Lecture 1, Intro

Introducing myself: I have been a faculty here since 2006, I am Canadian, with an accent and I am allowed to say zed. In this class, you will have to work to understand, and to understand to pass. We will also put an strong emphasis on oral communication of Mathematics, which should be useful to you in your career.

Important points as to how this class is run can be found in the syllabus. Please refer to it for logistical questions.

Radioactive decay

Let $N(t)$ be the number of radioactive atoms in a sample. They decay over time, meaning they change their chemical property. We want to describe how many atoms N are left that haven't decayed.

Over time (a time unit is however you decided to measure time) the rate at which atoms decay is proportional to the number of atoms left. Can you write a differential equation to reflect that?

Some things to keep in mind:
The number of atoms goes down over time.
You may need to introduce some notation, because the statement "is proportional to" is not very precise.

You should get something like:

$$\frac{dN}{dt} = -kN, \quad k > 0$$

Note that the units of k are: $[k] = \frac{1}{T}$.

How do you know? The units of $\frac{dN}{dt}$ are

$$\left[\lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t}\right] = \lim_{\Delta t \to 0} \left[\frac{N(t+\Delta t) - N(t)}{\Delta t}\right] = \frac{[N]}{t}$$

and both sides of an equation have the same units.

At the initial time $t = t_0$, we have a known number of atoms N_{t_0}. We denote this as $N(t_0) = N_0$

Claim: A solution satisfying equation (1) and initial condition (2) is: $N(t) = N_0 e^{-k(t-t_0)}$ (Note: only unit-free quantities can appear in complicated functions.)

Check: (in this class, this is the easy part)

$N(t_0) = N_0 e^{-k(0)} = N_0$

$\frac{dN}{dt} = N_s e^{-k(t-s)}(-k) = -kN$, as we wanted.

Half-life

Now that we have a solution, we can use it to learn stuff. The half-life is the time required for the amount of radioactive atoms to be reduced by half.

Let $t_0 = 0$. So we have $N(0) = N_0$

$N(t) = N_0 e^{-kt}$

When does $N(t) = N_0/2$? This defines the half-life, $t_{1/2}$: the time it take to reduce N to half the original value.

$\frac{N_0}{2} = N_0 e^{-kt_{1/2}}$ so

$$\frac{1}{2} = e^{-kt_{1/2}} \quad \text{and} \quad \ln(1/2) = -kt_{1/2} \quad \text{and} \quad t_{1/2} = \frac{\ln 2}{k}$$

Note that units work out!

Note also that N_0 does not matter. This is why the half-life is a useful concept.

This is how carbon dating works:

-Live matter gets its Carbon from air.

-In air, there are amounts of radioactive ^{14}C . After an organism's death, ^{14}C decays and does not get replaced.

-$t_{\frac{1}{2}}$ for ^{14}C is about 5700 years.

So what is k? Well $k = \frac{\ln 2}{t_{1.2}} = 0.0001216/\text{year}$.

Usually, we can find that there is only a fraction of ^{14}C left.

$\frac{N_{now}}{N_{original}} = p$. How can we date it?

$N_{now} = pN_0 = N_0 e^{-kt}$

What is t? $p = e^{-kt}$; so $\ln p = -kt$ and $t = -\ln p/k$.

In **Example 1.4**, $p = 0.916$, so what is t?
Plugging in we find that $t = -721.5$ years ago so in year ≈ 1300

What does it change in the process that the amount of ^{14}C varies in time?

What could you do if k varied in time?

Verifying that a given function is a solution

It is important to be able to verify whether a given function is or is not a solution to a differential equation problem. To do so, we need to determine if the given function and its derivatives:

1. Satisfy the differential equation itself
2. Satisfy the given initial or boundary conditions, where the function or its derivatives are evaluated at a point.

For example, consider the Initial value problem

$$y'' + 2y' + y = 2\cos t \text{ subject to } \quad y(0) = 0, \quad \text{and} \quad y'(0) = 2$$

Is the function $z(t) = \sin t$ a solution?

To verify, we compute as many derivatives as needed: $z'(t) = \cos t$ and $z''(t) = -\sin t$.

We can now plug in:

$$z'' + 2z' + z = -\sin t + 2\cos t + \sin t = 2\cos t$$

and that is the right-hand-side that we wanted.

Now for the initial conditions: $z(0) = \sin(0) = 0$, which is what we want.
Finally, $z'(0) = \cos(0) = 1$. That is not what it should be (it should be 2).

So $z(t) = \sin t$ is NOT a solution to this Initial Value Problem.

A note on notation

A function is a RULE that tells you how to handle an input.

Example: f square inputs, is a function.

We denote their inputs (real, $\mathbb{R}$) and outputs (real, $\mathbb{R}$)

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

Often we prefer to show the effect of a function:

$f(x) = x^2$

This helps with clarity. But recall that here x is purely symbolic.

So $f(x) = x^2$ and $g(y) = y^2$ are the SAME function.

In this class, we need to be careful of our notation:

Given $f(x)$, we may speak of its derivative with respect to x:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{df}{dx} = f'(x)$$

There is no confusion in using $'$ here because f has only one input.

Integration notation is more flexible. The $d()$ states what is the name of the variable being changed.

$$\int_a^b f(x)dx$$

Figure 1: Integration of a function f over a finite interval $x \in [a, b]$.

But this name is unimportant, so long as it is the same in $d()$ and $f()$:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(\ddagger)d\ddagger$$

This may get confusing when we introduce ANTIderivatives:

Let $F(x)$ be such that $F'(x) = \frac{dF}{dx} = f(x)$ (x is the same)

We can try to write this as an integral: $F(x) = \int_0^x f(x)dx + C$

But this is a bad notation, because x has 2 meanings.
Here x goes from 0 to x? That is really not clear.

So in the integral we use a DUMMY variable.

$$F(x) = \int_0^x f(\tilde{x})d\tilde{x} = \int_0^x f(t)dt$$

The book likes to add a tilde ($\sim$) to the usual variable of f.

Figure 2: Plot of $f(t)$ or $f(\tilde{x})$ from 0 to x. The area is the antiderivative $F(x)$ with $dF/dx = f(x)$.

Classification of Differential equations

We will classify D.E. based on:

1 - Number of variables (type)

2 - Highest number of derivation taken (order)

3 - Autonomous (or not)

4 - Linearity (or not)

5 - Constant coefficients (or not)

The point of this classification is to determine how to solve the D.E.

1 - Number of variables

In our class, we have only —> 1 variable —> ordinary D.E. (or O.D.E),
Example $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 7$

In Math 126 —> more than one variables —> partial D.E. (or PDE),
Example $u_t = Du_{xx} - u$

2 - Order

Look for the term with the most derivatives taken.
Count those derivatives —> that is the order:

$y'' = -\omega^2 y + \epsilon(y'^3)$ is 2^{nd} order

$u_{xxxx} + v(x)u = 0$ is 4^{th} order.

In general, an n^{th}-order D.E. looks like:

$$F(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}} ... \frac{dy}{dx}, y, x) = 0$$

We want the function $y(x)$ that satisfies it.

3 - Autonomous equation

If the D.E does not depend explicitly on the independent variable, it is autonomous:

$$F(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}} ... \frac{dy}{dx}, y) = 0$$

Example: $y'' + y = 0$ is autonomous.
but $y'' + (1 + \frac{1}{2}\sin x)y = 0$ is not autonomous.

4 - Linear or non-linear in $y(x)$

This one is a BIG deal.

- Linear equations have a good chance of being analytically tractable.

- Nonlinear equations... not so much.

Again, start from a D.E.

$$F(y^{(n)}, y^{(n-1)}, ...y', y, x) = L(y) = f(x)$$

This is a linear equation if the operator $L(y)$ is linear, so if $L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$.

In practice, it is linear if it looks like this exactly:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ...a_1(x)y' + a_0(x)y = f(x)$$

Moreover, if $f(x) = 0$, the equation is **Homogeneous**.

Finally, if $\frac{da_i}{dx} = 0$ for $i = 0, 1...n$ then the equation has constant coefficients (but $f(x)$ doesn't have to be 0).

Classify a few **examples**:

1) $u_t = (1 + \frac{x}{2})u_{xx}$

2) $y''y' + (y')^2 = 0$

3) $y'' + 2y' + 16y = \cos x$.

Types of solutions

We are looking for solutions in the form of functions $y(x)$.

Ideally, we can get an explicit solution:

$y(x) = f(x)$

Elementary functions are best, like $e^x, \sin x, x^2$ etc.

Sometimes, only an integral solution is available, or a special fucntion:

$$y(x) = \int_0^x e^{-t^2} dt \quad \text{or,} \quad y(x) = \text{erf}(x)$$

or an infinite series:

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!n!}$$

Sometimes only implicit solutions are found: $y(x) + \ln y(x) - x^2 = 0$

Sometimes, only numerical solutions are available: $y(x_i) = y_i$.
These are APPROXIMATE solutions.

Graph of an approximate solution obtained numerically.

Lecture 3: Fundamental Theorem of Calculus (FTC)

Recall the FTC:
Let $f : [a, b] \to \mathbb{R}$ be a continuous function over $a \leq x \leq b$ and define

$$G(x) = \int_a^x f(\tilde{x}) d\tilde{x}.$$

Then we have that

$$\frac{dG}{dx} = f(x)$$

and for any antiderivative $F(x)$ such that $\frac{dF}{dx} = f(x)$

$$\int_a^b f(\tilde{x}) d\tilde{x} = F(b) - F(a).$$

(What is the difference between $G(x)$ and $F(x)$ in the statement above? Why not just use one notation?)

This is handy to solve the simplest type of DE. If

$$\frac{dy}{dx} = f(x),$$

then $y(x) = F(x) + c$ or $y(x) = \int_0^x f(\tilde{x}) d\tilde{x} + C$

There are infinitely many solutions.

With an initial condition: $y(x_0) = y_0$, we can select a UNIQUE solution.

Try $\frac{dy}{dx} = x + 10 \sin x$ subject to $y(\pi) = 0$

General solution:

$$\begin{aligned} y(x) &= \int x + 10 \sin x dx + C \\ &= \frac{x^2}{2} - 10 \cos x + C \end{aligned}$$

We can find C by plugging in:
$y(\pi) = \frac{\pi^2}{2} - 10 \cos \pi + C = \frac{\pi^2}{2} + 10 + C = 0$, so $C = -\pi^2/2 - 10$

Or directly:

$$y(x) = \int_\pi^x t + 10 \sin t dt + 0 \quad \text{(this 0 is the value of } y \text{ when } x = \pi\text{)}$$

This second approach is useful if you cannot simplify the integral:

$$y' = e^{-x^2}, y(0) = y_0$$

$$y(x) = \int_0^x e^{-\tilde{x}^2} d\tilde{x} + y_0$$

Note that as $x \to \infty$, then $y \to y_0 + \int_0^\infty e^{-\tilde{x}^2} d\tilde{x} = y_0 + \sqrt{\pi}/2$

Unique solution

When can we expect a UNIQUE solution?

First, we define a solution, on an open interval I (like $a < t < b$)

$x(t)$ is a solution to $x' = f(x, t)$, with $x(t_0) = x_0$ if:

$x(t)$ is differentiable over an open interval I and $t_0 \in I$.

and $\frac{dx}{dt} = f(x, t)$ over I and $x(t_0) = x_0$.

Now we can state our Theorem (6.2)

Theorem: if $f(x, t)$ and $\frac{\partial f}{\partial x}$ are continuous over a region $a < x < b, c < t < d$, then for any x_0 such that $a < x_0 < b$ and t_0 with $c < t_0 < d$, the IVP has a unique solution over some interval I containing t_0.

(Note that this can be formulated more generally by requiring that f only satisfy the Lipschitz condition i.e $|f(x_1) - f(x_2)| < L|x_1 - x_2|$)

This theorem says that $x' = f(x, t)$, with $x(t_0) = x_0$ has a unique solution on SOME open interval. It might not be all of I.

Notes:
1) This guarantees a unique solution if conditions are met.
It is also possible to have a unique solution if they are not met.
2) This only guarantees a solution on some interval. Not everywhere., such as over all (c, d)

Try $x' = x^2$ with $x(0) = x_0$. Now try $x' = \sqrt{x}$ with $x(0) = 0$.

Qualitative solutions

For autonomous equations, it is relatively easy to describe a solution's behavior as $t \to \infty$.

Consider $\frac{dy}{dt} = f(y)$ and $y(t_0) = y_0$.

We can find equilibrium points y_e where $\frac{dy}{dt} = 0$ by solving $f(y_e) = 0$.

We then have that $y = y_e$ is a solution to the system.

An equilibrium point is said to be stable if a solution that starts nearby remains nearby:

There exist a positive number ϵ such that $|y_0 - y_e| < \epsilon$, implies that $|y(t) - y_e| < \epsilon$ for all times. So if you start close enough, you cannot escape the interval $(y_e - \epsilon, y_e + \epsilon)$

Stable solutions of an autonomous system

A stronger statement is that an equilibrium point can be ATTRACTIVE if $\exists \delta$ such that $|y_0 - y_e| < \delta$ implies $\lim_{t\to\infty} y(t) = y_e$
This is even more restrictive.

To determine how attractive an equilibrium point is, we only need to know the sign of $f(y)$:

We must have, for y_e to be attractive:
if $y > y_e$, (nearby) $f(y) < 0$, and
if $y < y_e$ (nearby) $f(y) > 0$
To be stable, we have the same conditions, but allow $f(y) = 0$ too:

Example: $\frac{dy}{dt} = (y-1)(y-2)$

Equilibrium points: $y_e = 1$ and $y_e = 2$. Classify them.

Attractive solutions of an autonomous system

Stability diagram

Example: Population model: $\frac{dp}{dt} = kp(1 - \frac{p}{M})\ k > 0, M > 0$

Solutions to $\frac{dy}{dt} = (y-1)(y-2)$

Population model

Lecture 4

Separable Equations

Remember this? If you can get a D.E. of the form:

$$\frac{dy}{dt} = f(y)g(t)$$

Then you can "separate" it: $\frac{1}{f(y)}dy = g(t)dt$

and integrate both sides: $\int \frac{1}{f(y)}dy = \int g(t)dt$
Note: these are mathematically ill-defined.

to get an implicit, general, solution: $F(y) = G(t) + C$
where $\frac{dF}{dy} = \frac{1}{f(y)}$, and $\frac{dG}{dt} = g(t)$.

Why does this work? Start from

$$F(y) = G(t) + C \qquad (1)$$

Take $\frac{d(1)}{dt}$ and use $\frac{dF}{dy}\frac{dy}{dt} = \frac{dG}{dt}$
so $\frac{1}{f(y)}\frac{dy}{dt} = g(t)$ and $y' = f(y)g(t)$.
Great! So it is well justified.

Example: Malthus population model: $\frac{dp}{dt} = kp$ with $k > 0$ and $p(0) = N$.

$\frac{dp}{p} = kdt$ so $\log p = kt + C$
and $p = e^{kt+C}$ or C_2e^{kt}
at $t = 0, p(0) = C_2e^0 = C_2 = N$
So $p(t) = Ne^{kt}$

Example: Logistic equation: $\frac{dp}{dt} = kp(1 - \frac{p}{M})$ with $k, M > 0$ and $p(0) = N$

$$\frac{dp}{p(1 - \frac{p}{M})} = kdt$$

Partial Fraction time $\frac{A}{p} + \frac{B}{(1-\frac{p}{M})} = \frac{1}{p(1-\frac{p}{M})} = \frac{A-A(\frac{p}{M})+Bp}{p(1-\frac{p}{M})}$

So $A = 1$, $\frac{-A}{M} + B = 0$ (No p in numerator)
and $B = \frac{1}{M}$.

We have

$$\int \frac{dp}{p} + \int \frac{1}{M}\frac{dp}{(1 - \frac{p}{M})} = \int kdt$$

$$\log p - \log(M - p) = kt + C$$

$$\log \frac{p}{M - p} = kt + C$$

at $t = 0, C = \log(\frac{N}{M-N})$.

This is an implicit solution.

We can get an explicit solution too: $p = M(\frac{N}{M-N})e^{kt} - p(\frac{N}{M-N})e^{kt}$

so $p(1 + (\frac{N}{M-N}e^{kt})) = (\frac{MN}{M-N})e^{kt}$ and

$p(t) = \frac{MNe^{kt}}{(M-N)+Ne^{kt}} = \frac{M}{(\frac{M-N}{N})e^{-kt}+1}$, which incidentally, goes to M as $t \to \infty$.

Note $[M] = [N]$=population
$[k] = \frac{1}{\text{time}}$

General solution for a first order linear equation

Now consider the very common, 1^{st} order, linear DDE:

$$a_1(t)x' + a_0(t)x = f(t)$$

Usually, $a_1 \neq 0$, and we write:

$$x' + p(t)x = q(t) \qquad (1)$$

Recall the product rule:

$$\frac{d}{dt}(\phi x) = x'\phi + \phi' x \qquad (2)$$

We would like our LHS to be the derivative of a product.
We can get there by multiplying by the proper ϕ:
$\phi \cdot (1) = \phi x' + p(t)\phi(t)x = \phi(t)(q(t)$

We need to have $p(t)\phi(t) = \frac{d\phi}{dt}$ to recover form (2).

Solving that ODE: $\int p(\tilde{t})d\tilde{t} = \int \frac{d\phi}{\phi}$

so $\phi(t) = e^{\int_0^t p(\tilde{t})d\tilde{t}}$ is a good choice . It is always the same Integrating Factor.

Try $e^{\int_0^t p(\tilde{t})d\tilde{t}} \cdot (1)$

$$e^{\int_0^t p(\tilde{t})d\tilde{t}}\frac{dx}{dt} + e^{\int_0^t p(\tilde{t})d\tilde{t}} \cdot p(t) \cdot x = e^{\int_0^t p(\tilde{t})d\tilde{t}}q(t)$$

and

$$\frac{d}{dt}\left(e^{\int_0^t p(\tilde{t})d\tilde{t}}\, x\right) = e^{\int_0^t p(\tilde{t})d\tilde{t}} q(t)$$

so

$$x(t) = e^{-\int_0^t p(\tilde{t})d\tilde{t}}\left(\int_0^t e^{\int_0^{\tilde{t}} p(s)ds} q(\tilde{t})d\tilde{t} + x_0\right).$$

Example: Newton's law of cooling $\frac{dT}{dt} = -k[T - A(t)]$
$k =$ Rate of heat transfer (May or may not be constant)
$A(t) =$ Ambient temperature.
This can also be solved exactly using the same technique

Exact equations

Some D.E. are what we call Exact. They come from an implicit form of the solution: $F(x, y) = 0$ or in general $F(x, y) = C$

Taking $\frac{d}{dx}$, we have: $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

So given

$$f(x, y) + g(x, y)\frac{dy}{dx} = 0, \qquad (*)$$

how do we tell if it is exact?
Recall that if the mixed derivatives are continuous, we have that:

$$F_{xy} = F_{yx}$$

So for $(*)$ to be exact, we must have $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

In that case, we can find $F(x, y)$ by combining:
$F(x, y) = \int f(x, y)dx + C_1(y)$
$F(x, y) = \int g(x, y)dy + C_2(x)$

Example: $x^3 + \frac{y}{x} + (y^2 + \ln x)y' = 0$
Here $f(x, y) = x^3 + \frac{y}{x}$ and $g(x, y) = y^2 + \ln x$,
Checking, we have $f_y = \frac{1}{x}, g_x = \frac{1}{x}$, so it is exact.

$F(x, y) = \int (x^3 + y/x)\, dx = \frac{x^4}{4} + y\ln x + C(y)$

$F(x, y) = \int (y^2 + \ln x)\, dy = \frac{y^3}{3} + y\ln x + C_2(x).$

So combining those two results, we get:

$$F(x, y) = \frac{x^4}{4} + y\ln x + \frac{y^3}{3} + C$$

which is our general solution.

You can also look for integrating factors to make an equation exact, but they cannot always be found.

Substitution

Sometimes changing variables can simplify an equation greatly. It is usually hard to find the right substitution.
YOU should be able to perform a GIVEN substitution

Consider $\frac{dy}{dx} = H(\frac{y}{x})$ such as $\frac{dy}{dx} = 2\frac{x}{y} + 3\frac{y}{x}$

Introduce $u = y/x$ so $y = ux$ and $y' = u'x + u$
Here $u(x)$ is our new dependent variable, the solution we want.

Recompute $\frac{dy}{dx} = x\frac{du}{dx} + u = H(\frac{y}{x}) = H(u) = 2/u + 3u$

So we have $\frac{du}{dx} = \frac{1}{x}(\frac{2}{u} + 2u)$ and $\frac{du}{\frac{1}{u}+u} = \frac{2dx}{x}$

$\frac{udu}{1+u^2} = \frac{2dx}{x}$ and $\frac{1}{2}\log(1+u^2) = 2\log x + C$

So $\log(1+u^2) = \log x^4 + C$ and $1 + u^2 = ke^{x^4}$
Finally, $u = (ke^{x^4} - 1)^{\frac{1}{2}}$ and $y(x) = x(ke^{x^4} - 1)^{\frac{1}{2}}$

Note:You may substitute for x too:
Example: $v = x^2$ or $x = \sqrt{v}$ IN $\frac{dy}{dx} = f(x,y)$

Then $\frac{dy}{dv} = \frac{dy}{dx}\frac{dx}{dv} = f(x,y)\frac{1}{2\sqrt{v}} = \frac{f(\sqrt{v},y)}{2\sqrt{v}}$, which may help.
Try it for

$$\frac{dy}{dx} = f(x,y) = 2xy.$$

Lecture 5

Second order, linear equations

We will now study equations of the form:

$$a_2(t)x'' + a_1(t)x' + a_0(t)x = f_0(t)$$

With initial conditions: $x(t_0) = x_0$, and $x'(t_0) = p_0$

There is an existence and unicity theorem, similar to the first order one.

For $a_2(t) \neq 0$, we usually write $x'' + p(t)x' + q(t)x = f(t)$
We start by considering the corresponding homogenous system

$$L(x) = x'' + p(t)x' + q(t)x = 0$$

This system has 2 solutions, x_1 and x_2. They must be independent of each other and to check that it is enough to check if they are multiple of each other:
$x_1 = Cx_2 \longrightarrow$ dependent.

For independent x_1 and x_2, our general solution is $x_g(t) = \alpha x_1(t) + \beta x_2(t)$

Note: $L(x_g) = \alpha L(x_1) + \beta L(x_2) = 0$ by linearity and because $x_1(t)$ and $x_2(t)$ are solutions to the homogeneous problem.

Bonus: Can you compute what is $L(t\ x_1)$? Sometimes that is useful, especially if x_1' is a multiple of x_2.

Constant Coefficients (review)

We start with the friendliest case, where the coefficients are constant. We usually denote them as:

$$ax'' + bx' + cx = 0$$

Try the following ansatz (guess):
$x(t) = e^{kt}$
$x'(t) = ke^{kt}$
$x''(t) = k^2e^{kt}$
Plugging in, we obtain the equation:

$$e^{kt}(ak^2 + bk + c) = 0$$

and the solutions are exponential functions where $k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

There are three cases to consider:

1. We have 2 real, distinct roots. This happens when $b^2 - 4ac > 0$.

   We then have $k_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $k_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
   and our general solution is $x_g(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$.

   The long-term behaviour depends on the sign of k_1, and k_2. Note that here $k_1 > k_2$.
   If $k_1 \leq 0$ (and therefore $k_2 \leq 0$ too), the system is stable, as the solutions DO NOT GROW.
   If $k_1 < 0$ (and therefore $k_2 < 0$ too), the system is still stable, and it is now attractive, as all solutions approach 0 with increasing time.
   Note: if $k_i = 0$, the $x(t) = C_i$ (a constant) is a solution.

2. Repeated (real) roots: This happens if $b^2 - 4ac = 0$.
   In this case, we have a single solution (so far) $x_1 = C_1 e^{k_1 t} = C_1 e^{(-b/2a)t}$.
   We are therefore missing a solution.
   Try $x_2(t) = te^{k_1 t}$. We then have
   $x_2'(t) = e^{k_1 t}(1 + k_1 t)$, and
   $x_2''(t) = e^{k_1 t}(2k_1 + k_1^2 t)$
   If we plug this into our original equations, we find:

   $$ax'' + bx' + cx = (ak_1^2 t + 2ak_1 + bk_1 t + b + ct)e^{k_1 t}$$

   but because k_1 is a root of our equation, $ak_1^2 t + bk_1 t + ct = (ak_1^2 + bk_1 + c)t = 0$.
   So we get the simplified expression

   $$(2ak_1 + b)e^{k_1 t} = 0$$

   but because $k_1 = -b/2a$, this is always true! It's magic!

   It means that $x_2(t) = te^{k_1 t}$ is also a solution, and our general solution is therefore

   $$x_g(t) = C_1 e^{k_1 t} + C_2 t e^{k_1 t}$$

   The long-term behaviour depends on the sign of k_1
   If $k_1 < 0$, the system is stable, and attractive
   If $k_1 \geq 0$, the system is unstable.

3. Complex roots: $b^2 - 4ac < 0$.

   To simplify our notation, let us introduce the real and imaginary parts of our roots:

   $$p = \frac{-b}{2a} \quad \text{and} \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}$$

Our roots are then $k_1 = p + i\omega$ and $k_2 = p - i\omega$ and the solution is

$$x_g(t) = \alpha e^{(p+i\omega)t} + \beta e^{(p-i\omega)t} = e^{pt}(\alpha e^{i\omega t} + \beta e^{-i\omega t})$$

However, the solution is still a real function. So here α and β may be complex too.

Recall a few things about complex numbers: Firstly $i^2 = -1$. Also,

$$(c + id)(e + if) = ce - df + i(ce + df)$$

Importantly: $e^{i\theta} = \cos\theta + i\sin\theta$.
You may obtain this result using Taylor Series, as you can see in Math 122.

So we have:

$$\alpha e^{i\omega t} + \beta e^{-i\omega t} = \alpha(\cos\omega t + i\sin\omega t) + \beta(\cos\omega t - i\sin\omega t) = (\alpha+\beta)\cos\omega t + i(\alpha-\beta)\sin\omega t$$

and we can therefore introduce new constants $A = \alpha + \beta$ and $B = i(\alpha - \beta)$ and have the general solution

$$x_g(t) = Ae^{pt}\cos\omega t + Be^{pt}\sin\omega t$$

The long term behavior depends on the sign of $p = \frac{-b}{2a}$:
$p > 0$ is unstable, $p < 0$ is attractive stable, and $p = 0$ is stable but not attractive.

We can try a few examples:
Example 1: $x'' + x' - 6x = 0$, with $x(0) = 1$ and $x'(0) = 2$.
Example 2: $x'' - 2x' = 0$, with $x(0) = 3$ and $x'(0) = 1$.
Example 3: $x'' + 2x' + x = 0$, with $x(0) = 0$ and $x'(0) = 1$.
Example 4: $x'' + 2x' + 5x = 0$, with $x(0) = 1$ and $x'(0) = 0$.

Lecture 6

Spring/Pendulum interpretation

In general, a spring satisfies Hooke's law:

$$F = -kx = m\frac{d^2x}{dt^2}, \quad \text{with } k, m > 0$$

which is often rewritten as

$$x'' + \frac{k}{m}x = x'' + \omega^2 x = 0$$

where we introduced the *natural frequency*: $\omega = \sqrt{k/m}$.

This has the general solution (which you need to know like the back of you hand):

$$x_g(t) = A\cos(\omega t) + B\sin(\omega t)$$

Note that it is sometimes convenient to write this in the EQUIVALENT form

$$x_g(t) = \alpha\cos(\omega t - \phi) = \alpha(\cos\phi\cos(\omega t) + \sin\phi\sin(\omega t))$$

and you can see that if $A = \alpha\cos\phi$ and $B = \alpha\sin\phi$, we get the same solution as above. From A and B, you can find α and ϕ through: $\alpha^2 = A^2 + B^2$ and $\tan\phi = B/A$.

This solution does not grow in time, so it is stable.

Shifted sinusoidal, with amplitude α and period $2\pi/\omega$.

If we add friction, we get a new expression for the force: $F = -kx - \mu x'$.
The additional term *OPPOSES* motion for $\mu > 0$.

The governing equation becomes, using the notation $\lambda = \mu/m$, and noting that $\lambda > 0$:

$$x'' + \lambda x' + \omega^2 x = 0, \quad \text{with} \quad \lambda > 0.$$

Looking for solutions of the form $x(t) = Ae^{rt}$, we find the roots:

$$r_{1,2} = -\frac{\lambda}{2} \pm \sqrt{\left(\frac{\lambda}{2}\right)^2 - \omega^2}$$

Case 1: If $\lambda/2 > \omega$, we find two real roots, both negative. This is called an OVERDAMPED system

Overdamped system

Case 2: If $\lambda = \omega/2$, we have a repeated root, This is called a CRITICALLY DAMPED system.
The solution is the of the form: $x_g(t) = Ae^{-(\omega/2)t} + Bte^{-(\omega/2)t} = (A + Bt)e^{-(\omega/2)t}$. Here $x = 0$ is an attractive and stable equilibrium.

Critically damped system.
The solution crosses the x-axis exactly once for almost all initial conditions.

Case 3: If $\lambda/2 < \omega$, we have two imaginary roots.
The solution is then

$$x(t) = e^{-\lambda/2\ t}\left[A\cos\left(\left(\omega^2 - \frac{\lambda^2}{4}\right)^{1/2} t\right) + \sin\left(\left(\omega^2 - \frac{\lambda^2}{4}\right)^{1/2} t\right)\right]$$

This is an attractive, stable solution, which is called UNDER-DAMPED.

Under-damped system. The solution oscillates with decreasing amplitude.

Non-homogeneous systems

We now consider the additional complications that arise when the system is not homogeneous. To continue our parallel with springs or pendula, this corresponds to having an external force applied to the mass:

$$ax'' + bx' + cx = f(t) \quad \text{for } f(t) \neq 0$$

First, we look for a general solution to the homogenous system, as described before:

$$x_h(t) = \alpha e^{r_1 t} + \beta e^{r_2 t}$$

But we will also need a PARTICULAR solution $x_p(t)$ that satisfies

$$ax_p'' + bx_p' + cx_p = f(t)$$

though we won't care about initial conditions yet.

Our general solution will be the sum of those two: $x_g(t) = x_h(t) + x_p(t)$
From THAT solution, we will be able to apply our boundary conditions and solve for the constants α and β.

Let's start with some examples:
Example 1: Consider the LHS $x'' + 9x$, and try the particular solution $x_p(t) = t^2$.
Then we have: $x_p' = 2t$ and $x_p'' = 2$. So plugging this in the LHS, we find the RHS: $x_p'' + 9x_p = 2 + 9t^2$.

So now suppose the original problem was: $x'' + 9x = 2 + 9t^2$.
We then know a particular solution: $x_p(t) = t^2$ (no free constant here though!).
The homogeneous solution, we should also know: $x_h(t) = \alpha \cos 3t + \beta \sin 3t$.
The general solution is therefore: $x_g(t) = \alpha \cos 3t + \beta \sin 3t + t^2$.

Given some initial conditions, we can find the free constants. Say $x(\pi) = 0$ and $x'(\pi) = 0$, we get:

$$\begin{aligned}
&x(\pi) = 0 = -\alpha + \pi^2 \\
&\quad \text{so} \quad \alpha = \pi^2 \\
&\quad \text{and} \quad x'(\pi) = 0 = -3\beta + 2\pi \\
&\quad \text{so} \quad \beta = \frac{2}{3}\pi \\
&\quad \text{and finally} \quad x(t) = \pi^2 \cos 3t + \frac{2}{3}\pi \sin 3t + t^2.
\end{aligned}$$

How do we find a particular solution?

For a given second order, constant coefficients linear equation that is non-homogeneous:

$$L[x] = f(t)$$

(here L is a linear differential operator that contains the operations performed on $x(t)$) we need a way to find a particular solution. Our general approach will be to make an educated guess, and improve it.

In general, we guess a linear combination of $f(t)$ and of derivatives of $f(t)$. Ideally, these are only a finite number of *possible* derivatives.

For example, if $f(t)$ is a **polynomial** of degree n, we will try a general polynomial of degree n.
Ex. 1: If $L[x] = x'' + 2x' + 4x = t^2$, we guess

$$x_p(t) = at^2 + bt + c$$

and solve for $a, b,$ and c by plugging into the original problem. Here we get:

$$\begin{aligned} (2a) + 2(2at + b) + 4(at^2 + bt + c) &= t^2 + 0t + 0 \\ (4a)\, t^2 + (4a + 4b)\, t + (2a + 2b + 4c) &= t^2 + 0t + 0 \end{aligned}$$

We then solve coefficient-by-coefficient, to get a system of linear equations (3 equations. and 3 unknowns)

For t^2	$4a = 1$
For t	$4a + 4b = 0$
For 1	$2a + 2b + 4c = 0$

and you can find that $a = 1/4$, $b = -1/4$, and $c = 0$. So our particular solution is

$$x_p(t) = \frac{1}{4}(t^2 - t)$$

You can (and should because it is an easy step) verify that it is correct by plugging in.

Generally

- If $f(t)$ is a polynomial of degree n try $x_p(t) =$ polynomial of degree n.
- If $f(t) = e^{\alpha t}$ try $x_p(t) = Ce^{\alpha t}$
- If $f(t) = \cos \omega t$ try $x_p(t) = A \cos \omega t + B \sin \omega t$.

- If $f(t) = \sin \omega t$ try $x_p(t) = A \cos \omega t + B \sin \omega t$.
- If $f(t) = e^{\alpha t} \cos \omega t$ try $x_p(t) = Ae^{\alpha t} \cos \omega t + Be^{\alpha t} \sin \omega t$.

Note that because our equation is linear, our solutions may be added together, so if $f(t)$ is a sum of simple functions, you can treat each simple function one at a time.

Below are some good sample problems:
Question 1: $x'' + \omega^2 x = \sin \alpha t$
Question 2: $x'' + 2x' + 10x = e^{-t}$
Question 3: $x'' + 3x' + 4 = 1 + e^t$

Lecture 7: Particular solutions and intro to resonance

How do we find a particular solution?

For a given second order, constant coefficients linear equation that is non-homogeneous:

$$ax'' + bx' + cx = L[x] = f(t)$$

(here L is a linear differential operator that contains the operations performed on $x(t)$) we need a way to find a particular solution. Our general approach will be to make an educated guess, and improve it.

In general, we guess a linear combination of $f(t)$ and of derivatives of $f(t)$. Ideally, these are only a finite number of *possible* derivatives.

For example, if $f(t)$ is a **polynomial** of degree n, we will try a general polynomial of degree n.
Ex. 1: If $L[x] = x'' + 2x' + 4x = t^2$, we guess

$$x_p(t) = at^2 + bt + c$$

and solve for $a, b,$ and c by plugging into the original problem. Here we get:

$$\begin{aligned}(2a) + 2(2at + b) + 4(at^2 + bt + c) &= t^2 + 0t + 0 \\ (4a)\, t^2 + (4a + 4b)\, t + (2a + 2b + 4c) &= t^2 + 0t + 0\end{aligned}$$

We then solve coefficient-by-coefficient, to get a system of linear equations (3 equations. and 3 unknowns)

$$\begin{array}{rl} \text{For } t^2 & 4a = 1 \\ \text{For } t & 4a + 4b = 0 \\ \text{For } 1 & 2a + 2b + 4c = 0 \end{array}$$

and you can find that $a = 1/4$, $b = -1/4$, and $c = 0$. So our particular solution is

$$x_p(t) = \frac{1}{4}(t^2 - t)$$

You can (and should because it is an easy step) verify that it is correct by plugging in.

Generally

- If $f(t)$ is a polynomial of degree n try $x_p(t) =$ polynomial of degree n.

- If $f(t) = e^{\alpha t}$ try $x_p(t) = Ce^{\alpha t}$
- If $f(t) = \cos \omega t$ try $x_p(t) = A\cos \omega t + B \sin \omega t$.
- If $f(t) = \sin \omega t$ try $x_p(t) = A\cos \omega t + B \sin \omega t$.
- If $f(t) = e^{\alpha t}\cos \omega t$ try $x_p(t) = Ae^{\alpha t}\cos \omega t + Be^{\alpha t} \sin \omega t$.

Note that because our equation is linear, our solutions may be added together, so if $f(t)$ is a sum of simple functions, you can treat each simple function one at a time.

Below are some good sample problems:
Question 1: $x'' + \omega^2 x = \sin \alpha t$
Question 2: $x'' + 2x' + 10x = e^{-t}$
Question 3: $x'' + 3x' + 4x = 1$

Resonance

We have seen strategies to look for a particular solution to forced systems

$$L[x] = ax'' + bx' + cx = f(t).$$

Our strategies involved guessing a linear combination of $f(t)$ and its derivatives. We now consider an important exception where this approach does not work, even for simple $f(t)$. **The exception occurs when $f(t)$ is a solution to the homogeneous problem.** This is called **RESONANCE**

For example, if we have: $x'' + 4x = \sin 2t$.
Our usual guess would be $x_p(t) = A\cos 2t + B\sin 2t$.
But that would fail because for any A or B, we would have $x_p'' + 4x_p = 0$.

Instead, we must consider our usual guess, *multiplied by t*:

$$x_p(t) = tx_h = t(A\cos 2t + B\sin 2t)$$

where x_h is a solution to the homogeneous problem.

Whenever we use a homogeneous solution multiplied by t, we have:
$x_p = tx_h$
$x_p' = tx_h' + x_h$
$x_p'' = tx_h'' + 2x_h'$
so that

$$ax'' + bx' + cx = a(tx_h'' + 2x_h') + b(tx_h' + x_h) + ctx_h = t[ax_h'' + bx_h' + cx_h] + 2ax_h' + bx_h$$

Note that the factor that t multiplies is exactly the original equation. Because x_h is a solution to the original problem, it is therefore zero. We can therefore match a forcing term that is a combination of x_h and x_h'.

Let's see an example, using the same equation as above: $x'' + 4x = \sin 2t$.
Our guess and its derivatives will be:

$$\begin{aligned}
x_p(t) &= t(A\cos 2t + B\sin 2t) \\
x_p'(t) &= (A\cos 2t + B\sin 2t) + t(-2A\sin 2t + 2B\cos 2t) \\
x_p''(t) &= 2(-2A\sin 2t + 2B\cos 2t) + t(-4A\cos 2t - 4B\sin 2t)
\end{aligned}$$

So when we plug this into our equation, we find:

$$\begin{aligned}
x_p'' + 4x_p &= (-4A\sin 2t + 4B\cos 2t) + t(-4A\cos 2t - 4B\sin 2t) + 4t(A\cos 2t + B\sin 2t) \\
&= -4A\sin 2t + 4B\cos 2t
\end{aligned}$$

So if we want the RHS to match $\sin 2t$, we must select $B = 0$ and $A = -1/4$. Our solution is therefore:

$$x_p(t) = \frac{-t}{4}\sin 2t$$

Yes, but whoop-de-doo, why the big deal about this special case?

The big deal is that even though our forcing stay of a fixed amplitude, our solution now grows in time! This resonance phenomena is well-known and has caused famous issues in engineering (look the Tacoma bridge) and has a lot of cool applications in physics.

We will generalize our result next.

In general, if we have an oscillator without friction, it can be forced at any frequency

$$x'' + \omega^2 x = a\cos\alpha t$$

We say that ω is the *natural* frequency of the system (the frequency of the homogeneous solution)
and that α is the forcing frequency.
Note that the same discussion works if the forcing is a $\sin$ or a $\cos$.

If the frequencies do NOT match, we have $\alpha \neq \omega$, and the homogeneous solution is:

$$x_p(t) = A\cos\omega t + B\sin\omega t = M\cos(\omega t - \phi)$$

with $M\cos\phi = A$ and $M\sin\phi = B$, or $A^2 + B^2 = M^2$ and $\tan\phi = B/A$.

The particular solution is

$$x_p = C\cos\alpha t, \quad \text{with} \quad C = \frac{a}{\omega^2 - \alpha^2}$$

So what happens if $\alpha \to \omega$? The amplitude of the particular solution tends to infinity (it blows up!). In the case $\alpha = \omega$, this form of the particular solution is no longer valid, and we need to multiply our cosine term by t, as we saw.

Realistically, there will always be some friction, or damping, in the system

$$x'' + \lambda x' + \omega^2 x = a \cos \alpha t.$$

The forced system is then not exactly resonant because the homogeneous solution has an exponential decay.

One can find a particular solution by guessing $x_p = A \cos \alpha t + B \sin \alpha t$. More precisely, one finds

$$x_p(t) = a \left(\frac{\omega^2 - \alpha^2}{(\omega^2 - \alpha^2)^2 + \lambda^2 \alpha^2} \cos \alpha t + \frac{\lambda \alpha}{(\omega^2 - \alpha^2)^2 + \lambda^2 \alpha^2} \sin \alpha t \right)$$

This is a complicated expression, but if we focus on the amplitude of the solutions, we see that it can get very large (but stay finite) if:
λ is small (too much friction removes the possibility of resonance)
α is close to ω (the forcing frequency still needs to nearly match the natural frequency)

Lecture 8

Higher order equations with constant coefficients

If we are faced with a higher order, constant coefficient, differential equation, such as

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + .. + a_1 x' + a_0 x = F(t)$$

what changes when we look for a solution $x(t)$?

The answer is that the principles are the same. In practice though, it can get messy very quickly.

So we would still do:

1. Solve the homogeneous equation by trying $x_h(t) = e^{rt}$, and looking for the n roots of the resulting polynomial.

   $$a_n r^n + a_{n-1} r^{n-1} + .. + a_1 r + a_0 = 0$$

   Note that if you allow complex roots and count multiplicity, there are always n roots to an nth degree polynomial (Fundamental theorem of algebra). Unfortunately, looking for roots can be hard.

2. Look for a particular solution using the same type of educated guess as for second order problems.

3. Use the n initial conditions to determine the coefficients multiplying the homogeneous solutions.

We will do one **example**:
$x''' - x = t$, with $x(0) = 2, x'(0) = (-1 - \sqrt{3})/2$, and $x''(0) = 2$.

First, we look for solutions of the homogeneous problem, $x_h(t) = e^{kt}$ by solving:
$k^3 - 1 = 0$.
We can see that $k = 1$ is a root. To find the rest, we first divide:

$$\frac{k^3 - 1}{k - 1} = k^2 + k + 1$$

and then use the quadratic formula: $k = \frac{-1}{2} \pm \sqrt{1/4 - 1} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$.

We can therefore write our homogeneous solution as

$$x_h(t) = C_1 e^t + C_2 e^{-t/2} \cos(\frac{\sqrt{3}}{2} t) + C_3 e^{-t/2} \sin(\frac{\sqrt{3}}{2} t)$$

Now for a particular solution, we guess $x_p(t) = \alpha t + \beta$. This implies $x_p' = \alpha$, $x_p'' = 0$ and $x_p''' = 0$.
Plugging in, we have:

$$x''' - x = 0 - \alpha t - \beta = t$$

So we must have $\beta = 0$ (matching the constants) and $\alpha = -1$, matching the coefficients of t. We find:

$$x_p(t) = -t$$

So altogether, our general solution and its derivatives are

$$\begin{aligned}
x_g(t) &= C_1 e^t + C_2 e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + C_3 e^{-t/2}\sin(\frac{\sqrt{3}}{2}t) - t \\
x_g'(t) &= C_1 e^t + ((\sqrt{3}/2)C_3 - (1/2)C_2)e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + ((-\sqrt{3}/2)C_2 - (1/2)C_3)e^{-t/2}\sin(\frac{\sqrt{3}}{2}t) - 1 \\
x_g''(t) &= C_1 e^t + (-(\sqrt{3}/4)C_3 + C_2/4 - 3/4C_2 - (\sqrt{3}/4)C_3)e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + \\
&\quad (\sqrt{3}/4C_2 + C_3/4 - (3/4)C_3 + \sqrt{3}/4C_2)e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)
\end{aligned}$$

That was... a lot of calculations. In any case, what we want are the values at $t = 0$, and we will set those equal to our initial conditions. So we have

$$\begin{aligned}
x_g(0) &= C_1 + C_2 = 2 \\
x_g'(0) &= C_1 - (1/2)C_2 + (\sqrt{3}/2)C_3 - 1 = \frac{-1-\sqrt{3}}{2} \\
x_g''(0) &= C_1 + \frac{1-\sqrt{3}}{4}C_2 + \frac{-\sqrt{3}-3}{4}C_3 = 2
\end{aligned}$$

If you solve this system you find that $C_1 = 1$, $C_2 = 1$, and $C_3 = -1$ (funny how well that works out).

So finally, our solution is:

$$x_g(t) = e^t + e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) - e^{-t/2}\sin(\frac{\sqrt{3}}{2}t) - t$$

Cauchy-Euler Equations

In general, we cannot find closed-form solutions when the coefficients of a differential equation are not constant. One significant exception are the equations of the Cauchy-Euler type:

$$at^2\frac{d^2x}{dt^2} + bt\frac{dx}{dt} + cx = f(t)$$

Note that here the UNITS of a, b, and c are all the same (usually unitless)

$$\left[t^2 \frac{d^2x}{dt^2}\right] = T^2 \frac{[x]}{[T^2]} = [x]$$

We can get from a Cauchy-Euler equation to an equation constant coefficients by substituting: $t = e^u$, or $u = \log t$. When the system is not-homogeneous, there is no other good way to look for a particular solution. However, for the homogeneous case, we can try the Ansatz: $x(t) = t^k$
What do we then get?

$$at^2(k)(k-1)t^{k-2} + bt(k)t^{k-1} + ct^k = 0$$

which simplifies to

$$t^k(a(k)(k-1) + b(k)c) = t^k\left[ak^2 + (b-a)k + c\right] = 0$$

This is another quadratic equation to solve for k. We can find two roots, k_1 and k_2 according to

$$k_{1,2} = \frac{a - b \pm \sqrt{(a-b)^2 - 4ac}}{2a}$$

Again, we have three cases:

1. 2 real roots, the solution is $x(t) = At^{k_1} + Bt^{k_2}$
2. Repeated (real) root: $x(t) = (A + B\log t)t^{k_1}$.
3. 2 complex roots $k = \alpha + i\beta$ requires us to know what $t^{\alpha+i\beta}$ means.

$$t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{i\beta\log t} = t^\alpha(\cos(\beta\log t) + i\sin(\beta\log t))$$

If we rearrange terms and use both the $\alpha + i\beta$ root and the $\alpha - i\beta$ root, we can find that our homogeneous solution is:

$$x(t) = C_1 t^\alpha \cos(\beta\log t) + C_2 t^\alpha \sin(\beta\log t)$$

Before we do examples, we consider what happens to the equation as $t \to 0$

$$at^2\frac{d^2x}{dt^2} + bt\frac{dx}{dt} + cx = 0$$

So as $t \to 0$, we must have either:

- $c = 0$, OR
- $x = 0$, OR
- $x' \to \infty$, OR
- $x'' \to \infty$

This is indeed confirmed by our solution. It shows that $t = 0$ is therefore not a suitable point for initial conditions, as only certain values of the function or its derivative are possible. This is because here $t = 0$ is NOT a REGULAR point. It is a SINGULAR point.

This is more evident if we divide the equation by t^2:

$$a\frac{d^2x}{dt^2} + \frac{b}{t}\frac{dx}{dt} + \frac{c}{t^2}x = 0$$

where we can see that there are issues when $t = 0$ because we would then divide by 0. We will return to this concept later in the class.

Now, let's look at some examples
Example 1: $2t^2x'' + 3tx' - x = 0$
We get the quadratic: $2k(k-1) + 3k - 1 = 0$
This is the same as $2(k^2 + (1/2)k - 1/2) = 2(k+1)(k-1/2) = 0$.
Therefore the roots are $k_1 = -1$ and $k_2 = 1/2$.
Our solution is thus: $x(t) = At^{-1} + Bt^{1/2}$.

Example 2: $t^2x'' + 3tx' + x = 0$
We get the quadratic: $k(k-1) + 3k + 1 = 0$
This is the same as $k^2 + 2k + 1 = 0$.
Therefore that the only root is $k_1 = -1$
Our solution is thus: $x(t) = At^{-1} + Bt^{-1}\log t = \frac{A}{t} + \frac{B\log t}{t}$.

We should check, to make sure:

$$\begin{aligned}
x(t) &= \frac{A}{t} + \frac{B\log t}{t} \\
x'(t) &= \frac{-A}{t^2} + \frac{B}{t^2} - \frac{B\log t}{t^2} \\
x''(t) &= \frac{2A}{t^3} + \frac{-2B}{t^3} + \frac{2B\log t}{t^3} - \frac{B}{t^3} \\
t^2x'' + 3tx' + x &= \frac{2A}{t} + \frac{-3B}{t} + \frac{2B\log t}{t} + \frac{-3A}{t} + \frac{3B}{t} - \frac{3B\log t}{t} + \frac{A}{t} + \frac{B\log t}{t} \\
&= 0A + 0B + 0B\log t
\end{aligned}$$

Example 3: $t^2x'' - tx' + 5x = 0$
We get the quadratic: $k(k-1) - k + 5 = 0$

This is the same as $k^2 - 2k + 5 = (k+1)^2 = 0$.
The roots are thus $k_{1,2} = 1 \pm \sqrt{1-5} = 1 \pm 2i$
Our solution is thus: $x(t) = At\cos(2\log t) + Bt\sin(2\log t)$.

We can check this one too:

$$\begin{aligned}
x(t) &= At\cos(2\log t) + Bt\sin(2\log t)\\
x'(t) &= A(\cos(2\log t) - 2\sin(2\log t)) + B(\sin(2\log t) + 2\cos(2\log t))\\
x''(t) &= A(\frac{-2}{t}\sin(2\log t) - \frac{4}{t}\cos(2\log t)) + B(\frac{2}{t}\cos(2\log t) - \frac{4}{t}\sin(2\log t))\\
t^2x'' - tx' + 5x &= A((-2t)\sin(2\log t) - (4t)\cos(2\log t)) + B((2t)\cos(2\log t) - (4t)\sin(2\log t)) +\\
&\quad -(A(t\cos(2\log t) - 2t\sin(2\log t)) + B(t\sin(2\log t) + 2t\cos(2\log t))) +\\
&\quad 5(At\cos(2\log t) + Bt\sin(2\log t))\\
&= A((2-2)t\sin(2\log t) + (-4-1+5)t\cos(2\log t)) +\\
&\quad B((-4-1+5)t\sin(2\log t) + (2-2)t\cos(2\log t)) = 0A + 0B = 0
\end{aligned}$$

Lecture 9

Review of Power Series

You may recall studying sequences: $\{s_1, s_2, ...s_n...\} = \{s_n\}_{n=1}^{\infty}$.

Typically, you have a general term $s_n = f(n)$. For example, you can generate a sequence from $s_n = \frac{n+2}{n}$: 3, 2, 3/2, 7/5, 4/3, 9/7...

Alternatively, you may have a recursive form:

$$s_n = \frac{s_{n-2}}{n} \quad \text{with starting points} \quad s_1 = 1, \quad s_2 = 2.$$

which gives $s_3 = 1/3,\ s_4 = 1/2,\ s_5 = 1/15,\ s_6 = 1/12$, and so on.

Formally, we say that the limit

$$\lim_{n\to\infty} s_n = S$$

exists if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n > N$, then $|s_n - S| < \epsilon$.

Sequence with a limit. You may study such formal definitions in details in Math 101.

We mostly use sequences in the context of Series, a special type of sequence. We define PARTIAL SUMS as

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + ... + a_n$$

We then say that

$$\sum_{k=0}^{\infty} a_k = S \quad \text{if and only if} \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = S$$

There are several tests that may be used to check for convergence of a Series: Integral test, root test, ratio test, alternating series test, ratio test.
We focus on the last two:

Alternating Series test:

$$\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k a_k$$

is a convergent Series (has a limit) if $\lim_{k\to\infty} a_k = 0$ and a_k are monotonic

Ratio test: For $\sum_{k=0}^{\infty} a_k$, consider the ratio of consecutive terms:

$$L = \left|\frac{a_{k+1}}{a_k}\right|$$

- If $L < 1$, the Series converges.
- If $L > 1$, the Series diverges.
- $If L = 0$, the test is inconclusive.

Example:

$$\sum_{k=0}^{\infty} (-1)^k \frac{k^3}{2^k}$$

This is an alternating series, and we have $a_k = \frac{k^3}{2^k}$.

We (should) know that

$$\lim_{k\to\infty} \frac{k^3}{2^k} = 0$$

Moreover, the general term is monotonic, as in it keeps getting smaller and smaller. Therefore, the Series must be convergent.

Is is *Absolutely* convergent? This is asking if

$$\sum_{k=0}^{\infty} \left|(-1)^k \frac{k^3}{2^k}\right| = \sum_{k=0}^{\infty} \frac{k^3}{2^k}$$

is convergent. We can now use the ratio test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^3}{2^{k+1}} \frac{2^k}{k^3} = \lim_{k\to\infty} \frac{(k+1)^3}{k^3} \frac{1}{2} = \frac{1}{2}$$

Because the limit is less than 1, the Series is convergent, so our original Series is Absolutely convergent.

We will focus here on **Power Series**, which have the form

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + ...$$

For these series, three things can happen (and only 3).

1. There exists $R > 0$ such that the Series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$ (when $|x - x_0| = R$, the convergence depends).
2. The Series converges only for $x = x_0$, and it converges to c_0. (I guess you can then say that $R = 0$).
3. The Series converges for every real x. (You can then say that $R = \infty$).

The number R is known as the radius of convergence.

For values of x where the Series converges, we can define the function

$$f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$$

Given a function $f(x)$, the radius of convergence of the power series equal to $f(x)$ can be found as the distance between x_0 and the closest point in the complex plane where the function is NOT analytic. This is studied in detail in Math 122, but you can use the idea here if you are looking for a power series of $f(x)$. You simply need to identify where $f(x)$ is not analytic, which is to say where it "gets in trouble", for example by dividing by 0 or taking the log of 0.

Let's look at a famous example:

$$f(x) = \sum_{k=0}^{\infty} r^k x^k = 1 + rx + r^2x^2 + ... + r^n x^n + ... \text{ when it converges}$$

We can define the partial sum $f_n(x) = \sum_{k=0}^{n} r^k x^k$.

Now here is an important trick about this series:

$$f_n(x) - (rx)f_{n-1}(x) = 1 + rx + r^2x^2 + ... + r^n x^n - (1 + rx + r^2x^2 + ... + r^{n-1}x^{n-1}) = 1 - r^n x^n$$

which always holds. We would like to take a limit as $n \to \infty$.
We can see from the RHS that to get convergence, we will need $|rx| < 1$. In that case, we get

$$\lim_{n\to\infty} f_n(x) - (rx)f_{n-1}(x) = 1$$

This means that our original function satisfies

$$f(x) - rxf(x) = 1 \quad \text{if } |rx| < 1.$$

Therefore, we can get a closed form for $f(x)$:

$$f(x) = \frac{1}{1 - rx} \quad \text{if } |x| < 1/|r| = R.$$

and for other values of x the function is not defined.

Note that you can rewrite similar function as multiple of the function above. For example:

$$\frac{2}{3 + 5x} = \left(\frac{2}{3}\right) \frac{1}{1 - (-5/3)x} = \left(\frac{2}{3}\right) \sum_{k=0}^{\infty} \left(-\frac{5}{3}\right)^k x^k \quad \text{if } |x| < 3/5$$

Here, you could also have found the radius of convergence by noting that the original function is singular at $x = -3/5$.

Lecture 10

Properties of Power Series

Power Series have some very convenient properties, especially in the context of differential equations.

Suppose we have $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ for $|x| < R$ (note that here we consider that our Series are centered at $x_0 = 0$, to simplify the notation). The following properties then hold, for $|x| < R$

1. $$f(x) \pm g(x) = \sum_{k=0}^{\infty} (a_k \pm b_k)\, x^k$$

2. $$f(x)g(x) = \sum_{k=0}^{\infty} (c_k)\, x^k \quad \text{with} \quad c_k = \sum_{n=0}^{k} a_n b_{k-n}$$

3. $$\frac{df}{dx} = \sum_{k=1}^{\infty} k\ a_k\ x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1}\, x^k$$

4. $$\int f(x)\, dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^{k+1} + C = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k}\, x^k + C$$

How can this be useful? Let's see an example:

Example Consider $f(x) = \frac{1}{(1+x)^2}$. Find a Power Series representation for $f(x)$ for $|x| < 1$.

$$\begin{aligned} f(x) &= \frac{d}{dx}\left(\frac{-1}{1+x}\right) \\ &= \frac{d}{dx} \sum_{k=0}^{\infty} (-1)^{k+1} x^k \\ &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{dx^k}{dx} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1} \\ &= \sum_{k=0}^{\infty} (-1)^k (k+1) x^k \end{aligned}$$

One more property we will take advantage of a lot is that:
If we have $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ for $|x| < R$, then
$f(x) = g(x)$ if and only if $a_k = b_k$ for all $k = 0, 1, ...\infty$.

This will allow us to match coefficients term-by-term. We have already done this for finite sums

$$c_0 + c_1 x + c_2 x^2 = 3 + 7x + 11x^2$$

implies that $c_0 = 3$, $c_1 = 7$, and $c_2 = 11$.
But now we will also be able to do this for infinite Series.

Taylor Series and how to find them

We will now review the most famous Power Series, which are the Taylor Series. We begin with a definition:

Definition: A function $f(x)$ is **ANALYTIC** at a point x_0 if $\exists R > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad \text{for } |x - x_0| < R$$

for some sequence of coefficients a_k. When that sequence exists, it is unique.

Here, to simplify the notation, we will look at $x_0 = 0$, but these results hold around any point and the formulas we give can all be shifted (for example, by defining a variable $u = x - x_0$).

Taylor Series

For an analytic function, the coefficients in the Power Series given above satisfy

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Importantly, because those coefficients are unique, if you happen to know the Power Series of $f(x)$, you may use them to find derivatives of $f(x)$.

Example: $f(x) = \frac{1}{1-x}$.
We have seen that

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1.$$

This implies that $a_k = 1 = \frac{f^{(k)}(0)}{k!}$.
We can therefore conclude that $f^{(k)}(0) = k!$, (which deserves an exclamation point, but that is confusing with factorials!)

Moreover, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $|z| < R$, then you may replace z by anything and obtain a valid expression.

Example: $f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, for any z.
We therefore have that

$$f(3x) = e^{3x} = \sum_{k=0}^{\infty} \frac{(3x)^k}{k!}.$$

and so we could find the Power series for e^{3x} without taking derivatives.

Second example: $f(x) = \frac{1}{1+x^2}$.
Using our geometric Series, we have

$$f(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} \quad \text{for } |x| < 1.$$

Recall that for a general analytic function, we can find the radius of convergence R by looking for the distance to the closest point, IN THE COMPLEX PLANE, where $f(x)$ is NOT analytic (which is also called a point where $f(x)$ is singular).

So in our last example, $f(x) = \frac{1}{1+x^2}$, we have singularities if $1 + x^2 = 0$, which occurs if $x = \pm i$.
The distance in the complex plane between i and the origin is: $|i| = i(\bar{i}) = i(-i) = 1$.
So our radius of convergence is confirmed to be one.

Lecture 11

Power Series solutions to differential equations

Suppose we have an equation of the form

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

where the coefficients are not necessarily constants. Note that here the coefficient of y'' must be ONE.

We will focus our attention on the point $x = 0$, and assume that this is where initial conditions are given. In general, this could be done at any point where we have initial conditions.

If the coefficients $p(x)$ and $q(x)$ are analytic at $x = 0$, we say that $x = 0$ is an ORDINARY point of the D.E.. We like those.

At an ordinary point, the solution $y(x)$ is also analytic. So we can write it as

$$y(x) = \sum_{k=0}^{\infty} a_k x^k.$$

"All" we need to figure out are the coefficients a_k (which are constants). Note that if we had initial conditions are a general point x_0, we would consider the general form of the Taylor Series:

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

So how do we find those coefficients? Let's try a familiar case and study it as a power Series

$$y'' + y = 0$$

Here $x = 0$ is an ordinary point, so we try $y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + ...$

First, we need to compute derivatives:

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... = \sum_{k=1}^{\infty} a_k (k) x^{k-1}$$

Note that the sum now starts at $k = 1$, not zero, because the derivative of a constant is 0. This will be key. We also have

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 ... = \sum_{k=2}^{\infty} a_k (k)(k-1) x^{k-2}$$

In these problems, we need to be VERY CAREFUL with our indices, including where they start.

In general, it is best to write all the terms of the equation as powers of the same form, say x^n. So we will rewrite some terms:

For $y(x)$, not much changes, we just set $n = k$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

For $y'(x)$, we set $n = k - 1$ and so $n + 1 = k$.

$$y'(x) = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n.$$

For $y''(x)$, we set $n = k - 2$ and so $n + 2 = k$.

$$y''(x) = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n.$$

This was tedious, but this process is common to all second order ODEs. So in the future we can start from here. We are now ready to plug into our original D.E. which was $y'' + y = 0$. We get:

$$0 = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + a_n]x^n$$

This series can only be 0 if all the coefficients of x^n are zeros themselves (an x^2 cannot cancel an x^3). So we must have that

$$a_{n+2}(n+2)(n+1) + a_n = 0 \quad \text{for } n = 0, 1, 2...$$

So we can get coefficients with bigger indices in terms of coefficients with smaller indices as

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \quad \text{or} \quad a_k = -\frac{a_{k-2}}{(k)(k-1)}$$

Note: This doesn't tell us how to start. So the first two constants, a_0 and a_1 are undetermined constants. We can find them if initial conditions are given.

So, what do we get? We can now find all coefficients in terms of a_0 and a_1:

$$\begin{array}{rr} a_0 \ \text{free} & a_1 \ \text{free} \\ a_2 = -\dfrac{a_0}{2 \cdot 1} = -\dfrac{a_0}{2} & a_3 = -\dfrac{a_1}{3 \cdot 2} = -\dfrac{a_1}{6} \\ a_4 = -\dfrac{a_2}{4 \cdot 3} = \dfrac{a_0}{4 \cdot 3 \cdot 2} & a_5 = -\dfrac{a_3}{5 \cdot 4} = -\dfrac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} \\ a_{2n} = \dfrac{(-1)^n a_0}{(2n)!} & a_{2n+1} = \dfrac{(-1)^{n+1} a_1}{(2n+1)!} \end{array}$$

Great, so we have formulas. What does this means? It means that we can plot approximations of the solution. Try $a_0 = 1$ and $a_1 = 2$ and build some series, we can plot them to see the results:

One solution using Power Series, using different number of terms

More theoretically, we can write:

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

so in fact $y(x) = a_0 \cos x + a_1 \sin x$. It is magical!

Perhaps more importantly, this gives us an excellent way to evaluate trig functions. How do you think calculators can evaluate $\sin(1)$? They (used to) use power series.

A few notes:

- We can determine the free constants a_0 and a_1 using initial conditions: $y(0) = a_0$ and $y'(0) = a_1$.
- We may only be able to get a recursive formula for the coefficients a_n. That works just fine.
- Most solutions to D.E.s are calculated approximately in the end. Here, we would use $y(x) = \sum_{n=0}^{N} a_n x^n$.
- Watch your indices!

Let's try another example. The Airy equation is:

$$y'' - xy = 0$$

What do we expect the solutions to look like? If $x = X \gg 1$, this looks like $y'' - Xy = 0$ for some constant X. This should give exponentials (one growing, one decaying so the sum is growing). If $-x = X \gg 1$, this looks like $y'' + Xy = 0$. This should give some oscillations. Note that no elementary function oscillates for $x < 0$ and grows exponentially for $x > 0$.

We try as above, a power series centered at $x = 0$. This is still an ordinary point because the coefficient of y is $q(x) = x$, an analytic function. So we get:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y''(x) = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n.$$

Plugging in, we find

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We would like all powers to look the same, so let $k = n+1$ for the second term. Note that this is a dummy index. We can also set $k = n$ in the first term, to get everything in terms of k:

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} a_{k-1}x^k = 0$$

Now we have to be careful, because only one sum has a term when $k = 0$. We will separate it out:

$$a_2(2)(1) + \sum_{k=1}^{\infty} \left[a_{k+2}(k+2)(k+1) - a_{k-1}\right] x^k = 0$$

So we have two cases. The constant term must be zero, so

$$2a_2 = 0 \quad \text{and so} \quad a_2 = 0.$$

All the other powers can give us a recursion relation

$$a_{k+2}(k+2)(k+1) - a_{k-1} = 0 \quad \text{so} \quad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}$$

or if you prefer, setting $n = k+2$

$$a_n = \frac{a_{n-3}}{(n)(n-1)}$$

Once again, and this is the norm, we do not have constraints on a_0 and a_1, they can be found using initial conditions. We get

$$\begin{array}{rrr} a_0 \ \text{free} & a_1 \ \text{free} & a_2 = 0 \\ a_3 = \dfrac{a_0}{3 \cdot 2} = \dfrac{a_0}{6} & a_4 = \dfrac{a_1}{12} & a_5 = 0 \\ a_6 = \dfrac{a_3}{6 \cdot 5} = \dfrac{a_0}{6 \cdot 5 \cdot 3 \cdot 2} & a_7 = \dfrac{a_4}{7 \cdot 6} = \dfrac{a_1}{7 \cdot 6 \cdot 4 \cdot 3} & a_8 = 0 \end{array}$$

What does it look like? Let's try.

Series solution of the Airy equation.

Lecture 12

Power Series of Singular points

We will now consider the case where $x = 0$ is NOT an ordinary point. It is then called a SINGULAR point. So in the equation

$$y'' + p(x)y' + q(x)y = 0$$

Either $p(x)$ or $q(x)$ or both are not analytic at $x = 0$.

We will focus on the REGULAR singular points. These are singular points, but where the singularity is not *too* bad.

More precisely, if $g(x) = xp(x)$ and $h(x) = x^2q(x)$ are both analytic at $x = 0$, then $x = 0$ is a regular singular point.
Note: It is possible that $g(x)$ and $h(x)$ are not defined at $x = 0$, but if it is possible to define them to get an analytic function, that is sufficient. The typical example is if $g(x) = x/x$.

Example:

$$p(x) = \frac{\cos x}{x} \quad \text{and} \quad q(x) = \frac{e^x}{x^2}$$

Then we get

$$g(x) = \frac{x \cos x}{x} = \cos x \quad \text{if } x \neq 0 \quad \text{and } h(x) = \frac{x^2 e^x}{x^2} = e^x \quad \text{if } x \neq 0$$

So here both $g(x)$ and $h(x)$ are analytic, and $x = 0$ is a regular singular point.

In such cases, we can multiply our initial equation by x^2 and get

$$x^2y'' + x^2p(x)y' + x^2q(x)y = x^2y'' + xg(x)y' + h(x)y = 0$$

with all the coefficients in the last equation being analytic at $x = 0$.

Note: If a point is singular but not regular, it is said to be IRREGULAR (no surprise there). These equations are a lot messier and are beyond Math 125 (they are covered in Math 223 though).

Frøbenius Method

The Frøbenius method seeks power Series solutions to differential equations around a regular singular point. It is similar to what we did before, with one key difference: The first term of the Series is not necessarily a constant, rather it can be a power of x of the form x^r for some complex number r to be determined. So we start from

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r} \quad \text{for } |x| < R \text{ for some number } R \text{ and for } r \in \mathbb{C}$$

We are guaranteed that at least one solution will be of this form. Usually, we can find two independent solutions of that form.

Note: The exponent r is NOT always an integer!

Let's see an example:

$$x^2 y'' - xy' + (x^2 + \frac{3}{4})y = 0 \quad \text{or equivalently} \quad y'' - \frac{y'}{x} + (1 + \frac{3}{4x^2})y = 0$$

So here, $p(x) = 1/x$ and $q(x) = (x^2 + 3/4)/x^2$, and neither are analytic at $x = 0$.
However, $g(x) = xp(x) = 1$ and $h(x) = x^2 q(x) = x^2 + 3/4$ are both analytic at $x = 0$.
So $x = 0$ is a regular singular point.

In this method, we will always compute $y(x)$, $y'(x)$ and $y''(x)$ the same way:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}; \quad y'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}; \quad y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}$$

Now we can plug this in our equation. It is easier to avoid fractions, so
we use the form $x^2 y'' + x^2 p(x) y' + x^2 q(x) y = x^2 y'' + x g(x) y' + h(x) y = 0$. Here we get:

$$x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} - x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} + (x^2 + 3/4) \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Rearranging to get the same exponents everywhere, we get

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} - \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} + \sum_{k=0}^{\infty} \frac{3a_k}{4} x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r} = 0$$

where in the last term, we use the substitution $n = k + 2$, and then relabelled n as k.

The first thing to do is to solve for r. This is done by looking at the term with the smallest power of x in the equation. Here, this is the term x^r, where $k = 0$. The equation then becomes:

$$(r)(r-1) a_0 x^r - (r) a_0 x^r + \frac{3a_0}{4} x^r = a_0 x^r \left[(r)(r-1) - (r) + \frac{3}{4} \right] = 0$$

Note that the last sum did not contribute, since it starts at $k = 2$.

We will always assume that $a_0 \neq 0$. This is because we need to start the power Series somewhere, so by definition, a_0 is the first term that is not vanishing. So we get an algebraic equation for r

$$(r)(r-1) - (r) + \frac{3}{4} = r^2 - 2r + \frac{3}{4} = 0$$

This can be factored to yield $r_1 = 1/2$ and $r_2 = 3/2$. This is good, we found two values of r! However, we will need to be careful because $r_2 - r_1$ is an integer and that requires a bit more care.

Let's start with $r_1 = 1/2$, and recall that $a_0 \neq 0$ is a free constant.

We now look at the case $k = 1$, which will still not have any contribution from the last sum:

$$x^{r_1+1} a_1 \left[(1 + r_1)(r_1) - (1 + r_1) + \frac{3}{4} \right] = 0$$

This simplifies to

$$x^{r_1+1} a_1 (r_1^2 - 1/4) = 0$$

However, we know that $r_1 = 1/2$ so the last term here is always 0. This means that a_1 will also be a free constant! We can get both independent solutions from $r_1 = 1/2$. Note that since there are only two independent solutions, when using $r_2 = 3/2$, we will recover a solution we can already find here for $r_1 = 1/2$ (since there are only two independent solutions, once you have found two of them, you are done).

We can now look at the general case where $k \geq 2$, which involves all terms within the sum:

$$x^{k+1/2} \left[(k + 1/2)(k + 1/2 - 1)a_k - (k + 1/2)a_k + \frac{3a_k}{4} + a_{k-2} \right] = 0$$

This simplifies to

$$x^{k+1/2} \left[a_k \left((k + 1/2)(k - 3/2) + \frac{3}{4} \right) + a_{k-2} \right] = x^{k+1/2} \left[a_k \left(k(k-1) \right) + a_{k-2} \right] 0$$

So we can get a recursive formula

$$a_k = (-1) \frac{a_{k-2}}{k(k-1)}.$$

We have had this formula before! This is what appears in the Series for both Sine and Cosine! We can rewrite it for the even terms as

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}$$

and similarly for the odd terms

$$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}.$$

So we get our general solution

$$y(x) = x^{1/2}\left[a_0 \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

or in other words $y(x) = x^{1/2}[a_0 \cos x + a_1 \sin x]$.

Note that if we looked at $r_2 = 3/2$, the Series would start with a term $x^{3/2}$ which is exactly the form of the term $x^{1/2}a_1 \sin x$. So we can get this second solution either by noticing that for $r_1 = 1/2$ we have that a_1 is a free constant, or by looking at the case $r_2 = 3/2$ separately. There are only two independent solutions, so we cannot get more independent solutions than the two we just found.

Bessel Equation

We will do one last example, the Bessel Equation. This comes up when solving the Laplace Equation in a circular geometry, for example to describe the oscillations of the membrane of a drum. It generally has a parameter, ν. The equation is:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{or} \quad y'' + \frac{y'}{x} + \frac{x^2 - \nu^2}{x^2}y = 0$$

We have that $p(x) = 1/x$ and $q(x) = \frac{x^2-\nu^2}{x^2}$, neither of which is analytic at $x = 0$.

However, $g(x) = xp(x) = 1$ and $h(x) = x^2q(x) = x^2 - \nu^2$ are both analytic at $x = 0$.

So $x = 0$ is a regular singular point, and we can use Frøbenius's method.

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}; \quad y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}; \quad y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Plugging in the Bessel equation, we get

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} - \nu^2 \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Rewriting everything with the same power (using $n = k$ and then $n = k + 2$ for the third sum), we get

$$\sum_{k=0}^{\infty}[(k+r)(k+r-1) + (k+r) - \nu^2]a_k x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r} = 0$$

We now look at the smallest power of x, which is for $k = 0$. The last sum does not contribute yet, since it starts at $k = 2$. We get:

$$((r)(r-1) + (r) - \nu^2)a_0 = (r^2 - \nu^2)a_0 = 0.$$

Recall that we assume that $a_0 \neq 0$ so we must have that $r = \pm\nu$.

In general, if $\nu \neq k/2$ for $k \in \mathbb{Z}$, then we have two independent solution.
If $\nu = k/2$, and $k \neq 0$, then $r_1 - r_2 = k$ will be an integer, and we have to be careful to find two independent solutions.

We will focus in this example on the case $\nu = 0$, for which we only have one value of r, which is $r = 0$.

So for $\nu = 0$, the equation is

$$x^2y'' + xy' + x^2y = 0$$

has solutions of the form $y(x) = \sum_{k=0} a_k x^k$, which are analytic functions.

We now look at the term involving x^1, for which the last sum still does not contribute. We get, recalling that $r = 0$ and $\nu = 0$:

$$[(1+r)(1+r-1) + (1+r) - \nu^2]a_1 = [(1)(1-1) + (1)]a_1 = a_1 = 0$$

So we conclude that $a_1 = 0$.

For higher powers of x, we get

$$[(k+r)(k+r-1)+(k+r)-\nu^2]a_k+a_{k-2}]x^{k+r} = [(k)(k-1)+(k)]a_k+a_{k-2}]x^k = [k^2a_k+a_{k-2}]x^k0$$

So we get that $a_k = -\frac{a_{k-2}}{k^2}$. Because $a_1 = 0$, we will only get terms with even indices.

Using the recursion formula, we can find a close formed formula (though this is not really important)

$$a_k = -\frac{a_{k-2}}{k^2} = \frac{a_{k-4}}{k^2(k-2)^2} = -\frac{a_{k-6}}{k^2(k-2)^2(k-4)^2}\cdots$$

and if we let $k = 2n$, we can simplify

$$a_{2n} = -\frac{a_{2(n-3)}}{(2n)^2(2(n-1))^2(2(n-2))^2} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}$$

and we can write our solution as

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} = a_0 J_0(x)$$

In general, *Bessel functions* of order ν are denoted as $J_\nu(x)$ (the ones that are analytic). Note that because only even powers appear, $J_0(x)$ is an even function itself.

This was only one solution. In general, to find a second solution $y_2(x)$ when we only found one solution $y_1(x)$, we need to look for solutions of the form

$$y_2(x) = y_1(x)\ln x + \sum_{k=0}^{\infty} b_k x^{k+r}$$

and solve for the coefficients b_k (this r is the same that was found before).

This is, in all honesty, pretty messy, but has been done and is well documented. It yields Bessel functions of the second kind, denoted $Y_0(x)$. Importantly, these functions are NOT analytic at $x = 0$, where they behave like a logarithm. They look like:

$$Y_0(x) = J_0(x) \ln x + \sum_{k=0}^{\infty} b_k x^k.$$

Lecture 13

System of 2 Differential Equations and Matrix exponential

It is always possible to convert a second order linear equation with constant coefficients into a system of two first order equations. From

$$y'' + ay' + by = 0,$$

we introduce $z_x = y$ and $z_y = y'$, the components of a vector $\vec{z}$. We get its derivative as

$$\frac{d\vec{z}}{dt} = \frac{d}{dt}\begin{pmatrix} z_x \\ z_y \end{pmatrix} = \begin{pmatrix} z_y \\ -bz_x - az_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} z_x \\ z_y \end{pmatrix}$$

We are now going to consider the more general case where any matrix A, of general form

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can appear on the right-hand-side, so that we have

$$\vec{z}\,' = \frac{d\vec{z}}{dt} = \mathrm{A}\vec{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_x \\ z_y \end{pmatrix}$$

So we consider linear, homogeneous systems of 2 first order equations.

We can still take advantage of the superposition principle, even for vector solutions, so we expect to find solutions up to a multiplicative constant:

$$\vec{z}(t) = c_1\vec{w}_1(t) + c_2\vec{w}_2(t)$$

with $\vec{w}_1$ and $\vec{w}_2$ both satisfying $\vec{w}_i\,' = \mathrm{A}\vec{w}_i$, for $i = 1$ or 2.

To find solutions $\vec{w}(t)$, we will try a similar guess as before, but now in vector form:

$$\vec{w}(t) = e^{\lambda t}\vec{v}$$

for some constant scalar λ and constant vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ to be determined.

What is the derivative of this guess?

$$\frac{d\vec{w}}{dt} = \frac{d(e^{\lambda t})}{dt}\vec{v} = \lambda e^{\lambda t}\vec{v} = \lambda\vec{w}$$

Because $\vec{w}$ is a solution to our original matrix DE, we get that

$$\frac{d\vec{w}}{dt} = \mathrm{A}\vec{w} = \lambda\vec{w}$$

Equivalently, because $e^{\lambda t}$ is a scalar, we obtain that $\mathrm{A}\vec{v} = \lambda\vec{v}$.

So that means that:
λ is an eigenvalue of A, and
$\vec{v}$ is an eigenvector of A.

In the most common, and simplest, case we have 2 independent eigenvectors, $\vec{v}_1$ and $\vec{v}_2$, each with an eigenvalue λ_1 and λ_2.
Notes:
1) $\lambda_1 = \lambda_2$ is possible. What matters is that we can get $\vec{v}_1$ independent from $\vec{v}_2$.
2) For $n \times n$ systems, we would be looking for n independent eigenvectors.
3) The eigenvalues of A completely determine the long-time behavior of the solution. If their real part is negative, the solutions approach zero (stable, attractive). If one real part if positive, the system is unstable.

Our general solution can then be written as

$$\vec{z}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}\vec{z}(0) = e^{\mathrm{A}t}\vec{z}(0)$$

where we introduced a bunch of notation that we will explain now.

First, we built a matrix P whose columns are the eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

$$\mathrm{P} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$$

Because the eigenvectors are independent, P is always invertible.

Then we build a diagonal matrix D that contains the eigenvalues along the diagonal:

$$\mathrm{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

We then have that

$$\mathrm{A}\ \ \mathrm{P} = \mathrm{A}\begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathrm{P}\ \ \mathrm{D}$$

So we have A P = P D, or A = P D P^{-1}.

We can use this notation to rewrite our solution. It is useful to think of the matrix P as one that allows a change of variable to a system where the equations are *decoupled* and so can be solved one at a time. Define

$$\vec{w} = \mathrm{P}^{-1}\vec{z} \quad \text{and equivalently} \quad \vec{z} = \mathrm{P}\vec{w}$$

We then have

$$\vec{w}\,' = \mathrm{P}^{-1}\vec{z}\,' = \mathrm{P}^{-1}\mathrm{A}\vec{z} = \mathrm{P}^{-1}\mathrm{P}\ \mathrm{D}\ \mathrm{P}^{-1}\vec{z} = \mathrm{D}\vec{w}$$

So if we write $e^{\mathrm{D}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$, we can write our solution for $\vec{w}$ as

$$\vec{w} = e^{\mathrm{D}t}\vec{w}(0)$$

Finally, we can return to $\vec{z}$ using $\vec{w} = \mathrm{P}^{-1}\vec{z}$.

$$\mathrm{P}^{-1}\vec{z} = e^{\mathrm{D}t}\mathrm{P}^{-1}\vec{z}(0) \quad \text{so} \quad \vec{z} = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}\vec{z}(0)$$

Finally, we can denote the coefficient of $\vec{z}(0)$ by the MATRIX EXPONENTIAL (which is itself a matrix) $e^{\mathrm{A}t}$

$$e^{\mathrm{A}t} = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}$$

so in the end we have $\vec{z}(t) = e^{\mathrm{A}t}\vec{z}(0)$.

Last bit of theory before looking at examples. This was not a very well justified way to define a matrix exponential. It just "looked right". A more rigorous approach would be to define it based on Taylor Series. Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We can use this definition for matrices, since we can take power of matrices just by multiplying a matrix by itself.

Here, we will need powers of A. They can be simplified using our notation:

$$\mathrm{A}^k = (\mathrm{P}\ \mathrm{D}\ \mathrm{P}^{-1})^k = (\mathrm{P}\ \mathrm{D}\ \mathrm{P}^{-1}\mathrm{P}\ \mathrm{D}\ \mathrm{P}^{-1}...\mathrm{P}\ \mathrm{D}\ \mathrm{P}^{-1}) = \mathrm{P}\ \mathrm{D}^k\mathrm{P}^{-1}$$

So we can compute the matrix exponential as

$$e^{\mathrm{A}t} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathrm{P}\ \mathrm{D}^k\mathrm{P}^{-1}t^k}{k!} = \mathrm{P}\left[\sum_{k=0}^{\infty} \frac{(\mathrm{D}t)^k}{k!}\right]\mathrm{P}^{-1} = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}$$

where the exponential of a diagonal matrix defined as above.

Summary

For a system of 2 DEs of the form $\vec{z}\,' = \mathrm{A}\vec{z}$, and if A has 2 independent eigenvectors:
- Denote the eigenvectors $\vec{v}_1$ and $\vec{v}_2$.
- Denote the corresponding eigenvalues as λ_1 and λ_2.
- The solution is $\vec{z}(t) = e^{\mathrm{A}t}\vec{z}(0) = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}\vec{z}(0) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$,
for some constants c_1 and c_2.

Examples: Find $e^{\mathrm{A}t}$ for the following matrices and determine the long term behavior of the solutions

$$\mathrm{A}_1 = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \quad \mathrm{A}_2 = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \quad \mathrm{A}_3 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

Lecture 14

Phase plane trajectories

We saw last time that for a system of 2 DEs of the form $\vec{z}\,' = \mathrm{A}\vec{z}$, the solution is
$\vec{z}(t) = e^{\mathrm{A}t}\vec{z}(0) = \mathrm{P}e^{\mathrm{D}t}\mathrm{P}^{-1}\vec{z}(0) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$,
for constants c_1 and c_2, eigenvalues λ_1 and λ_2, and independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

One good way to represent solutions to these problems in using the *phase plane*, which has one coordinate, z_x on the horizontal axis and the other, z_y, on the vertical axis. Here time t serves as a parameter, and the solutions are parametric curves in this plane.

We will explore the possible outlook of the phase planes, depending on the eigenvalues associated to A.

Real, distinct, eigenvalues

Best (simplest) case scenario: λ_1 and $\lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$.

We will suppose whenever possible that $\lambda_1 < \lambda_2$. We always use the same eigenvectors in our representations

Sample phase plane

In this case, one of the solutions is:
$\vec{z}(t) = \vec{v}_1 e^{\lambda_1 t}$ (you get this if $c_1 = 1$ and $c_2 = 0$.
If $\lambda_1 > 0$, this solution moves away from the origin along $\vec{v}_1$.
If $\lambda_1 < 0$, this solution moves toward from the origin along $\vec{v}_1$.

Another solution is $\vec{z}(t) = \vec{v}_2 e^{\lambda_2 t}$ (you get this if $c_1 = 0$ and $c_2 = 1$.
This behaves similarly, depending on the sign of λ_2.

Excluding zero eigenvalues (we will return to those), we can have 3 cases:

Case 1: $\lambda_1 < \lambda_2 < 0$
Stable node

Case 2: $0 < \lambda_1 < \lambda_2$
Unstable node

Case 3: $\lambda_1 < 0 < \lambda_2$
Saddle-point

Between eigenvectors, solutions slowly approach the direction of the eigenvector with the biggest eigenvalue, so $\vec{v}_2$.
For Case 1, every solution goes toward the origin.
For Case 2, every solution goes away from the origin.
For Case 3, every solution goes away from the origin, except those that start exactly along $\vec{v}_1$ (so with $c_2 = 0$).

What happens if an eigenvalue is 0?
In that case, the CONSTANT corresponding eigenvector is a solution: $\vec{z}(t) = c_i\vec{v}_i$ for $i = 1$ or 2. Solutions then only move over time in the direction parallel to the other eigenvalue.

Case 4: $\lambda_1 < \lambda_2 = 0$
Semi-stable node (inner comb)
Solutions tend to $c_2\vec{v}_2$

Case 5: $0 = \lambda_1 < \lambda_2$
Semi-unstable node (outer comb)
Solutions tend away from $c_1\vec{v}_1$

This may be a good time to mention that there are LOTS of cases. You shouldn't try to memorize them all. But you can UNDERSTAND them all.

If we have equal eigenvalues: $\lambda_1 = \lambda_2$, but still two independent eigenvectors, then all solutions are of the form $\vec{z}(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_1 t} \end{pmatrix}$.

Case 6: $\lambda_1 = \lambda_2 < 0$
Stable star

Case 7: $0 < \lambda_1 = \lambda_2$
Unstable star

All solutions stay along the vector connecting their starting point to the origin. They go inward or outward, depending on the sign of the eigenvalue.

Last case for today, what if both eigenvalues are 0?
Case 8: $\lambda_1 = \lambda_2 = 0$
The the solutions are of the form: $\vec{z}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. They never change in time!

Case 8: Constant solution (dots). It is stable.

Lecture 15

Complex Eigenvalues

What happens if we have complex eigenvalues? They will be of the form $\lambda_{1,2} = r \pm i\omega$. In that case, the eigenvectors will also be complex. Note also that here there are always 2 distinct eigenvalues.

Our solutions are then of the form:

$$\begin{aligned}
\vec{z}(t) &= c_1\vec{v}_1 e^{(r+i\omega)t} + c_2\vec{v}_2 e^{(r-i\omega)t} \\
&= e^{rt}\left[c_1\vec{v}_1 e^{i\omega t} + c_2\vec{v}_2 e^{i\omega t}\right] \\
&= e^{rt}\left[\vec{w}_1 \cos(\omega t) + \vec{w}_2 \sin(\omega t)\right]
\end{aligned}$$

for some constant vectors $\vec{w}_1$ and $\vec{w}_2$. We can do a bit more algebra and show that the constants can be chosen so that the solutions are of the form

$$\vec{z}(t) = Re^{rt}\left[\text{Re}(\vec{v_1})\cos(\omega t + \phi) + \text{Im}(\vec{v}_1)\sin(\omega t + \phi)\right].$$

where we wrote the result in terms of the real and imaginary parts the eigenvectors and of two free constants R and ϕ.

Here, the role of the eigenvectors is less critical (they only set the shape of the resulting trajectories). What **matters the most is the sign of r = Re(λ).**

In **Case 9**, we have $r > 0$ and the amplitude of motion of the solution will grow in time. The trigonometric functions ensure that there will be oscillations. In the phase plane, these growing oscillations appear as trajectories spiraling outward (unstable).

Case 9: Outward spiraling trajectories (unstable) for complex eigenvalues with positive real part.

To tell in which direction the spiral goes, the simplest method is to pick a point, say $\vec{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and compute the tangent vector at the point using the original matrix.

$$\frac{d\vec{z}}{dt} = \mathrm{A}\vec{z} = \mathrm{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

You can sketch this vector starting from the point $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and determine the direction of rotation.

In **Case 10**, we have $\mathrm{Re}(\lambda) = r < 0$. In this case, the spirals are inward, as the solution approaches the origin over time. The direction of spiraling can be determined as in the previous case. The solutions are then stable, and attractive.

Case 10: Inward spiraling trajectories (stable, attractive) for complex eigenvalues with negative real part.

Finally, in **Case 11**, we have that $\mathrm{Re}(\lambda) = r = 0$. The trajectories then form ellipses and these points are called centers. The solutions are then stable, but not attractive.

Missing eigenvector

The last type of situation that can arise is one where we do NOT have n independent eigenvectors. For a 2×2 system, this means that we only have one eigenvector. In that case, we cannot form the matrix P and we say that A is NOT diagonalizable. In that case, we will also have only one eigenvalue, which we call λ.

We will not go into all the details here regarding how to find the constant vectors involved into getting an exact solution. Rather, we will focus on the qualitative behavior of these systems.

Case 11: Elliptical trajectories (stable, not attractive) for complex eigenvalues with zero real part.

The simplest such case is the so-called Gauss-Jordan form

$$\mathrm{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} z_x' \\ z_y' \end{pmatrix} = \begin{pmatrix} \lambda z_x + z_y \\ \lambda z_y \end{pmatrix}$$

The last equation we can easily solve: $z_y = C_1 e^{\lambda t}$.

The first equation then become forced, with the forcing having the form of the homogeneous solution

$$z_x = \lambda z_x + C_1 e^{\lambda t}$$

This has the general solution $z_x = C_2 e^{\lambda t} + D t e^{\lambda t}$, where we can find that $D = C_1 - C_2 \lambda$.

So overall, the solution looks like

$$\vec{z}(t) = e^{\lambda t} \begin{pmatrix} C_2 + (C_1 - C_2 \lambda) t \\ C_1 \end{pmatrix}$$

This is true of all systems with a missing eigenvector: The solutions are exponentials multiplied by constants AND by linear functions in t. If there are more eigenvectors missing, the solutions get multiplied by higher powers of t (t^2, t^3, etc).

So at long times, we get:
- **Case 12**: If $\lambda > 0$, the solution goes away from the origin, parallel to the eigenvector $\vec{v}$.
- **Case 13**: If $\lambda < 0$, the solution goes towards the origin, parallel to the eigenvector $\vec{v}$.
- **Case 14**: If $\lambda = 0$, the solution goes away from the origin along straight lines parallel to the eigenvector $\vec{v}$, but the growth is only linear.

Missing eigenvector cases.

It is important to note that this case is rare. If you select random real numbers to form a matrix, the probability of having a missing eigenvector is zero. But it can happen in specifically chosen cases.

Lecture 16

Nonlinear systems

What happens if the equations are NOT linear?

We will consider here autonomous systems of two equations, but allow nonlinearity:

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y), \quad \text{or equivalently}\\ \frac{d\vec{x}}{dt} &= \vec{F}(\vec{x})\end{aligned}$$

with f and g potentially nonlinear.

How can we study such a system?

First, we consider its stationary (equilibrium) points (x^*, y^*) or $\vec{x}^*$ where we have:
$f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$ (or in other words $\vec{F}(\vec{x^*}) = \vec{0}$).

These are equilibrium points since $x(t) = x^*$ and $y(t) = y^*$ is a solution to out original system. So if you start at equilibrium, you stay at equilibrium.

Next, we would like to know what happens NEAR an equilibrium point. We introduce new variables:

$$\begin{aligned}x(t) &= x^* + \xi(t)\\ y(t) = &= y^* + \eta(t)\end{aligned}$$

and assume that $\xi(t)$ and $\eta(t)$ are small, so that $(x(t), y(t))$ stays close to equilibrium.

We now consider Taylor Series expansions of $f(x,y)$ and $g(x,y)$ around the equilibrium point (x^*, y^*). We will only go up to the linear term, and this is therefore called *linearizing*. The process is similar for both f and g, so we only show one.

$$\begin{aligned}f(x,y) &= f(x^*,y^*) + (x-x^*)\left.\frac{\partial f}{\partial x}\right|_{\vec{x}^*} + (y-y^*)\left.\frac{\partial f}{\partial y}\right|_{\vec{x}^*} + O((x-x^*)^2, (x-x^*)(y-y^*), (y-y^*)^2)\\ f(x,y) &= \xi\left.\frac{\partial f}{\partial x}\right|_{\vec{x}^*} + \eta\left.\frac{\partial f}{\partial y}\right|_{\vec{x}^*} + O(\xi^2, \xi\eta, \eta^2).\end{aligned}$$

It is important to recall that the partial derivatives here are EVALUATED at $\vec{x}^*$ and are therefore NUMBERS.

So if $\xi \ll 1$ and $\eta \ll 1$, we can neglect $O(\xi^2, \xi\eta, \eta^2)$ and obtain a LINEAR system

$$\begin{aligned}
\frac{dx}{dt} = \frac{d}{dt}(x^* + \xi) = \frac{d\xi}{dt} &= \left.\frac{\partial f}{\partial x}\right|_{\vec{x}^*} \xi + \left.\frac{\partial f}{\partial y}\right|_{\vec{x}^*} \eta \\
\frac{dy}{dt} = \frac{d}{dt}(y^* + \eta) = \frac{d\eta}{dt} &= \left.\frac{\partial g}{\partial x}\right|_{\vec{x}^*} \xi + \left.\frac{\partial g}{\partial y}\right|_{\vec{x}^*} \eta
\end{aligned}$$

So in matrix form, we get

$$\frac{d}{dt}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix} f_x(\vec{x^*}) & f_y(\vec{x^*}) \\ g_x(\vec{x^*}) & g_y(\vec{x^*})\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \mathrm{J}\begin{pmatrix}\xi \\ \eta\end{pmatrix}$$

where J is the *Jacobian* matrix, which contains all the first derivatives, evaluated at the equilibrium point

$$\mathrm{J} = \begin{pmatrix} f_x(\vec{x^*}) & f_y(\vec{x^*}) \\ g_x(\vec{x^*}) & g_y(\vec{x^*})\end{pmatrix} = \left.\frac{\partial(f,g)}{\partial(x,y)}\right|_{\vec{x}^*}$$

This approximation will be valid so long as the Jacobian is not the zero matrix. If it is the zero matrix, we cannot neglect the higher order terms, and they would need to be included in any analysis. We will not go there.

So from a general non-linear system, we can:

1. Identify stationary points (x^*, y^*).
2. Compute the corresponding Jacobian, EVALUATED at that location.
3. Classify the stability of each equilibrium point based on the eigenvalues of $\mathrm{J}(\vec{x}^*)$.
4. Draw a complete phase portrait (we didn't cover this yet).

Example:

$$\begin{aligned}
\frac{dx}{dt} &= x(8 - 4x - y) = f(x,y) \\
\frac{dy}{dt} &= y(3 - 3x - y) = g(x,y)
\end{aligned}$$

Here the equilibria (or stationary) points are:
setting $f(x,y) = 0$, we get $x = 0$ or $8 - 4x - y = 0$ (so $y = 8 - 4x$).
setting $g(x,y) = 0$, we get $y = 0$ or $3 - 3x - y = 0$ (so $y = 3 - 3x$).

Both functions must be zero at the same time, so we get:
$x = 0$ and $y = 0$, so $P_1 = (0,0)$
$x = 0$ and $y = 3 - 3x$ so $y = 3$ and $P_2 = (0,3)$.

$y = 0$ and $y = 8 - 4x$ so $x = 2$ and $P_3 = (2, 0)$
$y = 8 - 4x = 3 - 3x$ so $x = 5$ and $y = -12$. and $P_4 = (5, -12)$

We can compute the formula for the Jacobian in general. We will then apply it to each equilibrium point in turn:

$$\mathrm{J} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 8 - 8x - y & -x \\ -3y & 3 - 3x - 2y \end{pmatrix}$$

So we find at each point:

$$\mathrm{J}_1 = \mathrm{J}|_{P_1} = \mathrm{J}|_{(0,0)} = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$$

Here, we get $\lambda_1 = 3$, with $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and also $\lambda_2 = 8$, with $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
So locally, this is an unstable node.

Unstable node at (0,0)

$$\mathrm{J}_2 = \mathrm{J}|_{P_2} = \mathrm{J}|_{(0,3)} = \begin{pmatrix} 5 & 0 \\ -9 & -3 \end{pmatrix}$$

Here, we get $\lambda_1 = -3$, with $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and also $\lambda_2 = 5$, with $\vec{v}_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$
So locally, this is a saddle-point.

Saddle-point at (0,3)

$$J_3 = J|_{P_3} = J|_{(2,0)} = \begin{pmatrix} -8 & -2 \\ 0 & -3 \end{pmatrix}$$

Here, we get $\lambda_1 = -8$, with $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
and also $\lambda_2 = -3$, with $\vec{v}_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$
So locally, this is a stable node.

Stable node at (2,0)

$$J_4 = J|_{P_4} = J|_{(5,-12)} = \begin{pmatrix} -20 & -5 \\ 36 & 12 \end{pmatrix}$$

Here, we get $\lambda_1 = -12.72$, with $\vec{v}_1 = \begin{pmatrix} -0.57 \\ 0.82 \end{pmatrix}$
and also $\lambda_2 = 4.72$, with $\vec{v}_2 = \begin{pmatrix} 0.20 \\ -0.98 \end{pmatrix}$
So locally, this is a saddle-point

Saddle-point at at (5,-12)

We can put all this information together into a single phase portrait, where we connect individual equilibrium points by following trajectories, approximately.

Single, large, phase-portrait of a non-linear system

So if we look at $x(t)$ and $y(t)$ as populations, over long times the total population would approach $x = 2$ and $y = 0$, unless there are no x to begin with, in which case it would approach $x = 0, y = 3$.

This may be seen as a competition model:
Say that $x(t)$ is the population of rabbits, and $y(t)$ is the population of koalas.

In $x' = x(8 - 4x - y)$:
the first factor indicates that it takes rabbits to make rabbits,
the second factor indicates that resource availability limits growth, and rabbits consume more than koalas.

In $y' = y(3 - 3x - y)$:
the first factor indicates that it takes koalas to make koalas,
the second factor indicates that resource availability limits growth, and rabbits still consume more than koalas.

But rabbits grow in number even if the number of koalas has been maximized, so they win in the end.

Lecture 17

Population models

We saw last time an example of a population growth model. To understand the origin of these equations, we start with a single equation model, the *logistic* equation.

Consider a population of size $x(t)$. A simple model for its growth is

$$\frac{dx}{dt} = kx$$

where the growth rate (left) is proportional to the population size itself. Each member reproduces at a certain rate, independently of the rest of the population.

The solution to this equation is an exponential growth: $x(t) = x(0)e^{kt}$.
However, at long times, this cannot be realistic, as the population grows too quickly, and without bounds.

A better model is to have the growth rate k depend on the resources available, which in turn depend on the population size x. We let

$$\text{growth rate} = k(1 - \frac{x}{M})$$

where k is the maximum growth rate, when there is no competition. However, as the population approaches M, the resources available decrease, and when $x = M$, growth is no longer possible.

We then have

$$\frac{dx}{dt} = k(1 - \frac{x}{M})x$$

which we can solve by separation of variables

$$\frac{M\,dx}{(M-x)x} = k\;dt.$$

We can use partial fractions to rewrite the left-hand-side

$$\left(\frac{1}{x} + \frac{1}{M-x}\right) dx = k\;dt.$$

Integrating on both sides, we get

$$\ln x - \ln(M-x) = \ln\frac{x}{M-x} = kt + C.$$

Note that this is a good time to solve for C. If at time $t = 0$ we have $x = x_0$, then $C = \ln(x_0/(M - x_0))$. We can now solve for $x(t)$ explicitly

$$\begin{aligned}
\ln\left(\frac{x}{M-x}\right) &= kt + \ln\left(\frac{x_0}{M-x_0}\right) \quad \text{exponentiating both sides, we get} \\
\frac{x}{M-x} &= e^{kt}\left(\frac{x_0}{M-x_0}\right). \quad \text{We can now get rid of the fractions} \\
x(M-x_0) &= e^{kt}x_0(M-x) \quad \text{We now group the } x \text{ terms} \\
x(M-x_0+e^{kt}x_0) &= e^{kt}x_0M \quad \text{and we can finally solve for } x \\
x(t) &= \frac{Mx_0e^{kt}}{M+x_0(e^{kt}-1)}.
\end{aligned}$$

Does this seem right? At $t = 0$, we get $x(0) = Mx_0/M = x_0$, which is good.
As $t \to \infty$, we get $x(t \to \infty) = Mx_0/x_0 = M$, which is an equilibrium point, so that also makes sense.

Note that if we looked at the limit $t \to -\infty$, we get zero, which is another equilibrium point.

Solutions to the logistic equation.

More than one species

With 2 species, more complicated behaviors are possible. In our first example, strong competition led to only one species surviving.

In general, competition models are written as

$$\begin{aligned}
x' &= x(A - ax - by) \\
y' &= y(B - cx - dy)
\end{aligned}$$

with $A, B, a, b, c, d > 0$.

Note that if only one species is present at a time, both of these equations reduce to the logistic equation. The presence of either species reduces the resources for both species, which captures a relation of competition.
We then have the stationary points $(A/a, 0$ and $(0, B/d)$, as well as the origin $(0, 0)$.
However, there can also be another equilibrium point, which may or may not be feasible (with $x > 0$ and $y > 0$)

We will first consider a model of *weak competition*

$$\begin{aligned} x' &= x(4 - 2x - 2y) \\ y' &= y(9 - 6x - 3y) \end{aligned}$$

If you recall, for the case of strong competition, the fourth equilibrium point was not feasible as a population. Here, there will actually be a fourth equilibrium point that can be a population.

We get the following equilibria:
A: (0,0) **B**: (0,3) **C**: (2,0) **D**: (1,1)
and the general form of J is

$$\mathrm{J} = \begin{pmatrix} 4 - 4x - 4y & -2x \\ -6y & 9 - 6x - 6y \end{pmatrix}$$

At each equilibrium point, we get, respectively:
$\mathrm{J}_A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ with $\lambda_1 = 4, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 9, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, an **unstable node at (0,0)**.
$\mathrm{J}_B = \begin{pmatrix} -2 & 0 \\ -18 & -9 \end{pmatrix}$ with $\lambda_1 = -9, \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = -2, \vec{v}_2 = \begin{pmatrix} 7 \\ -18 \end{pmatrix}$, a **stable node at (0,3)**.
$\mathrm{J}_C = \begin{pmatrix} -4 & -4 \\ 0 & -3 \end{pmatrix}$ with $\lambda_1 = -4, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = -3, \vec{v}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, a **stable node at (2,0)**.
$\mathrm{J}_D = \begin{pmatrix} -2 & -2 \\ -6 & -3 \end{pmatrix}$ with $\lambda_1 = -6, \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, a **saddle-point at (1,1)**.

So what does all this mean?
If you start with a non-zero population of each species, you will not tend to the origin, because it it unstable.
You may end up at either $(0, 3)$ or $(2, 0)$, with only one species surviving, *depending on where you start.*
If you start very carefully along the *separatrix*, you could end up at $(1, 1)$. However, in practice that is not possible because any deviation from the exact location of that curve will take you to either B or C.

So this is a case of weak competition, where either species might end up dominating the other, depending on how many individuals you start with. The scenario where both species survive at the same time is unstable, however, and will not occur.

Phase portrait for weak competition

We now look at a model of *coexistence*, where would would like to allow both species to exist at the same time. For this to happen, the stability of the fourth equilibrium point would need to change (it would have to be stable).

Consider the following system:

$$\begin{aligned} x' &= x(4-2x-y) \\ y' &= y(9-3x-3y) \end{aligned}$$

It is very similar to the previous one, but the effect of a species on the other one is weaker, so the competition is not as fierce.

We get the following equilibria:
A: (0,0) **B**: (0,3) **C**: (2,0) **D**: (1,2)
and the general form of J is

$$\mathrm{J} = \begin{pmatrix} 4-4x-y & -x \\ -3y & 9-3x-6y \end{pmatrix}$$

At each equilibrium point, we get, respectively:
$\mathrm{J}_A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ with $\lambda_1 = 4, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 9, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, an **unstable node**, as before.
$\mathrm{J}_B = \begin{pmatrix} 1 & 0 \\ -9 & -9 \end{pmatrix}$ with $\lambda_1 = -9, \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 10 \\ -9 \end{pmatrix}$, a **saddle-point** (unlike before).

J$_C = \begin{pmatrix} -4 & -2 \\ 0 & 3 \end{pmatrix}$ with $\lambda_1 = -4, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 3, \vec{v}_2 = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$, a **saddle-point** also.
J$_D = \begin{pmatrix} -2 & -1 \\ -6 & -6 \end{pmatrix}$ with $\lambda_1 = -4 - \sqrt{10}, \vec{v}_1 = \begin{pmatrix} 1 \\ 2+\sqrt{10} \end{pmatrix}$ and $\lambda_2 = -4 + sqrt10, \vec{v}_2 = \begin{pmatrix} 1 \\ 2-\sqrt{10} \end{pmatrix}$, a **stable node**.

So what does this mean?
If you start with only one species, you end up at its equilibrium point where the other species is absent. In theory, this is an unstable equilibrium, but in practice, this is the only case where you may end at un unstable equilibrium point, because if a population is 0, it will not spontaneously start to grow, in general.
If you start with a non-zero population of both species, you will always end up at the equilibrium $(1, 2)$.

This is qualitatively different than before, and is an example of coexistence.

Phase portrait for coexistence

Philosophical question of the day: do you think humans that originate from different locations are competing or coexisting? Maybe it is changing over the course of history?

Lecture 18

Predator-prey model

We now consider a different type of model, where the presence of one species helps, and is essential to, the growth of the other species.

Consider that $x(t)$ is the size of the rabbit population, and
that $y(t)$ is the size of the dingo population (they eat rabbits).

By themselves, each species can be modeled simply:

$$\frac{dx}{dt} = kx, \quad k > 0, \text{ yields exponential growth.}$$

$$\frac{dy}{dt} = -sy, \quad s > 0, \text{ yields exponential decay in the absence of prey.}$$

How do the two species interact?
Rabbits are eaten by dingos, so larger y slows growth and k get replaced by $k(1 - ay)$, with $a > 0$.
Dingos feed on rabbits, so larger x speeds growth and $-s$ get replaced by $-s(1-bx)$, with $b > 0$.

So we get

$$\begin{aligned} x' &= kx(1 - ay) \\ y' &= -sy(1 - bx) \end{aligned}$$

Let's look at a specific example

$$\begin{aligned} x' &= x(1 - y) = x - xy \\ y' &= -4y(1 - \frac{x}{2}) = -4y + 2xy \end{aligned}$$

What equilibrium points are there?
We still have that $A = (0, 0)$ is an equilibrium point.
Also if $1 - y = 0$ and $1 - x/2 = 0$ we are at equilibrium, so $B = (2, 1)$ is an equilibrium.
Note that there are no other equilibria, as the rabbit population grows forever without dingos, and the dingo population goes to 0 without rabbits.

The general form of the Jacobian is

$$\mathrm{J} = \begin{pmatrix} 1 - y & -x \\ 2y & 2x - 4 \end{pmatrix}$$

So we find: $J_A = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$ with $\lambda_1 = -4, \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, a **saddle-point**.
$J_B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ with $\lambda_1 = 2i, \vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\lambda_2 = -2i, \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, a **center**.

We have not encountered centers so far. They indicate that close to the equilibrium point, solutions go in circles, here counter-clockwise, as you can see if you plug in a point. However, we don't know whether solutions stay in loops, approach, or go away from the equilibrium point from this analysis alone: The linear approximation yields closed curves, but it is possible that the non-linear terms cause spiraling.

So we need to look a bit more closely. For this consider $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4y(1-\frac{x}{2})}{x(1-y)}$$

This equation is actually separable! We are in luck. Otherwise, we might have to resort to numerical methods.

Here we can rewrite this as

$$\frac{(1-y)dy}{-y} = \frac{(4-2x)dx}{x}$$

This can be integrated, splitting each fraction into 2 parts:

$$y - \ln y = 4\ln x - 2x + C$$

or $C = y + 2x - \ln(x^4 y)$.
The best we can get is this implicit formula. However, this is enough to give us a graph, and one can confirm that the trajectories are closed loops (consider $x \to \infty$ and show that C cannot remain finite, thus showing that any finite C yields bounded solutions in x; repeat the argument for y).

Finally, we consider a more complete model, where too many preys will limit their own growth, logistic-style:

$$\begin{aligned} x' &= x(1-2x-y) = -2x^2 - xy + x \\ y' &= -2y(1-3x) = -2y + 6xy \end{aligned}$$

We find three stationary points:

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix},$$

The general form of the Jacobian is

$$J = \begin{pmatrix} 1-4x-y & -x \\ 6y & 6x-2 \end{pmatrix}$$

So we find: $J_A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ with $\lambda_1 = -2, \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, a **saddle-point**.
Here this corresponds predators dying off when there is not prey ($\lambda_1 = -2$) and to preys growing their population when left alone ($\lambda_2 = 1$).
$J_B = \begin{pmatrix} -1 & -1/2 \\ 0 & 1 \end{pmatrix}$ with $\lambda_1 = -1, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, another saddle-point.
This corresponds to preys reaching an equilibrium population if left alone, but not being able to stay there is predators are present.
$J_C = \begin{pmatrix} -2/3 & -1/3 \\ 2 & 0 \end{pmatrix}$ with $\lambda_1 = (-1 + i\sqrt{5})/3, \vec{v}_1 = \begin{pmatrix} (1 + i\sqrt{6})/5 \\ 1 \end{pmatrix}$ and $\lambda_2 = (-1 - i\sqrt{5})/3, \vec{v}_2 = \begin{pmatrix} (1 + i\sqrt{6})/5 \\ 1 \end{pmatrix}$, a stable equilibrium.
In this case, trajectories spiral in toward the equilibrium point. Trying a point above C shows that trajectories go left, so counter-clockwise.

So in the end, all non-zero populations will end up at C. However, there will be oscillations in the population levels, as is commonly observed. When there are few predators, prey population grows, so much so that predator population grows, so much so that prey population declines, so much so that predator population decline, etc.

Phase diagram of a more complete predator-prey model

Lecture 19

Boundary value problems

So far, when looking at second order equations, we always gave two ICs (at the same time t_0, for example). However, it is also possible to specify Boundary Values (BV) instead, at two different locations (the boundaries).

In general, you need 2 BV for a second order equation in order to get a unique solution (there are exceptions to this though).

Example: Consider

$$y'' - y = 0 \quad \text{subject to } y(1) = 2 \text{ and } y(-1) = 2.$$

Here our domain of interest will be the region between the boundaries, so $[-1, 1]$.
We know the general solution: $y(t) = C_1 e^{-t} + C_2 e^t$.
We have two unknown constants, and two equations, so we can try to solve:

$$\begin{aligned} y(1) &= C_1 e^{-1} + C_2 e = 2 \quad \text{so} C_1 = 2e - C_2 e^2 \\ y(-1) &= C_1 e + C_2 e^{-1} \end{aligned}$$

Substituting for C_1, we get: $2e^2 - C_2 e^3 + C_2 e^{-1} = 2$ so $C_2(e^{-1} - e^3) = 2 - 2e^2$.
So finally we find

$$C_2 = 2\left(\frac{1-e^2}{e^{-1}-e^3}\right) = 2e\left(\frac{1-e^2}{1-e^4}\right) = 2e\left(\frac{1}{1+e^2}\right) \quad \text{and } C_1 = 2e - \frac{2e^3}{1+e^2} = \frac{2e}{1+e^2}$$

So altogether,

$$y(t) = \frac{2e}{1+e^2}(e^{-t} + e^t) = \frac{4e}{1+e^2}\cosh t.$$

Most importantly, from the general solution, we could find a unique solution using the 2 boundary conditions.

Example: Consider

$$y'' - y = t^2 \quad \text{subject to } y(0) = 1 \text{ and } y(3) = 0.$$

We need a particular solution, so we guess $y_p(t) = at^2 + bt + c$ which has $y_p'' = 2a$.

Plugging in, we get

$$2a - at^2 - bt - c = t^2$$

Matching coefficients, we must have that $a = -1$, $b = 0$, and $c = -2$
so we get $y_p(t) = -t^2 - 2$.

Our general solution is now

$$y_g(t) = C_1 e^{-t} + C_2 e^t - t^2 - 2$$

Plugging in the boundary conditions, we get
$y(0) = C_1 + C_2 - 2 = 1$, so $C_1 = 3 - C_2$, and
$y(3) = C_1 e^{-3} + C_2 e^3 - 11 = 0$ which becomes $(3 - C_2)e^{-3} + C_2 e^3 - 11 = 0$
We can rewrite this last equation as

$$C_2(-e^{-3} + e^3) = 11 - 3e^{-3}$$

so we get $C_2 = \frac{11-3e^{-3}}{e^3-e^{-3}}$ and $C_1 = 3 - \frac{11-3e^{-3}}{e^3-e^{-3}}$.

Once again, what matters here is that, even for a non-homogeneous system, we can find a unique solution matching the two boundary conditions.

It is possible, and common, to have a boundary condition at infinity. Often, the BC takes the form:

$$\lim_{t\to\infty} y(t) = \alpha \quad \text{or} \quad \lim_{t\to\infty} y(t) \text{ is bounded}$$

For **example**: $y'' + 4y' - 12y = 0$ subject to $y(0) = 1$, and $\lim_{t\to\infty} y(t)$ is bounded.

Our roots satisfy:
$k^2 + 4k - 12 = 0$ so that $k = -6$ or $k = 2$ and
$y_g(t) = C_1 e^{-6t} + C_2 e^{2t}$
For the solution to remain finite as $t \to \infty$, we must have $C_2 = 0$.
We also have $y(0) = 1 = C_1$ so that our unique solution is
$y(t) = e^{-6t}$

Note: Imposing a condition at infinity does not always yield a solution, so you can not always impose any condition you like at infinity.

For **example**: $y'' + 4y' - 12y = 0$ subject to $y(0) = 1$, and $\lim_{t\to\infty} y(t) = 2$.
This has no solution, because our solution either goes to zero or to infinity as $t \to \infty$.

Example: $y'' + y = 0$ subject to $y(0) = 1$, and $\lim_{t\to\infty} y(t)$ is bounded
$y_g(t) = C_1 \sin t + C_2 \cos t$, which is always finite as $t \to \infty$, no matter what the constants are.
Using the other BC: $y(0) = 1 = C_2$ yields infinitely many solutions:
$y(t) = C_1 \sin t + \cos t$.

It is also possible to get infinitely many solutions with BC at finite points if the boundary conditions do not restrict the constants:
Example: $y'' + y = 0$ subject to $y(0) = 0$ and $y(2\pi) = 0$
$y_g(t) = C_1 \sin t + C_2 \cos t$ gives
$y(0) = C_2 = 0$ and $y(2\pi) = C_2 = 0$
so we get $y(t) = C_1 \sin t$, for any C_1.

So when specifying Boundary conditions, we have existence of the solution, but we do not have unicity. So we must be careful.

Homogeneous Boundary conditions

A very common type of Boundary Value Problem (BVP) is one where the BCs are homogeneous:
$y(A) = 0$ and $y(B) = 0$.

For a linear equation, this always implies that $y(t) = 0$ is a solution (the trivial solution). However, sometimes there are other solutions too.
Example: $y'' + y = 0$ subject to $y(0) = 0$ and $y(1) = 0$.
Certainly $y(t) = 0$ is a solution. Is there another?
The general solution is $y(t) = A\cos t + B\sin t$.
From $y(0) = 0$, we get that $A = 0$.
From $y(1) = 0$ we get that $B\sin 1 = 0$, so $B = 0$.
Therefore, in this case, there is only one solution: $y(t)$.

However, if we had imposed the second boundary solution at a different value of t, we might not need to have $B = 0$. Can you see where that would be?

Different locations for the BC may yield non-trivial solutions

Lecture 20

Eigenfunctions with homogeneous boundary conditions

A very common type of Boundary Value Problem (BVP) is one where the BCs are homogeneous:
$y(A) = 0$ and $y(B) = 0$.

For a linear equation, this always implies that $y(t) = 0$ is a solution (the trivial solution). However, sometimes there are other solutions too.
Example: $y'' + y = 0$ subject to $y(0) = 0$ and $y(1) = 0$.
Certainly $y(t) = 0$ is a solution. Is there another?
The general solution is $y(t) = A\cos t + B\sin t$.
From $y(0) = 0$, we get that $A = 0$.
From $y(1) = 0$ we get that $B\sin 1 = 0$, so $B = 0$.
Therefore, in this case, there is only one solution: $y(t)$.

However, if we had imposed the second boundary solution at a different value of t, we might not need to have $B = 0$. Can you see where that would be?

Different locations for the BC may yield non-trivial solutions

In general, we write such problems with a parameter, for example

$$y'' + \omega^2 y = 0 \text{ subject to } y(0) = 0 \text{ and } y(1) = 0$$

and we ask for which values of ω^2 does the system have a non-trivial solution.
The general solution is $y(t) = A\sin\omega t + B + \cos\omega t$.
From $y(0) = 0$, we get that $B = 0$.
From $y(1) = 0$, we get that $A\sin\omega = 0$.

This could imply that $A = 0$ (we don't want that) or that $\sin\omega = 0$.
This is possible if $\omega = n\pi$ for $n \in \mathbb{N}$.

So here we found infinitely many possible values of ω that yield a non-trivial solution.

This is in fact a (generalized) eigenvalue problem!
We define the operator L as

$$L[\phi] = \frac{d^2\phi}{dt^2}$$

for $\phi \in C^2[0,1]$ and $\phi(0) = 0$ and $\phi(1) = 0$. The function ϕ actually live in a vector space of functions twice continuously differentiable that satisfy the boundary conditions.

So here $\phi(t)$ are our vectors and L is our operator, which, along with its BC, plays the role of a matrix.

Now we look for eigen "vectors": $L[\phi] = \lambda\phi$ or $\phi'' = \lambda\phi$ with $\phi(0) = 0$ and $\phi(1) = 0$.
Our general solution is $\phi(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$.

One can verify that the only way to get a non trivial solution is to have $\lambda < 0$ so that the general solution oscillates (the sum of exponential can only cross the x-axis once.

To indicate that $\lambda < 0$, we introduce a real number ω and write $\lambda = -\omega^2$.

Then we have that $\phi(t) = A\sin\omega t$ satisfies $L[\phi] = -\omega^2\phi$ and $\phi(0) = 0$.
To satisfy $\phi(1) = 0$, we need to select $\omega = n\pi$, for $n \in \mathbb{N}$.

So the eigenvalues of L are $\lambda = -n^2\pi^2$, for $n \in \mathbb{N}$ and the eigenvectors are $\phi(t) = \sin(n\pi t)$. Both sets have infinitely many elements. In fact, these eigenvalues/eigenvectors have a number of nice properties (see Sturm-Liouville problems, in Math 126).

We will look at a few other examples.
Example: $\phi'' = -\omega^2\phi$ subject to $\phi(0) = 0$ and $\phi'(1) = 0$.
Satisfying the equation and the first BC still gives $\phi(t) = A\sin\omega t$.
The second BC yields $\omega A\cos\omega = 0$ so $\omega = (n - 1/2)\pi$ for $n \in \mathbb{N}$.
So we have eigenvalues $\lambda = -\omega^2 = -(n-1/2)^2\pi^2$ and eigenfunctions $\phi_n(t) = \sin((n - 1/2)\pi t)$.

We can also change the equation:
Example: $L[\phi] = \phi'' + 2\phi' = \lambda\phi$, subject to $\phi(0) = 0$ and $\phi(1) = 0$.
The general solution is of the form $\phi(t) = Ae^{rt}$ with the roots satisfying $r^2 + 2r - \lambda = 0$.
We find that $r = -1 \pm \sqrt{1+\lambda}$.

Once again, we need oscillations, so we must have $1 + \lambda < 0$ and set $-\omega^2 = 1 + \lambda$.

Our solution is then: $y_g(t) = Ae^{-t}\sin(\omega t) + Be^{-t}\cos(\omega t)$.
From $\phi(0) = 0$, we get that $B = 0$.
From $\phi(1) = 0$, we get $Ae^{-1}\sin\omega = 0$.

So once again $\omega = n\pi$ for $n \in \mathbb{N}$.
So we have eigenvalues $\lambda = -n^2\pi^2 - 1$ or $n \in \mathbb{N}$
and eigenfunctions $\phi(t) = e^{-t}\sin(n\pi t)$.

To finish, we look at the Bessel equation (of order zero):

$$L[\phi] = t^2\phi'' + t\phi' = -t^2\lambda\phi. \quad \text{subject to } \phi(1) = 0 \text{ and } \phi(0) \text{ is finite.}$$

We note that the eigenvalue can be moved to the BC via a change of variables. Consider the change of variables $x = \sqrt{-\lambda}t$. then we have

$$\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = \sqrt{-\lambda}\frac{d}{dx} \quad \text{and} \quad \frac{d^2}{dt^2} = -\lambda\frac{d^2}{dx^2}$$

Our equation then becomes

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + x^2\phi = 0$$

and the boundary conditions become: $\phi(x = 0)$ is finite and $\phi(x = \sqrt{\lambda}) = 0$.

The solution to this equation is denoted by the Bessel function $J_0(x)$. The eigenvalues are found as $\sqrt{\lambda} = k_n$ where k_n are the zeros of the Bessel function: $J_0(k_n) = 0$.

Returning to the original t variable, we get eigenvalues $\lambda = k_n^2$ and eigenfunctions $J_0(k_n t)$.

Why do we need eigenfunctions?

One use of these eigenfunctions is to create an expansion of a general function.

Say $\phi_n(t)$ are a set of eigenfunctions of $L[\phi] = \lambda_n\phi_n$. We can then write

$$f(t) = \sum_{n=1}^{\infty} a_n\phi_n(t)$$

Similarly, we can look for a solution to $L[y] = f(t)$ in the form of an expansion

$$y(t) = \sum_{n=1}^{\infty} c_n\phi_n(t)$$

Then it is easier to compute what L does to $y(t)$

$$L[y(t)] = \sum_{n=1}^{\infty} c_n L[\phi_n(t)] = \sum_{n=1}^{\infty} c_n\lambda_n\phi_n(t)$$

Now to solve $L[y] = f(t)$, we set

$$\sum_{n=1}^{\infty} c_n \lambda_n \phi_n(t) = \sum_{n=1}^{\infty} a_n \phi_n(t)$$

Matching coefficients, we get that $c_n = a_n/\lambda_n$. So our solution is

$$y(t) = \sum_{n=1}^{\infty} \frac{a_n \phi_n(t)}{\lambda_n}$$

where we can find the coefficients a_n if we know the original RHS $f(t)$. To see how, take Math 126!

Lecture 21

Conserved quantities

Many physical systems described by systems of differential equations can be studied by taking advantage of conserved quantities. These often do not have an obvious physical meaning, but they have a mathematical one.

A conserved quantity is a quantity that does not change in time. We briefly saw one such example in our predator-prey model:

$$\begin{aligned} x' &= x(1-y) \\ y' &= y(2x-4) \end{aligned}$$

which became

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{y(2x-4)}{x(1-y)}$$

and we found that $C = y + 2x - \ln(yx^4)$ is a constant. This means that $y + 2x - \ln(yx^4)$ is conserved over time. In general, this can help plotting trajectories (though in this case it is still hard).

The most famous conserved quantity is Energy.

Suppose we have a physical system with a conservative force $F(x)$, which is the derivative of a potential function $\phi(x)$:

$$\frac{d\phi}{dx} = -F(x).$$

In that case, Newton's law becomes:

$$\frac{dx}{dt} = v \qquad\qquad m\frac{dv}{dt} = F(x) = -\frac{d\phi}{dx}.$$

If we multiply the acceleration equation by the velocity v, we get

$$mv\frac{dv}{dt} = -\frac{d\phi}{dx}v = -\frac{d\phi}{dx}\frac{dx}{dt}$$

This can now be integrated over time

$$\begin{aligned} \int mv\frac{dv}{dt}dt &= \int -\frac{d\phi}{dx}vdt = \int -\frac{d\phi}{dx}\frac{dx}{dt}dt \\ \frac{1}{2}mv^2 &= -\phi(x) + E \end{aligned}$$

So we get a quantity, which we call E for energy, which is constant along a trajectory: $E = mv^2/2 + \phi(x)$.

Elliptical trajectories in the xv-plane conserving $E = (1/2)mv^2 + (1/2)kx^2$.

For example for a pendulum, $F(x) = -kx$, and $\phi(x) = kx^2/2$. We then have $E = (1/2)mv^2 + (1/2)kx^2$.

Trajectories are therefore ellipses in the $x - v$ plane. Changing E changes the size of the ellipses.

For a more complete version of the force, we should use $F(x) = -k \sin x$ and $\phi(x) \ = \ -k \cos x$. In that case,

$$E = \frac{1}{2}mv^2 - k \cos x$$

and the trajectories can be either closed of not, depending on the energy (you can go over the top of the pendulum or not)

Trajectories for $E = \frac{1}{2}mv^2 - k \cos x$ in the $x - v$ plane.

Chaos and dependence on initial conditions

You may have heard of Chaos Theory. It relates to systems of differential equations. However, for this phenomena to occur, you need to be in dimension at least 3, so we didn't encounter it in this class.

The term chaos refers to the dependence of a solution on the initial conditions. For example,

$$y'' + y + 0 \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = A$$

The solution to this system is $y(t) = A \sin t$.
So if we pick 2 similar initial conditions, A_1 and A_2, we can get two solutions,
$y_1(t) = A_1 \sin t$ and $y_2(t) = A_2 \sin t$.

The difference between these solutions over time can be computed:

$$|y_1(t) - y_2(t)| = |A_1 - A_2||\sin t|$$

This grows over time, but only slowly and never exceeds $|A_1 - A_2|$. This is not chaotic because the dependence on the initial condition is smooth (if you start a pendulum at almost the same speed, you get almost the same response). The only special case may be if A changes sign, because $A = 0$ is an equilibrium point. There, starting in opposite directions leads to solutions that grow apart more quickly, but still relatively slowly. (Note that this discussion is only relevant for solutions that don't go to infinity).

In chaotic systems, a small difference in initial conditions can quickly grow to be very large. This is a bit like what happens near an equilibrium point, but it is not limited to equilibria. Entire regions of the phase portrait can be such that small changes in IC result in larger changes over time. Even worse, trajectories in a chaotic system grow

Chaotic trajectories

apart **exponentially fast**.

This means that you have no hope of predicting long-term dynamics in these systems, because initial conditions are never known exactly.

Important examples of chaotic system includes:

- Double-pendulum (solutions grow apart in seconds)
  https://www.youtube.com/watch?v=U39RMUzCjiU
- Weather (solutions grow apart in a couple of weeks)
- Solar system (solutions grow apart in about a million years)