frac{\partial u_e}{\partial x}\right|_{x=1} &= \frac{1}{2} = \frac{1}{2} + C_1 \quad \text{using there other BC} \to \text{consistent} \\
u_e &= \frac{x^3}{6} + C_2 \quad \text{we must use conservation to find } C_2 \\
\int_0^1 u_e(x)\, dx &= \left.\frac{x^4}{24} + C_2 x\right|_0^1 = \frac{1}{24} + C_2 \\
\int_0^L f(x)\, dx &= \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} = \frac{8}{24} \quad \text{so} \\
C_2 &= \frac{1}{3} - \frac{1}{24} = \frac{7}{24} \quad \text{and}
\end{aligned}$$

$$u_e(x) = \frac{x^3}{6} + \frac{7}{24}$$

Today's plan

1. What does it mean to separate variables to solve a PDE?
2. When can you use that?
3. How does it work?
4. What are things to watch out for?

We want to take advantage of the linearity of the heat equation:

$$L(u) = u_t - k u_{xx} = 0$$

where we consider the case without sources, which makes this equation homogeneous.

The operator L is then linear: $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$.
Importantly, this means that if $L(u_1) = 0$ and $L(u_2) = 0$ (we have two solutions to our equation) then $L(c_1 u_1 + c_2 u_2) = 0$ also.
In other words linear combinations of solutions REMAIN solutions.
This will be very useful to us when trying to match initial and boundary conditions.

We begin with the "simplest" boundary conditions, which are homogeneous, given values of u (Dirichlet). Our problem is to solve

$$\begin{aligned} u_t &= k u_{xx} \quad \text{for } k > 0 \quad \text{subject to} \\ u(x=0,t) &= 0 \\ u(x=L,t) &= 0 \\ u(x,t=0) &= f(x) \end{aligned}$$

We now make a pretty big assumption: We guess that the solution will be a sum of SEPARABLE functions:

$$u_n(x,t) = \phi_n(x) G_n(t)$$

This will not always work, but we will see theory showing that this works remarkably often.
We proceed in a specific order:

1. We focus on the PDE first.
2. Then we will use the BC.
3. We will match the IC last.

First, we need to know what derivatives of u_n look like. We have

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= \phi_n(x)\frac{dG_n}{dt} = \phi_n G_n' \\ \frac{\partial u_n}{\partial x} &= \frac{d\phi_n}{dx}G_n(t) = \phi_n' G_n \\ \frac{\partial^2 u_n}{\partial x^2} &= \frac{d^2\phi_n}{dx^2}G_n(t) = \phi_n'' G_n\end{aligned}$$

We can now use this in our PDE:

$$u_t = ku_{xx} \quad \text{becomes} \quad \phi_n(x)\frac{dG_n}{dt} = k\frac{d^2\phi_n}{dx^2}G_n$$

The key step is that we now divide by $ku_n = k\phi_n G_n$ (note that you don't have to divide by k, but it is usual)

$$\frac{\phi_n(x)G_n'(t)}{k\phi_n G_n} = k\frac{\phi_n''(x)G_n(t)}{k\phi_n G_n} \longrightarrow \frac{G_n'(t)}{kG_n(t)} = \frac{\phi_n''(x)}{\phi_n(x)} = \lambda$$

So here is the whole point: **Each side depends on one variable ONLY**.
The only way a function of x (LHS) can equal a function of t (RHS) is for both to be constant. We call that constant λ.

In general, λ is constant but can be ANY COMPLEX number, at least a priori.
We need to figure out which values of λ might work. We solve in t first:

$$\frac{G_n'(t)}{G_n(t)} = k\lambda$$

so $G_n(t) = C_n e^{k\lambda t}$.
What does this do as $t \to \infty$? It blows up (tends to infinity) if $\mathrm{Re}(\lambda) > 0$.
Moreover, in this case if λ is not real, the function G_n cannot be real for any choice of the integration constant C_n. So we must have that $\lambda \in \mathbb{R}$ and $\lambda < 0$. To illustrates that, we write

$$\text{Let } \lambda = -\alpha^2, \quad \alpha \in \mathbb{R}.$$

So out functions of time are $G_n(t) = C_n e^{-k\alpha^2 t}$.

Now in x, we have

$$\frac{\phi_n''(x)}{\phi_n(x)} = \lambda = -\alpha^2 \quad \text{or} \quad \phi_n'' + \alpha^2\phi_n = 0$$

with our boundary conditions (they only apply in x): $\phi_n(0) = 0$ and $\phi_n(L) = 0$.

To solve this system, we can assume $\phi_n(x) = e^{rx}$ and get the characteristic equation

$$r^2 + \alpha^2 = 0$$

so we must have that $r = \pm i\alpha$. So our solutions are

$$\phi_n(x) = A\cos(\alpha x) + B\sin(\alpha x).$$

Now we employ our boundary conditions. An important thing to recall is that we are looking for non-trivial solutions. So here $\phi_n(x) = 0$ is a solution to our system, but it is not very interesting and we want to find other solutions. This is possible because α is still an UNDETERMINED constant.

Our BC state that:

$$\phi_n(0) = 0 = A\cos(0) + B\sin(0) = A$$

so simply $A = 0$. Also

$$\phi_n(L) = 0 = 0 + B\sin(\alpha L)$$

In this case, we don't want $B = 0$ (that is the trivial solution) so we must have

$$\sin(\alpha L) = 0.$$

This is only possible if $\alpha L = n\pi$ for an integer n. So we now have constraints on the constant α. It must satisfy

$$\alpha_n = \frac{n\pi}{L} \quad \text{for} \quad n \in \mathbb{N}$$

We will say that those α_n are the eigenvalues of this system (can you see why we chose that name? we will get back to this).

So overall, we found that

$$u_n(x,t) = C_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n \in \mathbb{N}.$$

So in fact, we found infinitely many solutions (in a countable way).

We/you should try the same problem for the boundary conditions $u_x = 0$ at $x = 0$ and at $x = L$.

All our solutions u_n satisfy the PDE, and the boundary conditions, because of how we found them. That means that if we sum them, the sum will also satisfy the PDE (because it is linear) and the BCs (because they are homogeneous). So our current solution guess is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

The only thing left is to satisfy the initial condition. For that, we will need to pick the coefficients C_n carefully.

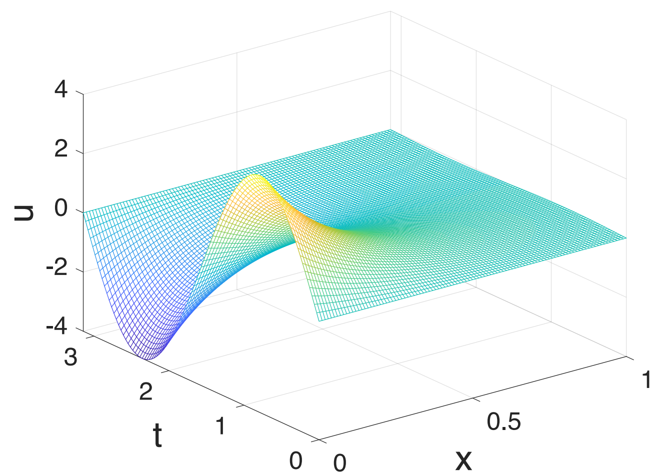
At time $t = 0$, we have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n\,(1)\sin\left(\frac{n\pi x}{L}\right)$$

If we are lucky, the coefficients C_n are easy to recognize.

Ex. 1: $f(x) = 4\sin(\frac{2\pi x}{L})$.

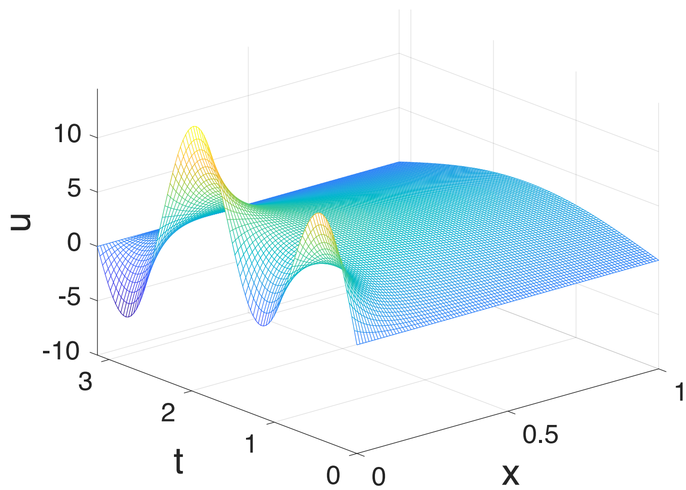
In that case: $C_1 = 0, C_2 = 4, C_3 = 0, ...C_n = 0$.



Solution to the heat equation, example 1 with $L = \pi$.

Ex. 2: $f(x) = 7\sin(\frac{\pi x}{L}) + 8\sin(\frac{4\pi x}{L})$.

In that case: $C_1 = 7, C_4 = 8$ and for all other n, we have $C_n = 0$.



Solution to the heat equation, example 2 with $L = \pi$.

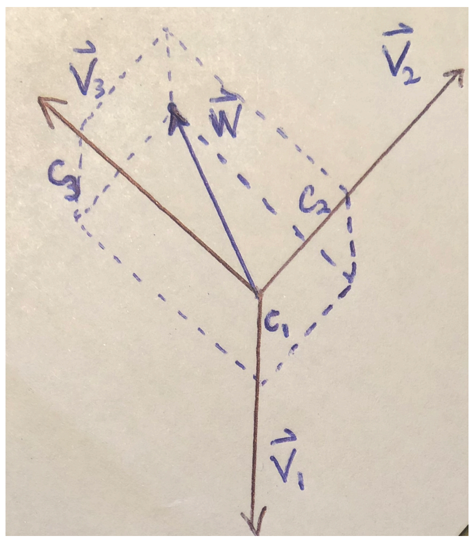
But what if we are not that lucky? What C_n satisfy $f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x/L)$?

Today's plan

1. What is orthogonal expansion for vectors?
2. How can we see functions as vectors.
3. How do this relate to solving PDEs with separation of variables.

We need to make a detour via vectors and linear algebra to understand how to use orthogonal expansions. We'll work in 3D.

Suppose I have an orthogonal basis in 3D, $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.
Suppose I also have a vector $\vec{w}$ that I want to write in terms of my basis.



Representation of a 3D vector in a general orthogonal basis

What we want is of the form

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \qquad *$$

and we are looking for the coefficients c_i. Remember that our basis is orthogonal, so $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, and $\vec{v}_2 \cdot \vec{v}_3 = 0$.
We will take advantage of this by taking a dot product of equation (*) with each basis vector in turn.

If we start with $\vec{v}_1$, we get

$$\vec{v}_1 \cdot \vec{w} = c_1\vec{v}_1 \cdot \vec{v}_1 + c_2\vec{v}_1 \cdot \vec{v}_2 + c_3\vec{v}_1 \cdot \vec{v}_3 = c_1\vec{v}_1 \cdot \vec{v}_1$$

So we can now solve for c_1, since all the other coefficients are no longer involved:

$$c_1 = \frac{\vec{v}_1 \cdot \vec{w}}{\vec{v}_1 \cdot \vec{v}_1} \quad \text{and similarly} \quad c_2 = \frac{\vec{v}_2 \cdot \vec{w}}{\vec{v}_2 \cdot \vec{v}_2}, \quad c_3 = \frac{\vec{v}_3 \cdot \vec{w}}{\vec{v}_3 \cdot \vec{v}_3}.$$

In fact, our coefficients are PROJECTIONS of $\vec{w}$ onto each basis vector.

But this only works if our original basis is orthogonal: $\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$.

In our case, we are trying to find how to write:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

So $f(x)$ is playing the role of $\vec{w}$ and $\sin\left(\frac{n\pi x}{L}\right)$ are the basis functions for various n. But are they orthogonal?

What does orthogonal even mean for functions??

We need to define a dot (or inner) product. In general, dot products must satisfy 3 properties: This will ensure that they behave in a useful way.

1. $f(x) \cdot f(x) \geq 0$ and $f(x) \cdot f(x) = 0$ if and only if $f(x) = 0$.
2. $f(x) \cdot g(x) = g(x) \cdot f(x)$, so it is symmetric.
3. $(af(x) + bg(x)) \cdot h(x) = af(x) \cdot h(x) + bg(x) \cdot h(x)$ which is to say it is linear in either argument, or bilinear.

We will use integration as our dot product:

$$f(x) \cdot g(x) = \int_0^L f(x)g(x)r(x)\,dx \quad \text{for some given function } r(x) > 0.$$

Importantly:

1. The interval over which we integrate is the domain of x for our problem (we don't care about other values of x).
2. $r(x) > 0$ over the whole interval is required to meet the first property of inner products.
3. For now, we will use $r(x) = 1$. So in our case $f \cdot g = \int_0^L f(x)g(x)\,dx$.

We can now speak of two functions being orthogonal. $f(x)$ and $g(x)$ are orthogonal if

$$f(x) \cdot g(x) = \int_0^L f(x)g(x)\,dx = 0.$$

You can, and should, use trig identities to show that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = 0 \quad \text{if } m \neq n \text{ and are natural numbers}$$

Moreover,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)^2\,dx = \frac{L}{2}$$

So indeed the functions $\sin\left(\frac{n\pi x}{L}\right)$ form an orthogonal set. This is really convenient for us, because we can use our projections to find the coefficients B_n in our solution.

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} B_n(1)\sin\left(\frac{n\pi x}{L}\right) \quad \text{gives us} \\
\int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\,dx &= \int_0^L \sin\left(\frac{k\pi x}{L}\right)\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\,dx \\
\int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\,dx &= \sum_{n=1}^{\infty}\int_0^L \sin\left(\frac{k\pi x}{L}\right) B_n \sin\left(\frac{n\pi x}{L}\right)\,dx \quad \text{as we will see in Chap. 3} \\
\int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\,dx &= \frac{LB_k}{2}
\end{aligned}$$

so we can find the coefficients as:

$$B_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) f(x)\,dx$$

So we now have a complete solution!

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

with

$$B_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

Let's pick a concrete initial condition. Let $L = \pi$ and $f(x) = x(\pi - x)$. We can find (with Wolfram alpha for example) that the coefficients are

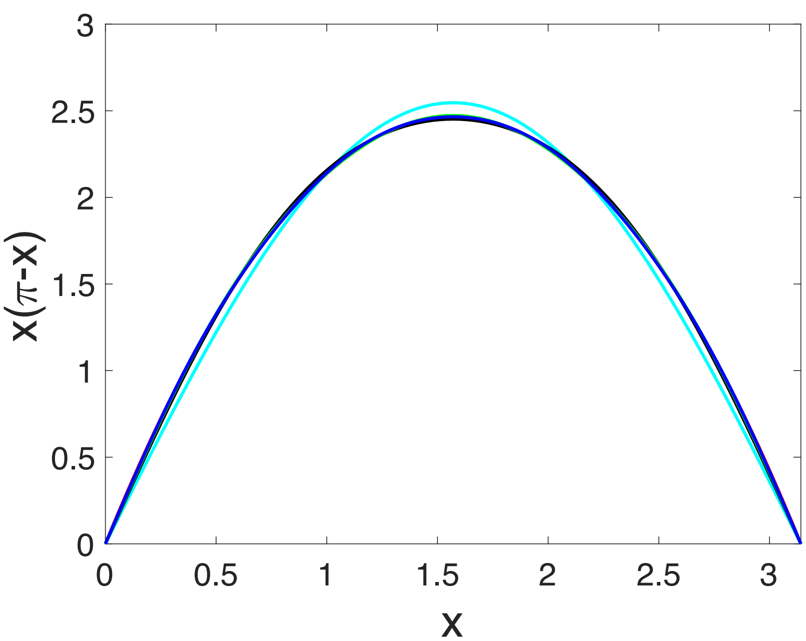
$$B_n = -\frac{2}{\pi}\frac{(\pi n\sin(\pi n) + 2\cos(\pi n) - 2)}{n^3} = \frac{4}{\pi}\frac{(1-(-1)^n)}{n^3}$$

so really $B_{2n} = 0$ and $B_{2n-1} = \frac{8}{\pi(2n-1)^3}$. In particular

$$B_1 = \frac{8}{\pi}, \quad B_3 = \frac{8}{27\pi}, \quad B_5 = \frac{8}{125\pi}, \quad B_7 = \frac{8}{343\pi}$$

We can plot this, for example in Matlab:

```
hold off
xs = 0:pi/100:pi;
plot(xs,xs.*(pi-xs),'r','LineWidth',2)
hold on
plot(xs,(8/pi)*(sin(xs)),'c','LineWidth',2)
plot(xs,(8/pi)*(sin(xs)+sin(3*xs)/27),'k','LineWidth',2)
plot(xs,(8/pi)*(sin(xs)+sin(3*xs)/27+sin(5*xs)/125),'g','LineWidth',2)
plot(xs,(8/pi)*(sin(xs)+sin(3*xs)/27+sin(5*xs)/125+sin(7*xs)/343),'b',
'LineWidth',2)
```



Increasingly accurate approximations using a sine series.

Zero-derivative BC

What happens if we change the boundary condition from $u = 0$ at the boundaries to $u_x = 0$ at the boundary?

If you recall, that BC corresponds to having no heat escaping from the domain (no flux at the boundaries). What does it change in our separation of variables process?

Only the functions of x will be affected. We now have to solve

$$\phi_n(x)'' + \alpha^2 \phi_n(x) = 0 \quad \text{subject to} \quad \left.\frac{d\phi_n}{dx}\right|_{0,L} = 0$$

Our general solution is the same, but there can only be cosines in the solution, since sine has a non-zero derivative when $x = 0$. WE have

$$\phi_n(x) = A\cos(\alpha x) \quad \text{and} \quad \phi_n'(x) = -A\alpha\sin(\alpha x)$$

and we need to select α so that $\phi_n'(L) = 0$. So we again have $\alpha = n\pi/L$ for $n \in \mathbb{Z}$, and we now also allow $\alpha = 0$.

Our general solution is then

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right)$$

Using trig identities, we can again show that the functions $\cos\left(\frac{n\pi x}{L}\right)$ form an orthogonal set. We also still have that

$$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\, dx = \frac{L}{2} \quad \text{if } n \neq 0$$

The case $n = 0$ is special, but easier. In the case the eigenfunction is simply $\phi_0(x) = 1$, and $\int_0^L (\phi_0)^2\, dx = L$.

We can obtain the coefficients to match an initial condition $f(x)$ in the same manner as before. We get very similar formulas, except for $n = 0$:

$$A_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{if } n \neq 0$$

and

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx.$$

Periodic boundary conditions

We consider a third type of boundary condition: periodicity. This is equivalent to solving the problem of the time-evolution of temperature when the domain is a ring of perimeter $2L$. The positions $x - L$ and $x + L$ are then identical, and therefore $u(x - L, t) = u(x + L, t)$ (our domain has size $2L$ in this case).

Our "boundary" conditions then become

$$u(L, t) = u(-L, t) \quad \text{and} \quad \left.\frac{du}{dx}\right|_{-L} = \left.\frac{du}{dx}\right|_{L}$$

Note that we don't have a value for u or its x-derivative at the boundary. We only know that u is periodic.

Once again, this only changes our functions of x. We now have to solve

$$\phi_n(x)'' + \alpha^2 \phi_n(x) = 0 \quad \text{subject to } \phi_n \text{ having period } 2L.$$

Our general solution remains

$$\phi_n(x) = A_n \cos(\alpha x) + B_n \sin(\alpha x)$$

From the condition $\phi_n(-L) = \phi_n(L)$, we have

$$A_n \cos(\alpha L) + B_n \sin(\alpha L) = A_n \cos(-\alpha L) + B_n \sin(-\alpha L)$$

which can be simplified because $\cos(x) = \cos(-x)$ and $\sin(x) = -\sin(x)$ o

$$A_n \cos(\alpha L) + B_n \sin(\alpha L) = A_n \cos(\alpha L) - B_n \sin(-\alpha L) \quad \text{or} \quad 2B_n \sin(\alpha L) = 0$$

We obtain a similar condition from applying the BC or the derivative,

$$2\alpha A_n \sin(\alpha L) = 0.$$

So once again we need $\sin(\alpha L) = 0$ and therefore $\alpha = n\pi/L$ for $n \in \mathbb{N}$. Note that here $\alpha = 0$ is also acceptable, giving a constant solution: $\phi_0(x) = 1$.

Our general solution now has more terms, as both the sine and cosine are present:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

The coefficients can be found as before, since $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are also orthogonal to each other. the coefficients are then

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and

$$A_n = \frac{1}{L} \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{if } n \neq 0$$

and

$$B_n = \frac{1}{L} \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{if } n \neq 0.$$

Note that here the factor in front of the integral has been halved compared to before, because our domain now has length $2L$.

You can see a summary of those coefficients on page 65 of the textbook. Learning them by heart is not useful, and you will never be asked to give those formulas in this class. However, you should understand how they were obtained (and you can be asked about THAT!).

Today's plan

1. What is a Fourier Series?
2. When can you use them?
3. What does this have to do with PDEs?

We have been using Fourier Series to match ICs in our last example. These are Series of the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

for $-L \leq x < L$. When this Series converges, we saw that the coefficients are

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

and

$$A_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{if } n \neq 0$$

and

$$B_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{if } n \neq 0.$$

But when DOES it converge? Does it always converge to $f(x)$?
Does it take many terms to get there, or to get close?
Basically, **does it work in any useful way?**

To answer this, we begin with a definition: A function $f(x)$ is **piecewise smooth** over a domain D if D can be broken into a finite number of intervals inside each of which:
- $f(x)$ is continuous and bounded.
- $f'(x)$ is continuous and bounded.

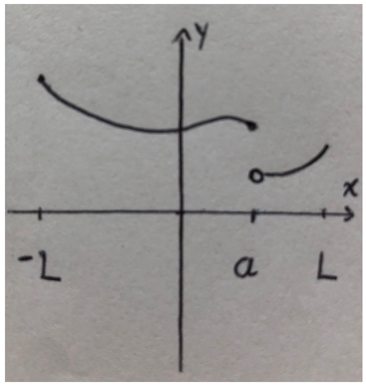
On the left, at the junction $x = a$, we need

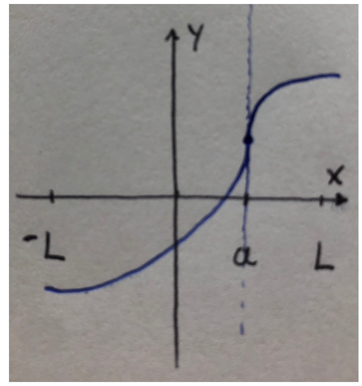
$$\lim_{x\to a^+} f(x) = f^+ \quad \text{to exist}$$

and

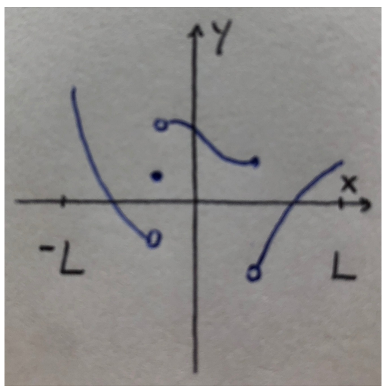
$$\lim_{x\to a^+} f'(x) = f_p^+ \quad \text{to exist}$$

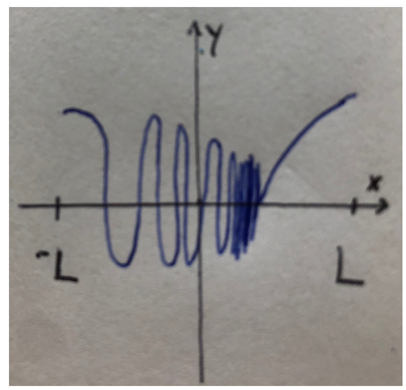
and we have a similar condition at the right end.





Examples of a piecewise smooth function (left) and a not piecewise smooth function (right)





Two more examples: piecewise smooth on the left, not piecewise smooth on the right.

Second definition: A function is said to be **periodic** with period $2L$ if for any $x \in \mathbb{R}$ it is true that

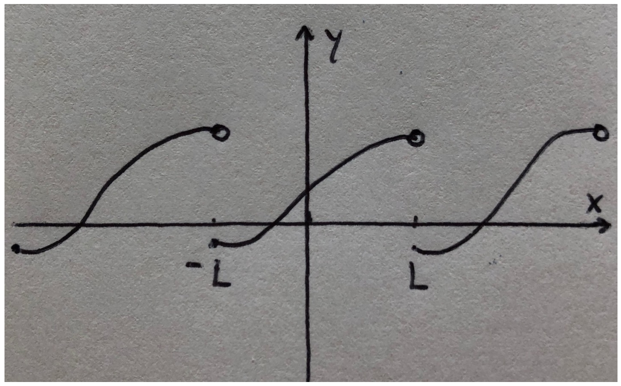
$$f(x) = f(x) + 2L$$

Periodic Extension

If a function is defined over an interval $[-L, L)$ only, we can construct its **periodic extension** by copying the part of the graph that is given over and over. So for any x, we can find $n \in \mathbb{Z}$ such that $-L \leq x + n(2L) < L$. The periodic extension, called $pef(x)$, of the function $f(x)$ is then, for that same n

$$pef(x) = f(x + n(2L)) \quad \text{for } n \in \mathbb{Z} \text{ such that } -L \leq x + n(2L) < L$$

Note that this is not continuous if $\lim_{x \to L^-} f(x) \neq f(-L)$.



A function constructed by periodic extension

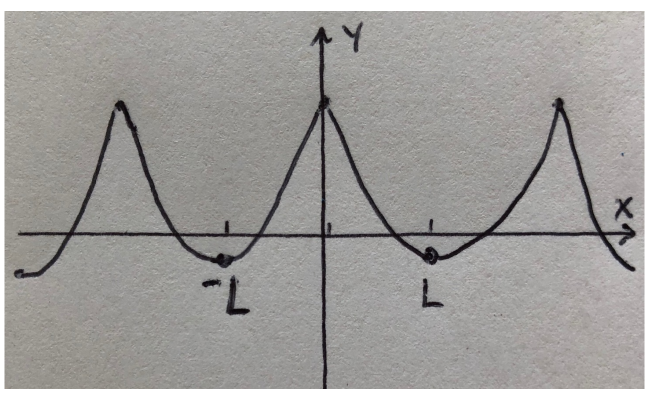
Even Extension

We can also introduce an **Even extension**. Recall that a function is said to be even if for any x, we have $f(x) = f(-x)$.

so if $f(x)$ is defined over $[0, L]$ only, we can construct its odd extension by symmetry about the y-axis

$$Eef(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x < 0 \end{cases}$$

We can then extend this using a periodic extension as well. Note that an even extension



A function constructed by even extension, and then periodic extension.

is continuous if $f(x)$ is continuous.

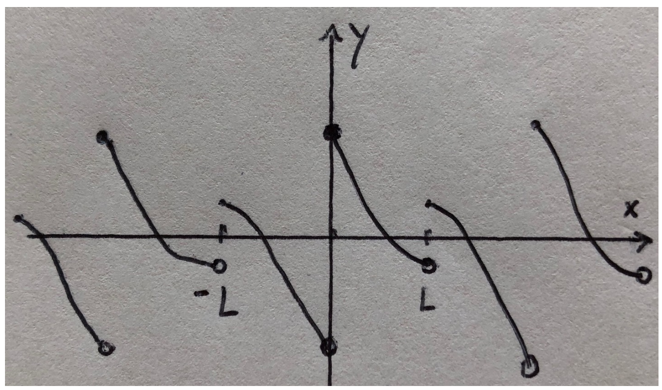
Odd Extension

We can also introduce an **Odd extension**. Recall that a function is said to be odd if for any x, we have $f(x) = -f(-x)$.

so if $f(x)$ is defined over $[0, L]$ only, we can construct its odd extension by symmetry about the x-axis followed by symmetry about the y-axis

$$Oef(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ -f(-x) & \text{if } -L \leq x < 0 \end{cases}$$

We can then extend this using a periodic extension as well. Note that an odd extension is



A function constructed by odd extension, and then periodic extension.

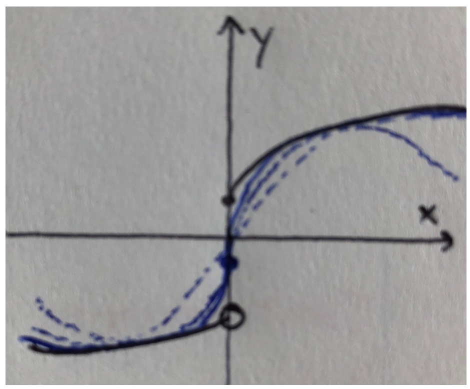
not continuous if $f(0) \neq 0$. For the periodic extension to be continuous, we also need that $f(L) = 0$.

Now we can talk about our big results: Fourier's theorem.

Fourier's Theorem

If $f(x)$ has a piecewise smooth periodic extension with period $2L$, denoted by $pef(x)$, then

$$A_0+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f(x) & \text{if } pef(x) \text{ is continuous at } x \\ \frac{1}{2}\left(\lim_{x_0\to x^+} f(x_0) + \lim_{x_0 \to x^-} f(x_0)\right) & \text{otherwise} \end{cases}$$



Convergence of Fourier Series

Note: If the original function $f(x)$ is not continuous, the Fourier Series doesn't converge to a continuous function, but it tries for the next best thing, aiming for the middle of the discontinuities.

We did not really answer what Fourier Series have to do with PDEs. You may recall that our solution to the heat equation may look like a Fourier Series for the periodic case. We will see this in more details in the coming lectures.

U N
. U N
. U T
So for now, we P T.

Today's plan

1. What is the difference between a Fourier Series and a Sine or Cosine series?
2. When can you get a Sine or Cosine Series?
3. What does that have to do with PDEs?

As the name implies, Sine and Cosine Series are Series that involve only one type of trigonometric function. As a result, the overall Series shares some properties of the function it is made of.

Fro example, consider a Sine Series. We consider a function $f(x)$ defined on the interval $[0, L]$, and we expand it as a Sine Series:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Note that here the domain size L corresponds to the distance between zeros of the function with the smallest index $n = 1$ (it is not the same as the period of that same function, it's actually half of the period).

This is not exactly a Fourier Series, but we can turn it into one. We consider the ODD extension of $f(x)$ to $[-L, L]$, and then the periodic extension of the resulting function, which we call $pef(x)$. The Sine Series then has the same convergence properties as the regular Fourier Series applied to $pef(x)$.

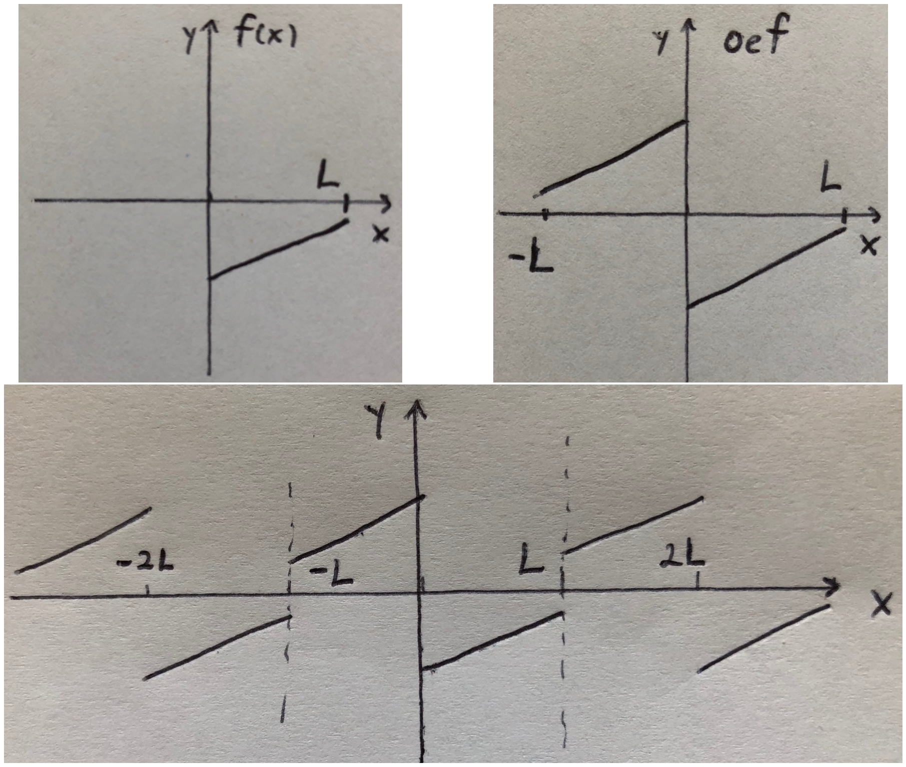
An important point to note is that if $f(x)$ is defined over $[0, L]$, its odd extension will only be continuous if $f(x)$ is continuous itself and if $f(0) = 0$ (by odd symmetry) and $f(L) = 0$ (so that we can have $f(-L) = -f(L) = f(L)$.

For Cosine Series, we need to consider first the EVEN extension of the original function, and then the periodic extension of that. In that case, if $f(x)$ is continuous, its even, periodic extension will be too.

Why do we care so much about continuity? Because when a function is not continuous, its Fourier Series does not converge to the original function at the points of discontinuity.

Examples:
1) $f(x) = 3$, We look for a Fourier, Sine, and Cosine Series.

1. For a Fourier Series, we consider that $f(x) = 3$ is defined over $[0, 2\pi)$.
   In fact, the periodic extension remains the very same, and we have that $f(x) = 3$ is valid for any x.
   The coefficients are:
   $$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \ \cos(nx) \ dx = 0$$



Original function over $[0, L]$, odd extension over $-[L, L]$, periodic odd extension.

and

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 3 \ \sin{(nx)} \ dx = 0$$

But the constant coefficient is different:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 3 \, dx = 3$$

So the Fourier Series is actually just 3, the original function itself. So things *are* simple.

2. For a Cosine Series, consider that $f(x) = 3$ is defined over $[0, \pi]$. The even extension remains just $f(x) = 3$, now valid over $[-\pi, \pi]$. So the coefficients are:

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} 3 \ \cos{(nx)} \ dx = 0$$

and

$$A_0 = \frac{1}{\pi} \int_{0}^{\pi} 3 \, dx = 3$$

So the Cosine Series is also just 3.

3. For a Sine Series, consider that $f(x) = 3$ is defined over $[0, \pi]$. The odd extension is not continuous, as shown below. The coefficients are:

$$B_n = \frac{2}{\pi} \int_0^\pi 3 \ \sin(nx) \ dx = \left. \frac{-6}{\pi} \frac{\cos(nx)}{n} \right|_0^\pi = \frac{-6}{\pi} \left(\frac{\cos(n\pi) - \cos(0)}{n} \right)$$

This can be further simplified to

$$B_n = \frac{6}{\pi} \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

So if we let $n = 2k - 1$, we find $B_k = \frac{12}{\pi(2k-1)}$.
So finally, our Sine Series is

$$3 = \sum_{k=1}^{\infty} \frac{12}{\pi(2k-1)} \sin((2k-1)x)$$

Note that at $x = 0$, the Sine Series is equal to 0 (the average between $f(x)$ and its odd extension). Similarly, we also get 0 at $x = \pi$.
Note also that in between the function is approaching 3. But it is not doing great at the ends eh? We will get back to that.

Odd extension of $f(x) = 3$ defined over $[0, \pi]$

We will do one more example of a Sine Series: $f(x) = x^2(\pi - x)^2$ over the interval $[0, \pi]$. Note that this was chosen so that the odd extension is continuous. In fact, it is also differentiable everywhere. The formula for the coefficients remains:

$$B_n = \frac{2}{\pi} \int_0^\pi x^2(\pi - x)^2 \sin(nx) \, dx = \frac{2}{n^5}(\pi^2 n^2 - 12)(\cos(n\pi) - 1)$$

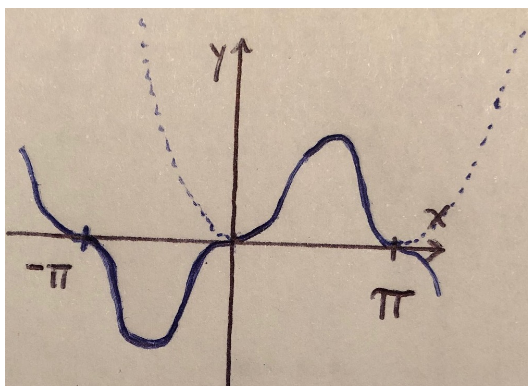
Note that this integral does not require any particular trick (so it is not *hard*) but it is long. This is perfect for computer programs.
Once again, we only get non-zero results for odd coefficients. Letting $n = 2k-1$, we have

$$B_{2k-1} = \frac{4}{(2k-1)^5}(12 - \pi^2(2k-1)^2)$$

and the Series is

$$x^2(\pi - x)^2 = \sum_{k=1}^{\infty} \frac{(48 - 4\pi^2(2k-1)^2)}{(2k-1)^5} \sin((2k-1)x)$$

Note that here, we have convergence to the original function everywhere, including at the end points.



Odd extension of $f(x) = x^2(\pi - x)^2$ defined over $[0, \pi]$

Today's plan

1. How can you manipulate Fourier Series?
2. When do you have to be careful (when does it not work?)
3. How best can you approximate discontinuous functions.

Term-by-term operations

In many instances, you can differentiate or integrate a Fourier series term-by-term. The key here is to verify that the RESULT of that operation is piecewise smooth.

So if you have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } x \in [0, L]$$

and you know that $f'(x)$ is piecewise smooth over $[0, L]$, then

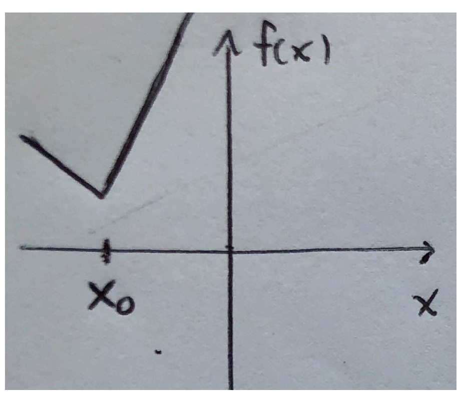
$$f'(x) = \sum_{n=1}^{\infty} B_n \left[\sin\left(\frac{n\pi x}{L}\right)\right]' = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right)$$

Note that in this case the behavior at the boundary changes.

But for which $f(x)$ will $f'(x)$ be piecewise smooth?

When $f'(x)$ is continuous, nothing special happens and you can integrate it to obtain $f(x)$.

When $f(x)$ is continuous but not differentiable at a point x_0, and has well-defined slopes on either side.



A function with a derivative having a jump discontinuity at $x = x_0$.

Here, $f'(x)$ is not defined at $x = x_0$. But we have that

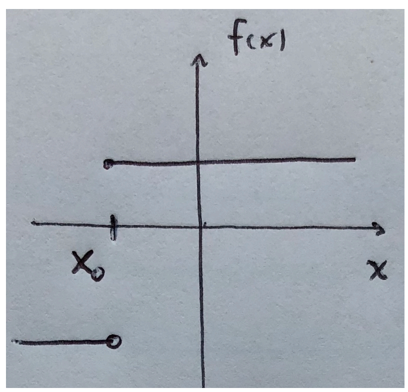
$$\lim_{x \to x_0^+} f'(x) \quad \text{exists, and} \qquad \lim_{x \to x_0^-} f'(x) \quad \text{exists}$$

The actual value of $f'(x)$ at x_0 is unimportant, and the results is that $f'(x)$ is piecewise smooth. more precisely, we have the definitions

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ f'(x_0^+) &= \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \\ f'(x_0^-) &= \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

Note that, importantly, we subtracted the same value $f(x_0)$, in all cases.

Things are different if $f(x)$ is not continuous: In that case,



Step-function, with a jump discontinuity.

$$f'(x_0^+) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \infty$$

This does not result in a piecewise smooth derivative, and we may not differentiate term-by-term the Fourier series of a discontinuous function.

For example, consider the Sine series of $f(x) = 3$ that we saw before over $[0, \pi]$. Recall that the odd extension is not continuous. We had

$$3 = \sum_{k=1}^{\infty} \frac{12}{\pi(2k-1)} \sin((2k-1)x)$$

Also, the derivative of $f(x) = 3$ is $f'(x) = 0$, which we can obtain as a Cosine series (all coefficients are simply zero). However, if we try to differentiate term-by-term, we get

$$f'(x) = \sum_{k=1}^{\infty} \frac{12(2k-1)}{\pi(2k-1)} \cos((2k-1)x) = \sum_{k=1}^{\infty} \frac{12}{\pi} \cos((2k-1)x)$$

This is not the zero function. In fact, this is not even convergent, as the coefficients do not even become smaller as k increases.

However, we had for $f(x) = x^2(\pi - x)^2$ over $[0, \pi]$ that the Sine series was

$$x^2(\pi - x)^2 = \sum_{k=1}^{\infty} \frac{(48 - 4\pi^2(2k-1)^2)}{(2k-1)^5} \sin((2k-1)x)$$

In this case, the derivative exists and is even continuous:

$$f'(x) = [x^2(\pi - x)^2]' = 2x(\pi - x)^2 - 2x^2(\pi - x) = 2x(\pi - x)[(\pi - x) - x] = 2x(\pi - x)(\pi - 2x)$$

We can differentiate term-by-term in this case, so we will have that

$$2x(\pi - x)(\pi - 2x) = \sum_{k=1}^{\infty} \left[\frac{(48 - 4\pi^2(2k-1)^2)}{(2k-1)^5} \sin((2k-1)x) \right]' = \sum_{k=1}^{\infty} \frac{(48 - 4\pi^2(2k-1)^2)}{(2k-1)^4} \cos((2k-1)x)$$

This last series is convergent, and the coefficients decay like $\sim 1/k^2$.

Aside: representing discontinuous functions

One way to think of functions with a jump discontinuity is as a limit of progressively steeper functions. If we have the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

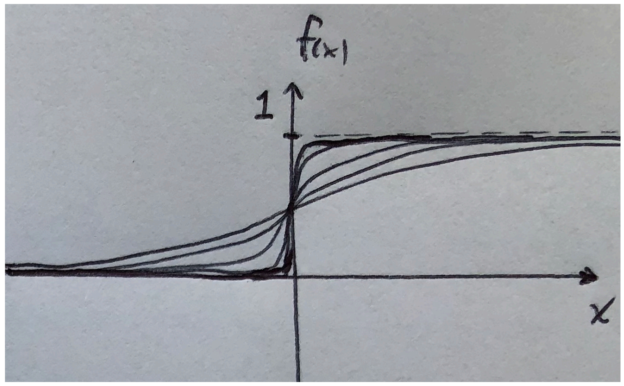
and

$$f_k(x) = \frac{\tanh(kx) + 1}{2}$$

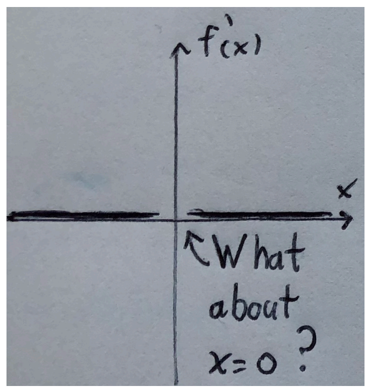
then we have

$$\lim_{k \to \infty} f_k(x) = H(x)$$

What about the derivative of the functions we just looked at? Well $H'(x) = 0$ everywhere, except at $x = 0$ where it is not defined.



Steeper and steeper $\tanh(kx)$ functions converging to a step function.



Derivative of a step function

We also have

$$f'_k(x) = \frac{k\ \text{sech}^2(kx)}{2}$$

As $k \to \infty$, we get what is called a δ function

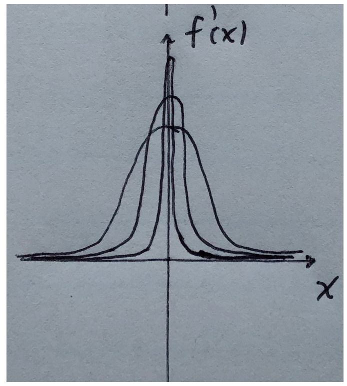
$$\lim_{k\to\infty} f'_k(x) = \delta(x)$$

which is actually not a function at all (it is a distribution). It satisfies two big properties:

$$\delta(x) = 0 \quad \text{if } x \neq 0$$
$$\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1 \quad \text{for any } \epsilon > 0$$

This is a very useful distribution, especially because

$$\int_{-L}^{L} f(x)\delta(x - x_0)\,dx = f(x_0) \quad \text{for any continuous function } f(x).$$



Derivative of a steeper and steeper $\tanh$ functions

However, the delta function does not have a Fourier series... sorry!

Integration of Fourier series

Integration is much simpler. If $f(x)$ is piecewise smooth, then

$$F(x) = \int_0^x f(s)\, ds$$

is also piecewise smooth. In fact, it is continuous. Note that at a discontinuity, the actual value of $f(x)$ has no effect on its integral.

So for any piecewise smooth $f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{implies} F(x) = \int_0^x f(s)\, ds = \sum_{n=1}^{\infty} \frac{-B_n L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + C$$

We use our same example of the Sine series of $f(x) = 3$ over $[0, \pi]$.

$$3 = f(x) = \sum_{k=1}^{\infty} \frac{12}{\pi(2k-1)} \sin((2k-1)x)$$

so the EVEN continuation of its integral $F(x) = 3x$ satisfies

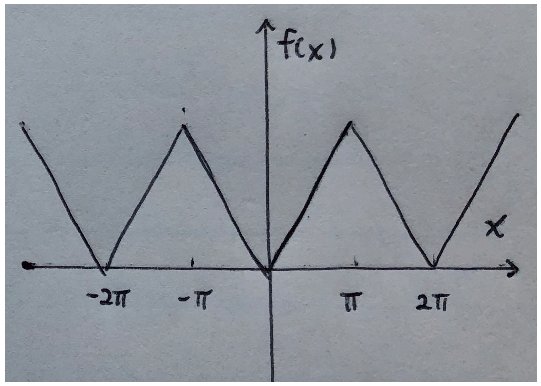
$$3x = F(x) = \sum_{k=1}^{\infty} \frac{-12}{\pi(2k-1)^2} \cos((2k-1)x) + C$$

To find the constant, you can integrate the sum and plug in x, or (usually it is easier to) use the regular formula for the constant, using the average value of $F(x)$:

$$C = A_0 = \frac{1}{\pi} \int_0^{\pi} 3x\, dx = \frac{3}{2}\pi$$

so altogether

$$3x = F(x) = \frac{3\pi}{2} - \sum_{k=1}^{\infty} \frac{12}{\pi(2k-1)^2} \cos((2k-1)x)$$



Even periodic extension of $f(x) = 3x$ over $[0, \pi]$.

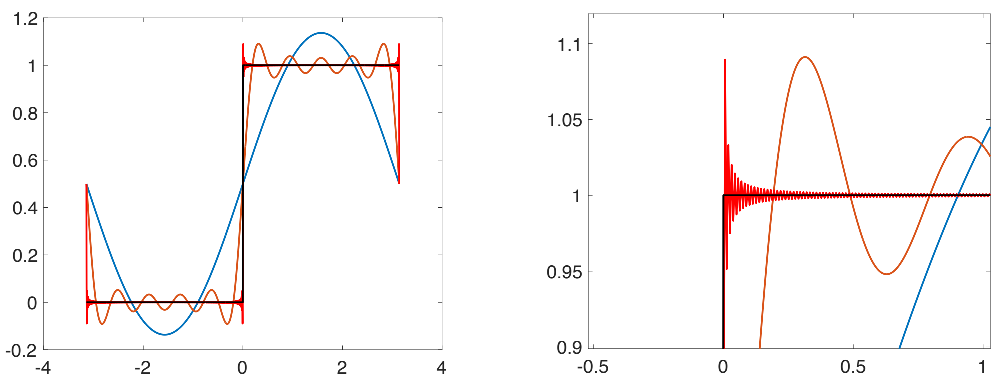
Today's plan

1. What is Gibbs' phenomena?
2. Do the coefficients in a Fourier Series always decay as the index gets bigger?
3. What controls the decay rate (when there is one)?
4. Does it matter?

We saw in our examples that near jump discontinuities, our Fourier Series don't do such a good job. However, the theorem we saw predicted convergence, so how are both possible?

The issue is that what we looked at were FINITE series, not the full infinite series. However, in practice, we ALWAYS look at finite series. So even if the infinite series converges, it is important that the finite series approaches the function with an error that gets smaller as more terms are included. But what error do we mean?

As the number of terms in our series, N, grows we will have near a discontinuity that
1) The region that it not approximating the function well shrinks in width.
2) The height of the overshoot DOES NOT shrink.



Convergence (?) of a Fourier Series near a discontinuity.

This overshoot at discontinuities is about 9% of the height of the jump, and is there for ANY finite series. This is known as Gibbs' phenomena.

It is a problem, at least sometimes, that this overshoot does not go away. So usually, Fourier series are not used to approximate discontinuous functions.

Fortunately for us, the heat equation is very forgiving and will smooth small features very quickly.

In general, we only ever use finite series. So it is important to know the size of what we are neglecting. How big the the "tail" of the series or REMAINDER, the part that we don't compute?

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{N} C_n \sin\left(\frac{n\pi x}{L}\right) + \text{Remainder}(N)$$

The size of the remainder is usually similar to that of C_{N+1}. So if we know how C_n depends on n, we can estimate the remainder.

For any piecewise smooth function $f(x)$, let "p" be the smallest integer for which:
$f^{(p)}(x)$ is discontinuous, and $f^{(p-1)}(x)$ is continuous.

If $f(x)$ itself is discontinuous, we say that $p = 0$. If $f(x) \in C^{\infty}$, we say that $p \to \infty$.

The coefficients of a Fourier Series are then bounded by

$$|C_n| < \frac{\gamma}{n^{p+1}} \quad \text{for some constant } \gamma$$

Moreover, if $p \to \infty$, we have

$$|C_n| < e^{-\alpha n} \quad \text{for some constant } \alpha > 0$$

Note that in some cases, all the coefficients for $n > N$ are 0, which is even better than the bounds given here.But you have to reach that N first. These decay rates are particularly important when using numerical methods to solve PDEs.

For example, we found a few series so far:
Sine series for $f(x) = 3$:

$$f(x) = 3 = \sum_{k=1}^{\infty} \frac{12}{\pi(2k-1)} \sin((2k-1)x)$$

Does this match with the rule give above? Yes, as here the periodic odd extension of $f(x)$ is discontinuous, so $p = 0$.

The Sine Series is of $f(x) = x^2(\pi - x)^2$ is

$$x^2(\pi - x)^2 = \sum_{k=1}^{\infty} \frac{(48 - 4\pi^2(2k-1)^2)}{(2k-1)^5} \sin((2k-1)x)$$

Here we have

$$f'(x) = 2x(\pi - x)(\pi - 2x) = 2x(2x^2 - 3\pi x + \pi^2) = 4x^3 - 6\pi x^2 + 2\pi^2 x$$

and

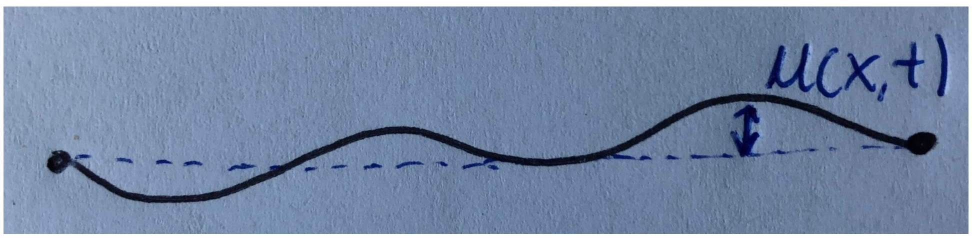
$$f''(x) = 12x^2 - 12\pi x + 2\pi^2$$

The even extension of $f'(x)$ is continuous, but the odd extension of $f''(x)$ is not, since $f''(0) = 2\pi^2 \neq 0$. Therefore $p = 2$.

Today's plan

1. What is the wave equation?
2. What does it describe?
3. What kind of boundary conditions can it have?

The wave equation is used to describe... waves! We will start in one dimension, so we will consider a vibrating string. Its height will be denoted by $u(x,t)$

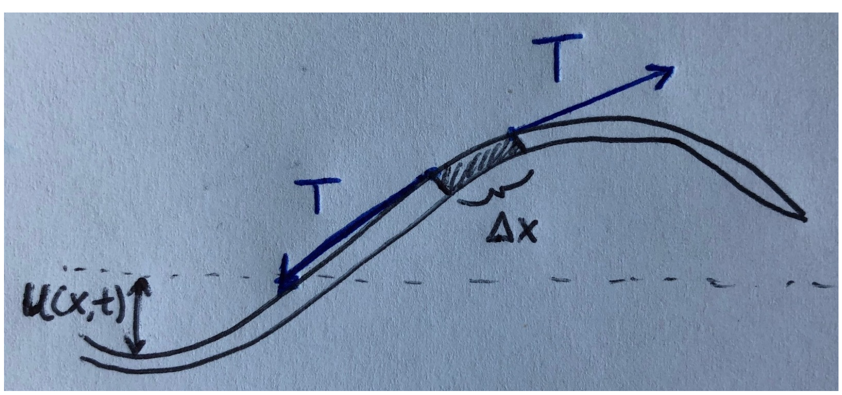


Schematics of a vibrating string

We will assume that when the string it at rest (flat), we have $u = 0$.

The string is subject to Newton's law $F = ma$. Here the force we focus on is the TENSION of the string. This is a force pulling on the string with a constant magnitude. However, its direction may vary. There could also be a load, like a weight, pulling on the string, which we will return to later.

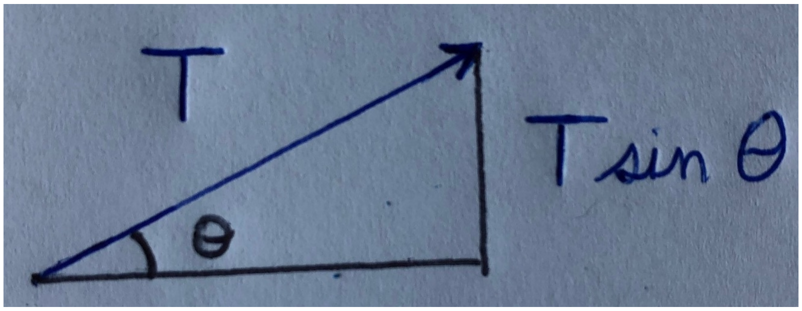
We will focus on a short piece of the string, located at a position x and with length Δx.



More precise schematic of a vibrating string.

Our small portion of string has mass $m = \rho\Delta x$, where ρ is the linear density (mass per unit length).
The vertical acceleration of the string is $a = \frac{\partial^2 u}{\partial t^2}$.

We will now concentrate on the vertical component of the tension. The magnitude of the tension, T, is constant. But the direction in which it is acting is not. What is its vertical component?



Components of the tension

The vertical component is $F = T\sin\theta$. To relate this to u, we use the slope

$$\tan\theta = \frac{\partial u}{\partial x} = u_x = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

So if we solve for $\sin\theta$ we find

$$\sin\theta = \frac{u_x}{\sqrt{1+u_x^2}} \approx u_x \quad \text{for small angles}$$

So the force acting on our piece of string is:

$$F = T\sin(\theta(x+\Delta x)) - T\sin\theta(x) \approx Tu_x(x+\Delta x) - Tu_x(x)$$

Now we could consider an additional load, which we would often write as

$$\text{Load} = \Delta x\ \rho(x)Q(x,t)$$

where, for example, Q could be the gravitational acceleration.

So Newton's law, $F = ma$ becomes in our case:

$$T[\sin\theta(x+\Delta x) - \sin\theta(x)] + \rho\Delta x Q(x,t) = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

Now we make use of our small slope approximation and get

$$T[u_x(x+\Delta x) - u_x(x)] + \rho\Delta x Q(x,t) = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

If we divide by $\rho\Delta x$, we find

$$\frac{T}{\rho}\frac{u_x(x+\Delta x)-u_x(x)}{\Delta x}+Q(x,t)=\frac{\partial^2 u}{\partial t^2}$$

Finally, we take the limit of $\Delta x \to 0$ to get our WAVE EQUATION

$$\frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}+Q(x,t)=\frac{\partial^2 u}{\partial t^2}$$

Usually, we denote $T/\rho = c^2$, where c has the units of speed.

$$c^2\frac{\partial^2 u}{\partial x^2}+Q(x,t)=\frac{\partial^2 u}{\partial t^2}$$

Overall, the units involved in this derivation are (with M=mass, L=length, T=time):
$[\rho] =$ M/L
$[T] =$ F = ML/T^2
$[T/\rho] = c^2$ = L^2/T^2, so
$[c] =$ L/T, a speed
$[u_{xx}]$ =L^2/L=1/L
$[u_{tt}] =$ L/T^2
$[Q] =$ L/T^2
All the units must match in our equation. Do they?

Boundary Conditions

There are two commonly occurring types of BC.

Clamped boundaries are ones where the level of the string is fixed, usually at zero:

$$u(x = 0, t) = A \quad \text{and} \quad u(x = L, t) = B$$

Free boundaries are ones where the vertical force applied at either end is prescribed, usually to be zero

$$Tu_x(x = 0, t) = F_0 \quad \text{and} \quad Tu_x(x = L, t) = F_L$$

When the force is zero, it corresponds to having a string that can freely move up or down (while tied to a metal ring on a pole, for example). This forces the string to be flat at the ends.

Initial Conditions

To completely prescribe our system, we also need to provide some initial conditions. Since we now have a second order equation in time, we will need to provide two initial conditions. Usually, this takes the form:

$$\begin{aligned} u(x, t = 0) &= f(x) \quad \text{the initial position} \\ u_t(x, t = 0) &= g(x) \quad \text{the initial velocity} \end{aligned}$$

Today's plan

1. Can we use our same method of separation of variables for the Wave Equation?
2. What stays the same?
3. What changes?

Let's try our favorite method again: separation of variables. To begin with, we set the load, $Q(x,t)$ to 0. We will also ouse fixed, homogeneous boundary conditions

$$u_{tt} = c^2 u_{xx} \quad \text{subject to } u(x=0,t) = u(x=L,t) = 0$$

Note that if we have BCs other than zero, we could subtract an equilibrium solution (which would be a linear function of x and the remaining problem would be of the form considered here.

So we assume a separable solution, $u_n(x,t) = G_n(t)\phi_n(x)$ and substitute that into our system.

We obtain

$$G_n''\phi_n = c^2 G_n \phi_n''$$

and dividing by $c^2 u_n$, we can separate variables:

$$\frac{G_n''\phi_n}{c^2 G_n \phi_n} = \frac{c^2}{c^2}\frac{G_n\phi_n''}{G_n\phi_n} = \lambda.$$

In space (in x), this is the same equation and BC that we had for the wave equation. So the solution is once again that λ must be real and negative. We had

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

In time, we therefore have to solve

$$G_n'' + c^2\left(\frac{n\pi}{L}\right)^2 G_n = 0$$

This also has a trigonometric solution, and the general solution is

$$G_n(t) = A_n \cos\left(\frac{n\pi}{L}c\,t\right) + B_n \sin\left(\frac{n\pi}{L}c\,t\right)$$

Note that here $n \geq 1$, as for $n = 0$, you have $\phi_0 = 0$.

Again, because the system is linear, the sum of solutions is a solution. We thus get as our most general solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} c\, t\right) \sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi}{L} c\, t\right) \sin\left(\frac{n\pi x}{L}\right)$$

We still don't know the coefficients A_n and B_n. These will be determined by the initial conditions.

Suppose we are given as initial conditions that $u(x, t = 0) = f(x)$ and $u_t(x, t = 0) = g(x)$. How do we use them?

For the initial position, $f(x)$, we can simply plug in $t = 0$ into our solution. This leaves:

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Conveniently, the only the coefficients A_n appear. We therefore have a sine series for A_n, so that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

For the initial velocity, we first need to differentiate with respect to time,. If $g(x)$ is piecewise smooth, we can do this term-by-term.

$$g(x) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This too is a sine series, but with coefficients $\frac{n\pi c}{L} B_n$. The coefficients are therefore

$$B_n = \frac{2}{L} \int_0^L g(x) \frac{L}{n\pi c} \sin\left(\frac{n\pi x}{L}\right)\, dx$$

and with that, we finally have the whole solution!

But what does it actually look like?

Let's try some concrete examples. Consider that $L = \pi$. Also, use as an initial position $f(x) = 4 \sin 3x$, and an initial velocity $g(x) = 7 \sin 2x$
(these choices will make it easy to find the sine series, but capture the essence of what is going on).

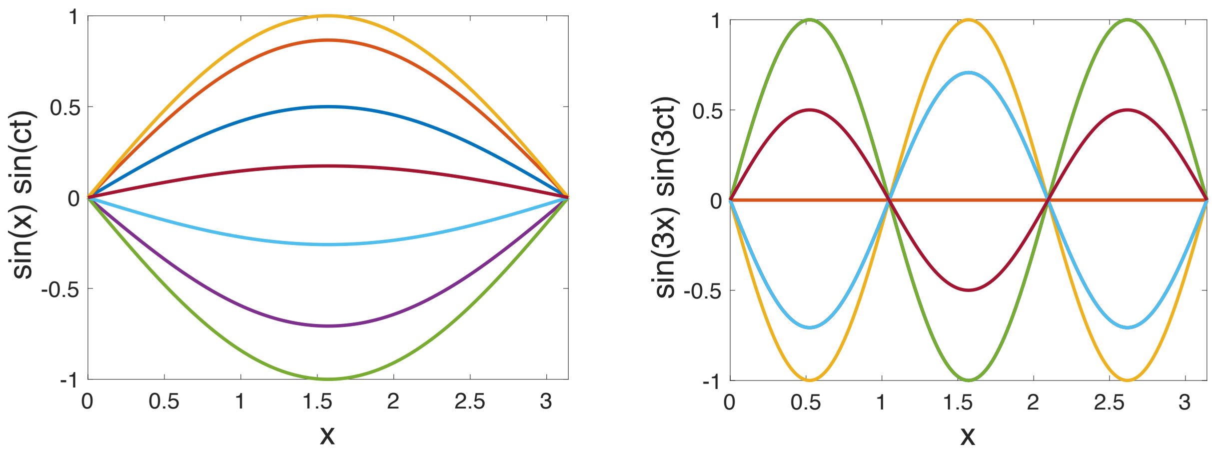
We find from the initial condition that $A_3 = 4$ and all other $A_n = 0$.

We find from the initial velocity that $B_2 = 7/2c$ and all other $B_n = 0$.

So altogether, we have

$$u(x,t) = 4\sin 3x\ \cos 3ct + \frac{7}{2c}\sin 2x\ \sin 2ct$$

The eigenmodes found here have a special interpretation. They are harmonics, or STANDING WAVES. The solutions therefore look like the eigenmodes (sine or cosine) changing in amplitude over time.



Eigenmodes seen as standing waves changing amplitude in time.

The "nodes" are points where $u = 0$. For harmonics, they do not move in time. However, when the solution is a sum of harmonics, the nodes will move in time.

The frequency of oscillation is $\frac{n\pi c}{L}$, which increases with n. In reality, there is usually some damping in the system as well, so that oscillations decay over time. This takes the form:

$$u_{tt} = c^2 u_{xx} - \eta u_t$$

As a result the time dependence includes an exponential damping

$$\sin(nx)\sin(nct)e^{-\eta n^2 t}$$

so that high frequency modes (large n) become small very quickly. The harmonics surviving the longest are the ones with small n (which is why bass travels further than high-pitched notes).

Today's plan

1. What does having an infinite domain change?
2. Can we still do separation of variables if $L \to \infty$? Not directly...
3. What can we do instead?

In an infinite domain, our usual approach fails. If we apply separation of variables to

$$u_{tt} = c^2 u_{xx}$$

using $u_n = \phi_n(x)G_n(t)$ as $L \to \infty$, we can't find any non-trivial solution to

$$\phi_n'' + \lambda\phi_n = 0$$

that goes to zero at the "ends" of the domain.

This doesn't mean that there are no solutions, simply that the method of separation of variables can't find them. We need something else.

Consider a function of one variable: $R(\xi)$. We want to use this, but set the variable to a special combination of x and t: $\xi = x + ct$.
So try the function

$$u(x,t) = R(x + ct)$$

into to the wave equation. When is it a solution to the PDE?

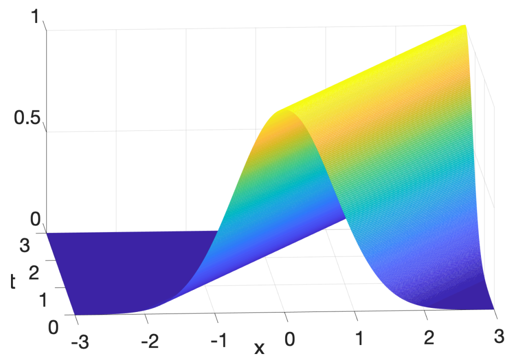
So long as $R(\xi)$ is twice differentiable, $R(x + ct)$ is a solution to the wave equation. Can you think of another solution using this first one as a guide?

A second solution can be found in the form $S(x - ct)$! we now have two family of solutions, which are valid everywhere. What do they look like? What does it do to have arguments $x + ct$ or $x - ct$?

Let's try a function $f(x) = 1 - x^2$ and see what it does in time when its argument is $x - ct$. As time passes, if you want to keep the same value of the function, like if you want to follow the maximum, the position x must change.
To keep the function value unchanged from $f(x_0)$ at time 0, you must use

$$x - ct = x_0 \quad \text{or equivalently} \quad x = x_0 + ct.$$

So you must move to the right, with a speed c. The curve is thus a traveling wave (it keeps its shape), moving to the right, with speed c.
Similarly, if the argument is $x + ct$, the curve is a traveling wave toward the left. Does this remind you of anything we did before?



Wave traveling to the right solution

How can we satisfy some given initial conditions with our solution of the form

$$u(x,t) = R(x+ct) + S(x-ct)$$

Suppose we have

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \quad \text{with} \quad G(x) = \int g(x)\,dx \ \text{ an antiderivative} \end{aligned}$$

If we plug in our form of the solution, we get

$$\begin{aligned} u(x,0) &= f(x) = R(x) + S(x) \\ u_t(x,0) &= g(x) = cR'(x) - cS'(x) \ \text{ so } \ \frac{G(x)}{c} = R(x) - S(x) \end{aligned}$$

We can now solve for R and S. If you add the two equations above and divide by 2, you find

$$R(x) = \frac{f(x)}{2} + \frac{G(x)}{2c}$$

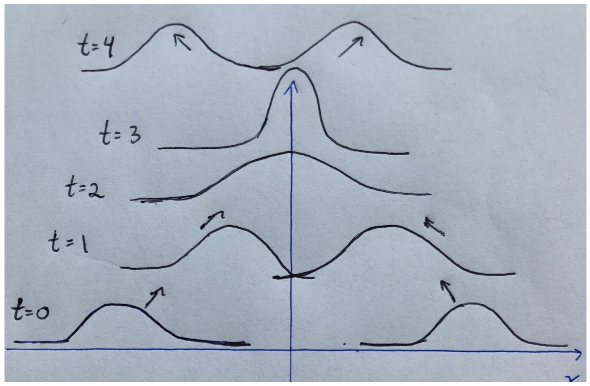
and if you subtract the two equations above and divide by 2, you find

$$S(x) = \frac{f(x)}{2} - \frac{G(x)}{2c}$$

Putting it all together, we get what is known as D'Alembert's solution

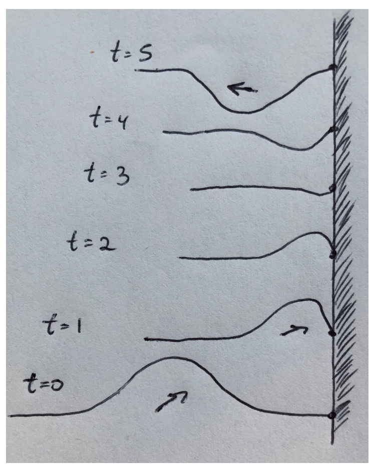
$$u(x,t) = \frac{f(x+ct)}{2} + \frac{G(x+ct)}{2c} + \frac{f(x-ct)}{2} - \frac{G(x-ct)}{2c}$$

So our general solution is the sum of two traveling waves, one going left, and one going right. Both waves travel with speed c.



Sum of two travelling waves

This idea also works in the presence of boundaries, but you then need to incorporate the reflection of waves. In general, this appears as though a similar but different wave was sent in the domain from outside the domain. This is known as the method of images.

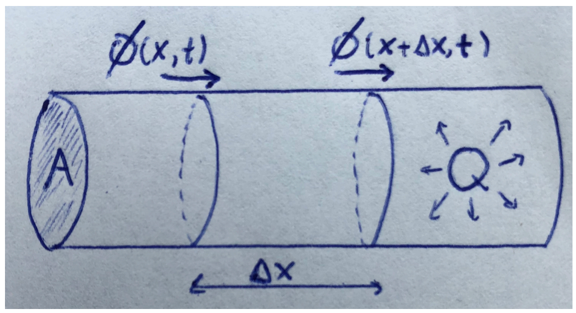


A wave reflected by a boundary where the function takes a fixed value.

Today's plan

1. What changes about the heat equation in higher dimension?
2. What stays the same about the heat equation in higher dimension?

Recall our 1D derivation of the heat equation We had that the heat rate of change in time



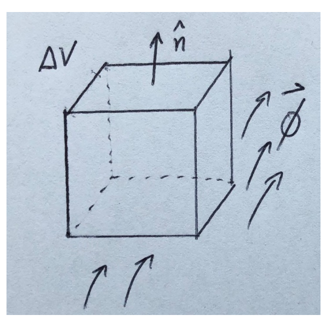
Heat flux in a rod.

is minus the derivative of the heat flux Φ:

$$\rho C_p \frac{\partial u}{\partial t} = -\frac{\partial \Phi}{\partial x}$$

where we had $\Phi = -K\frac{\partial u}{\partial x}$.

We want to use the same principle, but instead of using a segment, we will use a small solid, like a cube.



Heat flux into a cube.

Our cube has volume ΔV and surface area ΔS made of 6 faces. We denote the outer normal by $\hat{n}$.

The heat flux is now a vector field $\vec{\Phi}(x, y, z)$ giving the direction and rate at which the heat is transferred.

So over a time Δt, the heat flux OUT of our volume is

$$F = \int\int_{\Delta S} \vec{\Phi} \cdot \hat{n} \, dS \, \Delta t = -(\text{ Heat}(t+\Delta t) - \text{ Heat}(t) \,)$$

where the integral is taken over the 6 faces and all those contributions are added up.

Recalling the divergence theorem, we can rewrite the integral as

$$F = \int\int\int_{\Delta V} \text{div } \vec{\Phi} \, dV \;\; \Delta t.$$

In the limit of a small volume, the integrand above will be basically constant over the volume (Riemman sum with only one term) and we get the approximation

$$F \approx \text{div} \vec{\Phi}|_{\text{center}} \;\; \Delta V \;\; \Delta t$$

We also have that the heat is

$$\text{Heat}(t) = \rho C_p \Delta V \; u(t).$$

So putting it all together, we have

$$-\rho C_p \,(\, u(t+\Delta t) - \; u(t) \,) = \Delta t \; \Delta V \text{div } \vec{\Phi}$$

and finally if we lake the limit of small time step and volume, we can find

$$\rho C_p \, \frac{\partial u}{\partial t} = -\text{div } \vec{\Phi}$$

This is basically the same result we obtained in 1D, but the flux is now a vector, and the spatial derivative has been replaced by a divergence.

Once again, we need to determine the flux, this time in vector form. We still want

1. Heat to flow from hot to cold.
2. Heat to flow faster for larger temperature differences.
3. No heat flow when the temperature is uniform.

The simplest vector flux satisfying all three conditions is based on the gradient of the temperature.

$$\vec{\Phi} = -K \nabla u(x, y, z, t).$$

This give us our heat equation in any dimension:

$$\rho C_p \, \frac{\partial u}{\partial t} = \text{div } (K \nabla u) = \nabla \cdot (K \nabla u)$$

In the most common case where K is constant, this simplifies to

$$\rho C_p \frac{\partial u}{\partial t} = K \text{div}\,(\nabla u) = K\nabla^2 u$$

A few remarks:

1. ∇u is a coordinate independent physical vector (not always one derivative of each component).
2. div = $\nabla\cdot$ is a coordinate independent operator (not always the sum of a derivative of each variable).
3. As a result, ∇^2 is also coordinate independent.
4. We can still add a source/sink, in a manner similar to what we did in 1D.
5. Boundary conditions will be of a similar form as before: either u is given or the flux, $-K\frac{\partial u}{\partial \hat{n}}$, is given.

Setting the heat equation in higher dimensions

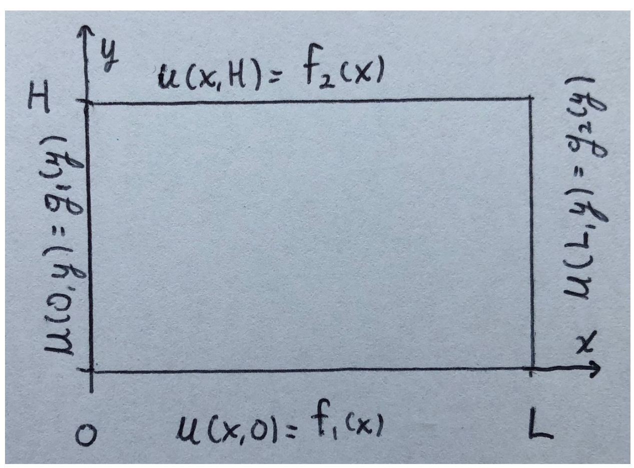
As you can imagine, it is more work to solve the diffusion equation in higher dimensions than in lower dimension. We will therefore simplify matters a little. Assume first that ρ, C_p and K are constant.

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + Q = \kappa(u_{xx} + u_{yy}) + Q$$

where $\kappa = K/\rho C_p$.

For our domain, we will use a rectangle: $0 \leq x \leq L$ and $0 \leq y \leq H$. We will specify the value of the temperature at each boundary:

$$\begin{aligned} u(x,0) &= f_1(x) \\ u(x,H) &= f_2(x) \\ u(0,y) &= g_1(y) \\ u(L,y) &= g_2(y) \end{aligned}$$



Our domain and boundary conditions

Today's plan

1. What is Laplace's equation?
2. Can we solve it by separation of variables?
3. How it is different from the heat or wave equations?
4. How many named equations are there??

We derived last time a higher dimensional version of the heat equation:

$$u_t = \kappa \nabla^2 u + Q$$

The first step to finding a solution to the heat equation is to look for an equilibrium solution, to which our final solution will tend as $t \to \infty$. For the moment, we will not consider a source term, and therefore, we first need to solve what is known as **Laplace's equation**:

$$\begin{aligned}
0 &= \kappa(u_{xx} + u_{yy}) \quad \text{subject to} \\
u(x,0) &= f_1(x) \\
u(x,H) &= f_2(x) \\
u(0,y) &= g_1(y) \\
u(L,y) &= g_2(y)
\end{aligned}$$

General approach: one non-homogeneous term at a time

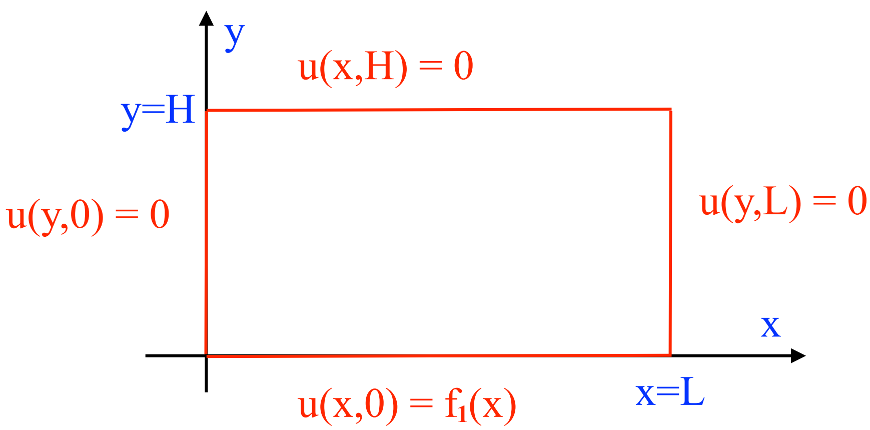
To solve this problem, we will actually consider 5 simpler problems. There are 4 terms that may be non-zero in this system without changing the nature of the equation: each of the boundary conditions. We will solve in turn a system where only one of them is non-zero at a time, and where all the others have been set to zero. More precisely:

1. u_1 will solve our system with no source, and all BC zero except for $u(x,0) = f_1(x)$.
2. u_2 will solve our system with no source, and all BC zero except for $u(x,H) = f_2(x)$.
3. u_3 will solve our system with no source, and all BC zero except for $u(0,y) = g_1(y)$.
4. u_4 will solve our system with no source, and all BC zero except for $u(L,y) = g_2(y)$.

Once we have all these sub-solutions, we will obtain our final solution by taking $u = u_1 + u_2 + u_3 + u_4$. This is possible because the original equation is linear, so the sum of solutions is also a solution. Moreover, the sum u will satisfy the sum of the BC, which here adds up to the original problem.

Let us consider u_1 first. Note that because they are so similar, we won't solve for u_2, u_3 and u_4 explicitly. Our system is (using u instead of u_1 to simplify the notation)

$$\begin{aligned} 0 &= u_{xx} + u_{yy} \quad \text{subject to} \\ u(x,0) &= f_1(x) \\ u(x,H) &= 0 \\ u(0,y) &= 0 \\ u(L,y) &= 0 \end{aligned}$$



Domain and BC for u_1.

Laplace's equation is an elliptic equation, as the corresponding algebraic equation is $x^2 + y^2 = C$. In general, this means that the entire solution depends on all the boundary conditions. You can think of such systems as ones where the information travels infinitely quickly in space, so the whole domain knows about what is going on instantaneously.

We will try separation of variables again: $u(x,y) = X(x)Y(y)$. Our equation is then:

$$\begin{aligned} 0 &= X''Y + XY'' \quad \text{or} \\ 0 &= \frac{X''}{X} + \frac{Y''}{Y} \quad \text{or} \\ -\frac{X''}{X} &= \frac{Y''}{Y} = \lambda \quad \text{a constant} \end{aligned}$$

Note that we placed the minus sign with X, the variable for which we have 2 homogeneous BC, for convenience. We solve that system first:

$$X'' + \lambda X = 0 \quad \text{subject to} \quad X(0) = X(L) = 0.$$

We find a well-known solution:

$$X(x) = C \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

We can now move to the equation in y, which we haven't solved recently:

$$Y'' - \left(\frac{n\pi}{L}\right)^2 Y = 0$$

The corresponding algebraic equation is $r^2 - \left(\frac{n\pi}{L}\right)^2 = 0$ so we find $r = \pm\frac{n\pi}{L}$. Our general solution is thus

$$Y(y) = C_1 e^{n\pi y/L} + C_2 e^{-n\pi y/L}$$

We use the homogeneous boundary condition first: $Y(H) = 0$ to find :

$$0 = C_1 e^{n\pi H/L} + C_2 e^{-n\pi H/L} \quad \text{so} \quad C_1 = -C_2 e^{-2n\pi H/L}$$

So we get a one-parameter family of solution. We will denote the remaining constant as C_n:

$$Y_n(y) = C_n \left(-e^{n\pi(y-2H)/L} + e^{-n\pi y/L}\right)$$

Finally, we are ready to put all those solutions together,

$$u_1(x,y) = \sum_{n=1}^{\infty} C_n \left(-e^{n\pi(y-2H)/L} + e^{-n\pi y/L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

and try to satisfy the remaining boundary condition

$$u_1(x,0) = f_1(x) = \sum_{n=1}^{\infty} C_n \left(-e^{-2n\pi H/L} + 1\right) \sin\left(\frac{n\pi x}{L}\right)$$

But we know this problem! This is a Sine series for $f_1(x)$, with an extra factor. So we have

$$C_n \left(-e^{-2n\pi H/L} + 1\right) = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

So finally, we find for u_1 that

$$\begin{aligned} u_1(x,y) &= \sum_{n=1}^{\infty} C_n \left(-e^{n\pi(y-2H)/L} + e^{-n\pi y/L}\right) \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \\ C_n &= \frac{2}{L(1 - e^{-2n\pi H/L})} \int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \end{aligned}$$

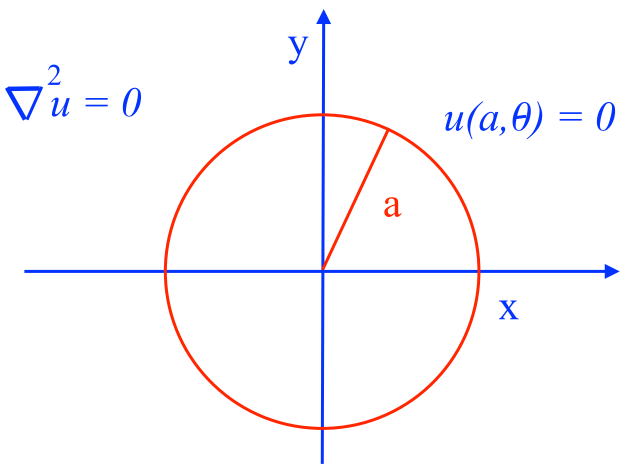
In general, we will have one variable with oscillations, here x which has the two homogeneous BCs, and one variable with exponential growth or decay.

As an exercise, you should try to find u_4, and verify that you get:

$$\begin{aligned} u_4(x,y) &= \sum_{n=1}^{\infty} F_n \left(e^{n\pi x/H} - e^{-n\pi x/H}\right) \sin\left(\frac{n\pi y}{H}\right) \text{ with} \\ F_n &= \frac{2}{H(e^{n\pi L/H} - e^{-n\pi L/H})} \int_0^H g_2(y) \sin\left(\frac{n\pi y}{H}\right) dy \end{aligned}$$

Today's plan

1. What can we do if the domain is a disk?
2. Can we solve it by separation of variables?



Laplace's equation over a disk

We will now consider a different domain: a disk of radius a. We will therefore have only one boundary, the circle of radius $r = a$.

As you can imagine, polar coordinates are well suited for this problem. So we need to convert our equation to polar coordinates. Recall that we have:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

We need to know the following derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \qquad \frac{\partial \theta}{\partial x} = \frac{-y\cos^2\theta}{x^2} = \frac{-\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r}$$

Importantly, we also need to know the unit vectors conversion:

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta \qquad \hat{\theta} = \hat{x}(-\sin\theta) + \hat{y}\cos\theta \ \text{ or } \begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

This relation can also be inverted (this easier to do in matrix form) to find

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} \ \text{ or } \ \hat{x} = \hat{r}\cos\theta + \hat{\theta}(-\sin\theta) \qquad \hat{y} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

Note that here the unit vectors change with position. They are independent of r, as changing the radius keep the orientations of $\hat{r}$ and $\hat{\theta}$ the same. But they change with θ. We have

$$\frac{\partial\hat{\theta}}{\partial\theta} = \hat{x}(-\cos\theta) + \hat{y}(-\sin\theta) = -\hat{r}$$

and

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{x}(-\sin\theta) + \hat{y}\cos\theta = \hat{\theta}$$

We are now ready to convert the gradient, divergence, and Laplace operators. In Cartesian coordinates, we had:

$$\text{grad} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

This converts in polar coordinates to, using the chain rule,

$$\begin{aligned}
\text{grad} &= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right)\hat{x} + \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}\right)\hat{y} \\
\text{grad} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\hat{x} + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\hat{y} \\
\text{grad} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(\hat{r}\cos\theta + \hat{\theta}(-\sin\theta)) + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(\hat{r}\sin\theta + \hat{\theta}\cos\theta) \\
\text{grad} &= \hat{r}(\cos^2\theta + \sin^2\theta)\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial}{\partial\theta} \\
\text{grad} &= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}
\end{aligned}$$

Thankfully, there were trig identities! Note that this kind of conversion can be done in general for other coordinate systems, but usually things don't simplify.

We can now convert the Lapacian, which we need in Laplace's equation:

$$\begin{aligned}
\nabla^2 &= \nabla\cdot\nabla = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\left(\hat{r}\frac{\partial}{\partial r}\right) + \hat{\theta}\cdot\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\hat{\theta}\cdot\hat{\theta}\frac{\partial}{\partial r} + \hat{\theta}\cdot(-\hat{r})\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial\theta^2} \\
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}
\end{aligned}$$

So we finally have our Laplacian in polar coordinates.

$$0 = \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

A simple and useful way to check if we did not make a mistake along the way is to make sure the units work out. Recall that angles are no units, and differentiating is like dividing, in terms of units. So we usually have that the Laplacian corresponds to dividing the a length squared (in fact, I found I made a mistake in those very notes that way).

We will now have our boundary condition at $r = a$ in the form $u(r = a, \theta) = f(\theta)$. This corresponds to fixing the temperature at the edge of the domain for the heat equation, or fixing the height of a membrane for the wave equation. Note that we could have had a BC on $\frac{\partial u}{\partial r}$ instead.

We will also assume that u remains finite over the domain. If we want to maintain that u is differentiable, we also need to have $f(\theta)$ be smooth, and PERIODIC. This last requirement is new for us.

Separation of variables in polar coordinates

We will try our trusted separation of variables method (there are other ways too, such as Poisson's formula and boundary integral methods). We assume $u(r, \theta) = \phi(\theta)R(r)$. We will have that $\phi(\theta)$ is periodic with period 2π.

Plugging in, we get

$$\begin{aligned} 0 = \nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= R''\phi + \frac{1}{r}R'\phi + \frac{1}{r^2}R\phi'' \quad \text{so dividing by } u/r^2 \text{ we get} \\ 0 &= r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\phi''}{\phi} \quad \text{and therefore} \end{aligned}$$

$$\lambda = -\frac{\phi''}{\phi} = r^2\frac{R''}{R} + r\frac{R'}{R}$$

We first consider the equation in θ

$$\phi'' + \lambda\phi = 0$$

which has a periodic general solution when λ is positive:

$$\phi(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta).$$

For the period to be 2π, we need $\lambda = n^2$. We therefore find, for $n \geq 1$

$$\phi(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

Note that $\phi(\theta) = A_0$ is also a solution.

In r, we have a more complicated (and less familiar) equation:

$$r^2 R'' + rR' - n^2 R = 0.$$

This is a special ODE, where every term has a power in r which is the same as the number of derivatives. These are called equi-dimensional, or Euler or Cauchy equation (but don't worry about the name). The solutions will be of the form: $R(r) = r^p$, for some power p. We have

$$R(r) = r^p \qquad R'(r) = pr^{p-1} \qquad R''(r) = p(p-1)r^{p-2}$$

Plugging those into our ODE, we get

$$0 = r^2 p(p-1)r^{p-2} + rpr^{p-1} - n^2 r^p = r^p\left(p^2 - p + p - n^2\right) = r^p\left(p^2 - n^2\right)$$

So we find that $p = \pm n$. Our general solution is therefore, if $n > 0$

$$R(r) = D_1 r^n + D_2 r^{-n}$$

and in the case $n = 0$, we have $R(r) = D_0 + D_2 \log r$.

We need our solution to remain finite at the origin, where $r = 0$. So all the coefficients D_2 must be 0 to avoid having a function that goes to infinity. We have

$$R(r) = D_n r^n \quad \text{for } n = 0, 1, 2, ...$$

Putting our solution together, we get

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

The coefficients will come from the boundary condition. At $r = a$, we have:

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

This is exactly a Fourier Series! So we can find the coefficients as before with:

$$a^n A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta$$

$$a^n B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, d\theta$$

and

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta$$

So we have a complete solution to Laplace's equation on a disk.

Can we plot some examples?

Today's plan

1. How can we solve the heat and wave equations in higher dimensions?
2. Does separation of variables still work?
3. What changes?
4. Which one is the Helmholtz equation?

We are now ready to tackle the heat and wave equations in higher dimension. We will do both in parallel, as many things are the same for both. So we want to solve:

$$\begin{aligned}&\frac{\partial u}{\partial t} = K\nabla^2 u \text{ with}\\ &u(x,y,t=0) = u_0(x,y)\\ &+B.C.\end{aligned} \qquad\qquad \begin{aligned}&\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \text{ with}\\ &u(x,y,t=0) = u_0(x,y)\\ &u_t(x,y,t=0) = v_0(x,y)\\ &+B.C.\end{aligned}$$

For both equations, we will start by separating time and space variables, by assuming $u(x,y,t) = H(t)\phi(\vec{x})$.
Here $\vec{x}$ is the position vector, in whichever coordinates we like (such as (x,y) or (x,y,z) or (r,θ)).

We get:

$$H'\phi = KH\nabla^2\phi \qquad \text{or} \qquad H''\phi = c^2 H\nabla^2\phi$$

and dividing by u we find

$$\frac{H'\phi}{H\phi} = K\frac{H\nabla^2\phi}{H\phi} \qquad \text{or} \qquad \frac{H''\phi}{H\phi} = c^2\frac{H\nabla^2\phi}{H\phi}$$

so we can separate the time variable from the space variables

$$\frac{H'}{KH} = \frac{\nabla^2\phi}{\phi} = -\lambda \qquad \text{or} \qquad \frac{H''}{c^2 H} = \frac{\nabla^2\phi}{\phi} = -\lambda$$

In time, we get the same equation, and solution, as before:

$$H(t) = Ce^{-\lambda K t} \qquad \text{or} \qquad H(t) = A\cos(\sqrt{\lambda}tc) + B\sin(\sqrt{\lambda}tc)$$

and all the λ are determined by the spatial problem.
Note that so far, this is exactly the same as when there was only one spatial variable (and that is good!).

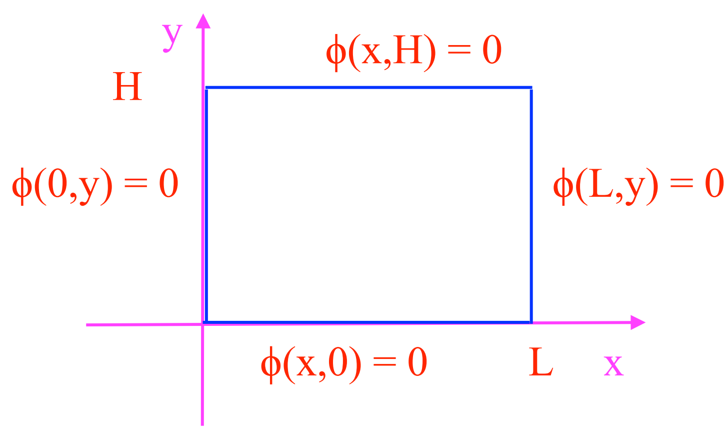
In space, we have a generalization of the problem we encountered before. This is the same equation for the heat or wave equation:

$$\nabla^2\phi + \lambda\phi = 0.$$

This is known as the Helmhotz equation. It must now be solved! How to do that depends on the shape of the domain.

Helmholtz equation over a rectangle

We solve first $\nabla^2\phi + \lambda\phi = 0$ over a rectangle of size $L \times H$. We consider the B.C. that $\phi = 0$ over the whole boundary. Once again, we separate variables: $\phi(x,y) = X(x)Y(y)$.



Helmholtz equation, rectangular domain, Dirichlet B.C.

plugging in, we get:

$$X''Y + XY'' + \lambda XY = 0$$

and dividing by ϕ,

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

where μ is another constant. Because we have 2 spatial dimensions, we will have two families of possible constants (or eigenvalues).

In X, we have the now familiar equation:

$$X'' + \mu X = 0 \quad \text{with} \quad X(0) = X(L) = 0$$

This has solution

$$X(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \mu_n = \left(\frac{n\pi}{L}\right)^2.$$

In y, we have

$$Y'' + (\lambda - \mu)Y = 0 \quad \text{with} \quad Y(0) = Y(H) = 0$$

This has solutions only if $\lambda > \mu$. we find that $\lambda - \mu = \left(\frac{m\pi}{H}\right)^2$ and

$$Y(y) = D_m \sin\left(\frac{m\pi y}{H}\right) \quad \text{with} \quad \lambda_{m,n} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

We can now assemble our solutions. We have, for **the heat equation**

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{m,n} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]Kt} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

with "only" the coefficients $C_{m,n}$ left to find. These will come from the initial condition, and from the orthogonality of the eigenfunctions we found. We have

$$\int\int_D \phi_i \phi_j \, dA = 0$$

for any eigenfunctions corresponding to different eigenvalues, so long as the B.C. are homogeneous.

So at $t = 0$, we have

$$u_0(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{m,n} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

Using orthogonality, we find:

$$\begin{aligned}
&\int_0^L \int_0^H u_0(x, y) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \, dydx \\
&\quad = \int_0^L \int_0^H \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{m,n} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \, dydx \\
&\quad = C_{j,k} \int_0^L \int_0^H \sin^2\left(\frac{j\pi x}{L}\right) \sin^2\left(\frac{k\pi y}{H}\right) \, dydx \\
&\quad = C_{j,k} \frac{L}{2} \frac{H}{2}
\end{aligned}$$

So we can find the coefficients as

$$C_{m,n} = \left(\frac{2}{L}\right)\left(\frac{2}{H}\right) \int_0^L \int_0^H u_0(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) \, dydx$$

Note that the first part of this formula is the inverse of the integral of the eigenfunction squared. The integral itself is always taken over the spatial domain under consideration.

We have now finished solving the Heat Equation in 2 spatial dimensions!

For the **wave equation**, we have one more set of constants to find. Our solution is:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{m,n}\cos(\sqrt{\lambda_{m,n}}tc) + B_{m,n}\sin(\sqrt{\lambda_{m,n}}tc))\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

with $\lambda_{m,n} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$.

For the coefficients $A_{m,n}$, the formula is the same as that of the coefficients of the heat equation (at $t = 0$, $B_{m,n}$ contributes nothing). For the coefficients $B_{m,n}$, we use the second initial condition:

$$u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{m,n}\sqrt{\lambda_{m,n}}c\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) = v_0(x,y).$$

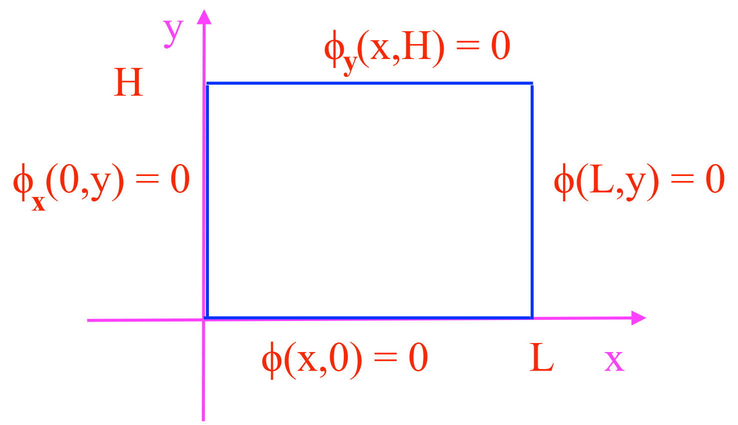
The same approach can be used with v_0. So we can find the coefficients as

$$B_{m,n}\sqrt{\lambda_{m,n}}c = \left(\frac{2}{L}\right)\left(\frac{2}{H}\right)\int_0^L\int_0^H v_0(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)\ dydx$$

or if you prefer

$$B_{m,n} = \left(\frac{2}{L}\right)\left(\frac{2}{H}\right)\frac{1}{\sqrt{\lambda_{m,n}}c}\int_0^L\int_0^H v_0(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)\ dydx$$

Different Boundary Conditions



Our rectangular domain, with different boundary conditions

When giving Flux boundary conditions, the conditions are always given on the NORMAL derivative: $\hat{n}\nabla\phi = \frac{\partial\phi}{\partial\hat{n}}$. To see a more complicated case, we will use different types of boundary conditions (flux or value) at different boundary conditions. Consider

$$\begin{aligned}\frac{\partial\phi}{\partial x}(x=0,y) &= 0\\ \phi(x=L,y) &= 0\\ \phi(x,y=0) &= 0\\ \frac{\partial\phi}{\partial y}(x,y=H) &= 0\end{aligned}$$

Our separated equation is the same

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu \quad \text{or} \quad \frac{Y''}{Y} = \mu - \lambda = -\gamma$$

but the BC are different. In x, we have

$$X'' + \mu X = 0 \quad \text{subject to} \quad X'(0) = 0 \;\text{ and }\; X(L) = 0$$

Our general solution is still

$$X(x) = A\cos(\sqrt{\mu}x) + B\sin(\sqrt{\mu}x) \;\text{ with }\; X'(x) = -A\sqrt{\mu}\sin(\sqrt{\mu}x) + B\sqrt{\mu}\cos(\sqrt{\mu}x)$$

So at $x = 0$, we have $0 = B\sqrt{\mu}$ so $B = 0$ or $\mu = 0$.
At $x = L$, we have $0 = A\cos(\sqrt{\mu}L)$.
We therefore need

$$\sqrt{\mu}L = n\pi + \frac{\pi}{2} \;\text{ and }\; \mu = \left(\frac{\pi(n+\frac{1}{2})}{L}\right)^2$$

Note that $\mu = 0$ forces $A = 0$, which is a trivial solution.
In y, we have

$$Y'' + \gamma Y = 0 \quad \text{subject to} \quad Y(0) = 0 \;\text{ and }\; Y'(H) = 0$$

From $Y(0) = 0$, we find that $Y(y) = C\sin(\sqrt{\gamma}y)$.
From $Y'(H) = 0$, we must have

$$\sqrt{\gamma}C\cos(\sqrt{\gamma}H) = 0$$

This can only give a non-trivial solution if

$$H\sqrt{\gamma} = \pi/2 + m\pi \;\text{ or }\; \gamma = \left(\frac{\pi(\frac{1}{2}+m)}{H}\right)^2.$$

Note that we have to use a different index (m instead of n) because the function in y is not related to the function in x. Also, we can express the original eigenvalue λ as

$$\lambda = \pi^2\left(\left(\frac{(\frac{1}{2}+n)}{L}\right)^2 + \left(\frac{(\frac{1}{2}+m)}{H}\right)^2\right)$$

Putting is all together, our eigenfunction is therefore

$$\phi_{m,n}(x,y) = C_{m,n}\cos\left(\frac{\pi(n+\frac{1}{2})x}{L}\right)\sin\left(\frac{\pi(m+\frac{1}{2})y}{H}\right)$$

Today's plan

1. Can you separate variable for Helmholtz on a disk?
2. What is the Bessel equation?
3. How do you solve it?

We begin by giving some general properties of the eigenfunctions of Helmholtz equation. This can be thought of as an eigenfunction problem, and many of those properties come the study of the Sturm-Liouville problem (chap 5, which we skipped).

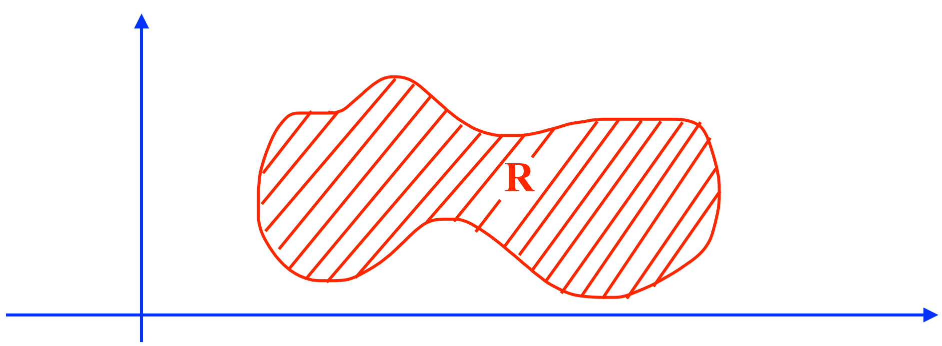
We want to study

$$\nabla^2\phi + \lambda\phi = 0 \text{ over a domain } R\text{, with given BC}$$

Here the domain must be simply connected (one piece), and can be of any dimension. We also need the BC to be homogeneous, of the form

$$a(\vec{x})\phi + b(\vec{x})\frac{\partial\phi}{\partial\hat{n}} = 0$$

Note that the coefficients a and b may depend on location, but not on the function ϕ. This setup is used for both the heat equation and the wave equation.



Domain under consideration

The solutions then have the following properties:

1. The eigenvalues λ are real ($\lambda \in \mathbb{R}$).
2. There are an infinity of λ, with one that is the smallest but no largest one.
3. There may be more than one eigenfunction per eigenvalue, but usually there is only one.
4. Eigenfunctions of different eigenvalues are orthogonal: if $\lambda_i \neq \lambda_j$, then $\phi_i \perp \phi_j$, i.e.$\int\int_R \phi_i \phi_j \, dA = 0$.
5. The series below is convergent

$$f(\vec{x}) = \sum_{\lambda_i} a_i \phi_i(\vec{x}) \text{ with } a_i = \frac{\int\int_R \phi_i f(\vec{x}) \, dA}{\int\int_R \phi_i^2 \, dA}$$

   Note that if an eigenvalue is repeated, the corresponding coefficients will have a different formula because the eigenfunction will be different.

Helmholtz equation on a disk

We can check that the rectangular domain we looked at before confirms the properties listed above. Let's see now what happens on a disk.

For context, we start by considering the wave equation on a disk of radius a, which can be thought of a tracking the height of the membrane of a vibrating drum

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

subject to (in polar cooordinates)

$$\begin{aligned} u(r, \theta, t = 0) &= \alpha(r, \theta) \\ u_t(r, \theta, t = 0) &= \beta(r, \theta) \\ u(r = a, \theta, t) &= 0 \end{aligned}$$

We will again separate variables: $u(r, \theta, t) = H(t)\phi(r, \theta)$ and get, after plugging in

$$\frac{H''}{c^2 H} = \frac{\nabla^2 \phi}{\phi} = -\lambda$$

and we find $H(t) = A\cos(\sqrt{\lambda}ct) + B\sin(\sqrt{\lambda}ct)$.

In space, we get our Helmholtz equation: $\nabla^2\phi + \lambda\phi = 0$. In polar coordinates, that is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0$$

Once again, we try to separate variables: $\phi(r,\theta) = f(r)g(\theta)$. Note that here $g(\theta)$ must be periodic with period 2π.

Plugging in, we find

$$g(f'' + \frac{1}{r}f') + \frac{1}{r^2}fg'' + \lambda fg = 0.$$

If we divide by $\phi = fg$ and multiply by r^2, we get:

$$\frac{r^2f'' + rf'}{f} + \frac{g''}{g} + \lambda r^2 = 0.$$

We can separate r and θ again

$$\frac{r^2f'' + rf'}{f} + \lambda r^2 = -\frac{g''}{g} = \mu$$

where we introduced another constant μ.

The equation in θ is familiar:

$$g'' + \mu g = 0.$$

To get a period of 2π, we must have $\mu = n^2$, and the eigenfunctions are:

$$g_n(\theta) = C_n\cos(n\theta) + D_n\sin(n\theta).$$

In r, however, we get a new equation:

$$\frac{r^2f'' + rf'}{f} + \lambda r^2 = n^2 \text{ or } r^2f'' + rf' + (\lambda r^2 - n^2)f = 0$$

with its boundary condition $f(a) = 0$. We also want $|f| < \infty$ for $r < a$ (the drum doesn't burst).

This is not an easy equation to solve. In fact, it has no elementary solution (no sine, cosine, exponential, polynomial... solution).

However, it has been VERY well studied. It is known as the **Bessel** equation. For example, you can solve it with Series solutions around $r = 0$.

Usually, the constant λ is moved from the equation to the BC using the change of variable $z = \sqrt{\lambda}r$. We then have:

$$\frac{d}{dr} = \frac{dz}{dr}\frac{d}{dz} = \sqrt{\lambda}\frac{d}{dz}, \text{ and } \frac{d^2}{dr^2} = \lambda\frac{d^2}{dz^2}$$

Using this change of variables, we get

$$\frac{\lambda}{\lambda}\, z^2 \frac{d^2 f}{dz^2} + \frac{\sqrt{\lambda}}{\sqrt{\lambda}}\, z \frac{df}{dz} + (z^2 - n^2) f(z) = 0$$

so

$$z^2\, \frac{d^2 f}{dz^2} + z\, \frac{df}{dz} + (z^2 - n^2) f(z) = 0$$

with boundary condition $f(z = \sqrt{\lambda} a) = 0$.
This is the Bessel equation of order n.

The solutions of the Bessel equation are, shockingly, called Bessel functions. There are two kinds (as we have a second order equation):
$J_n(z)$ is the Bessel function of the first kind.
$Y_n(z)$ is the Bessel function of the second kind.
So a general solution is: $\gamma_1 J_n(z) + \gamma_2 Y_n(z)$, or using r:

$$f(r) = \gamma_1 J_n(\sqrt{\lambda} r) + \gamma_2 Y_n(\sqrt{\lambda} r)$$

Ok, that was convenient, we just gave the unknown solutions a name... For this to be useful, we need to have some information about the solutions. So let's start by looking at the Bessel functions.

Bessel functions behaviors

What do solutions of the Bessel equation look like?

$$z^2 f'' + z f' + (z^2 - n^2) f = 0$$

We can look for them using a power series centered at $z = 0$. Unfortunately, the coefficient of f'' is zero when $z = 0$, so we need to look for solutions of the form:

$$f(z) = \sum_{m=0}^{\infty} a_m z^{m+p}$$

for some undetermined p. We need to find p, and then we expect 2 solutions. We can differentiate term-by-term (noting that the sums always start at 0)

$$f'(z) = \sum_{m=0}^{\infty} a_m (m+p) z^{m+p-1} \quad \text{and} \quad f''(z) = \sum_{m=0}^{\infty} a_m (m+p)(m+p-1) z^{m+p-2}$$

Plugging into the Bessel equation, we get:

$$z^2 \sum_{m=0}^{\infty} a_m (m+p)(m+p-1) z^{m+p-2} + z \sum_{m=0}^{\infty} a_m (m+p) z^{m+p-1} + (z^2 - n^2) \sum_{m=0}^{\infty} a_m z^{m+p} = 0$$

and rearranging terms:

$$\sum_{m=0}^{\infty} a_m(m+p)(m+p-1)z^{m+p} + a_m(m+p)z^{m+p} + a_m z^{m+p+2} - n^2 a_m z^{m+p} = 0$$

The coefficient of every power on the left side must be zero, to match the RHS. So we start with the lowest power, which is p, for $m = 0$:

$$a_0[(m+p)(m+p-1) + (m+p) - n^2]z^{m+p} = 0 \quad m = 0$$

We assume that $a_0 \neq 0$ (otherwise the series doesn't really start there) and get the condition that

$$[p(p-1) + p - n^2] = p^2 - n^2 = 0$$

So we find $p \pm n$. We found our two values of p, except when $n = 0$. If $n \neq 0$, we find that the first term of the series, which will be the dominant term for small z, is:

$$J_n(z) \sim z^n \quad \text{near } z = 0\text{, for } n > 0.$$

and

$$Y_n(z) \sim z^{-n} \quad \text{near } z = 0\text{, for } n > 0$$

For the case $n = 0$, the first solution has $p = 0$, and so the first term of the series is a constant. The second solution is $\log z$ times a series that starts with a constant (why $\log z$? Because it works when you plug it into the equation).

$$J_0(z) \sim C \quad \text{near } z = 0$$

and

$$Y_0(z) \sim \log z \quad \text{near } z = 0$$

We could continue and find all the terms of the series. For the moment, we have all the information we need. In particular, we see that all the solutions $Y_n(z)$ tend to infinity as $z \to 0$. So those solutions cannot contribute to our system where $z = 0$ is within our domain. Note that if we want to solve the wave equation outside away from the origin, like in an annulus, we would need to keep them.

Our solution to the Helmholtz equation within the disk is therefore, putting everything together,

$$\phi(r,\theta) = \sum_{n=0}^{\infty} J_n(\sqrt{\lambda_n} r)\ [C_n \cos(n\theta) + D_n \sin(n\theta)]$$

and we know $J_n(r) \sim r^n$ near $r = 0$. Also, note that for $n = 0$, there is no θ dependence. Also, note that here n counts the number of oscillations as you go around the drum.

We now have to determine the values of λ_n that satisfy the boundary condition $J_n(\sqrt{\lambda_n} a) = 0$. To do so, we first look at a few Bessel functions (for example in matlab).

We note a few things:

1. They all oscillate.
2. If $n \neq 0$, the all go through the origin (consistent with our analysis).
3. As $n \to \infty$, the oscillation wavelength decreases.
4. The amplitude of the oscillations decay as $z \to \infty$. The amplitude decays like $z^{-1/2}$.

The fact that the Bessel function oscillate means that there are infinitely many zeroes of the Bessel function, so infinitely many solutions to $J_n(\sqrt{\lambda_n}a) = 0$. We will use the notation that z_{n_j} is the the j^{th} root of $J_n(z)$. We therefore have

$$J_n(\sqrt{\lambda_n}a) = 0 \text{ if } \sqrt{\lambda_{n,j}}a = z_{n_j} \text{ for some } j\text{, or } \lambda_{n,j} = \frac{z_{n_j}^2}{a^2}$$

So we found infinitely many eigenvalues, and we will need to sum over all of them. Recall that λ is related to the time frequency of the oscillations. So the frequency you hear from a drum is determined by the zeroes of a Bessel function. The lowest frequency comes from the oscillations that do not depend on θ (only going up and down) and so are the zeros of $J_0(z)$.

The eigenfunctions we get are therefore of the form:

$$\phi(r,\theta) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} J_n(\frac{z_{n_j}}{a}r)\ [C_n\cos(n\theta) + D_n\sin(n\theta)]$$

Wave equation solution

Finally, we can put together our entire solution to the wave equation. It requires two sums:

$$u(r,\theta,t) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} J_n\left(\frac{z_{n_j}r}{a}\right)\ [C_{n_j}\cos(n\theta)+D_{n_j}\sin(n\theta)][A_{n,j}\cos\left(\frac{z_{n_j}ct}{a}\right)+B_{n,j}\sin\left(\frac{z_{n_j}ct}{a}\right)]$$

You can see a nice illustration of what these modes look like as they oscillate at:

`https://en.wikipedia.org/wiki/Vibrations_of_a_circular_membrane` where, in their notation the mode $u_{m,n}$ corresponds to our mode (n, j).

To determine the coefficients, we need to use the initial conditions, and the nice property that the eigenfunctions of any Helmholtz equation satisfy an orthogonality relation. It takes the form:

$$\int_0^a J_n\left(\frac{z_{n_j} r}{a}\right) J_n\left(\frac{z_{n_k} r}{a}\right)\ r\ dr = 0 \ \text{ if } j \neq k$$

so we can finally determine the coefficients. We find, for the cosine coefficient in time and in θ:

$$A_{n_j} C_{n_j} = \frac{\int_0^{2\pi}\ d\theta \int_0^a r\ dr J_n\left(\frac{z_{n_j} r}{a}\right)\ \cos(n\theta)\ u_0(r,\theta)}{\int_0^{2\pi} \cos^2(n\theta)\ d\theta\ \int_0^a r\ dr[J_n\left(\frac{z_{n_j} r}{a}\right)]^2}$$

and for the cosine coefficient in time and sine in θ:

$$A_{n_j} D_{n_j} = \frac{\int_0^{2\pi}\ d\theta \int_0^a r\ dr J_n\left(\frac{z_{n_j} r}{a}\right)\ \sin(n\theta)\ u_0(r,\theta)}{\int_0^{2\pi} \sin^2(n\theta)\ d\theta\ \int_0^a r\ dr[J_n\left(\frac{z_{n_j} r}{a}\right)]^2}$$

The results are similar for B_{n_j} using the initial condition on the time derivative, $v_0(r, \theta)$, for the cosine coefficient in time and cosine in θ:

$$B_{n_j} C_{n_j}\left(\frac{z_{n_j} c}{a}\right) = \frac{\int_0^{2\pi}\ d\theta \int_0^a r\ dr J_n\left(\frac{z_{n_j} r}{a}\right)\ \cos(n\theta)\ v_0(r,\theta)}{\int_0^{2\pi} \cos^2(n\theta)\ d\theta\ \int_0^a r\ dr[J_n\left(\frac{z_{n_j} r}{a}\right)]^2}$$

and for the cosine coefficient in time and sine in θ:

$$B_{n_j} D_{n_j}\left(\frac{z_{n_j} c}{a}\right) = \frac{\int_0^{2\pi}\ d\theta \int_0^a r\ dr J_n\left(\frac{z_{n_j} r}{a}\right)\ \sin(n\theta)\ v_0(r,\theta)}{\int_0^{2\pi} \sin^2(n\theta)\ d\theta\ \int_0^a r\ dr[J_n\left(\frac{z_{n_j} r}{a}\right)]^2}$$