Lecture 1, Introduction to fluid dynamics

Goal of the course: Develop a general understanding of fluid flows through physical and mathematical descriptions.

We begin with an introduction to some setups involving fluid dynamics. It is a VERY broad subject. First, a fundamental question: **What is a fluid?**

Generally, matter exhibits a fluid behavior if an arbitrarily small force results in a displacement (non-zero motion). Most fluids always exhibit this behavior, but some matter is more complicated and does so only in certain circumstances. *This definition includes both liquids and gases.*

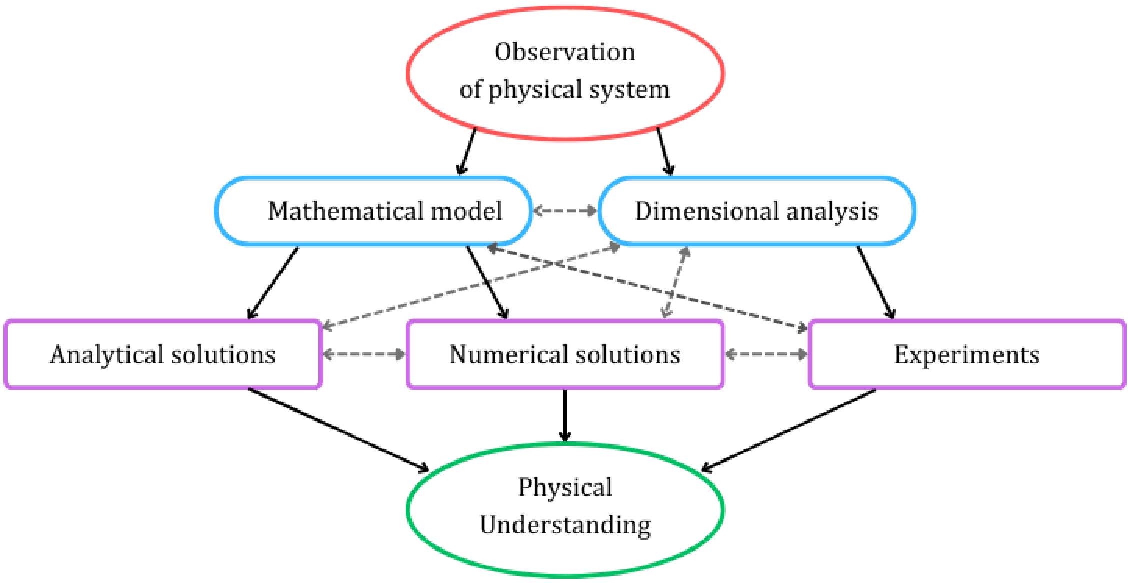
Examples of fluids are everywhere:

- Household: Soaps/bleaches, cooking, mixology, food items.
- Biological: Swimming, flying, cellular internal motion, breathing, blood flow, biofilm.
- Engineering: Oil recovery and treatment, heating or cooling systems (steam), wind effect on buildings, turbines, reactors.
- Aerodynamical: Transports, rockets, sport balls.
- Environmental: Groundwater, river systems, forest fires, chemical spills, climate change.
- Geophysical: Atmospheric flows, ocean currents, magma flow, mudslides, star dynamics, galaxies.

Fluid dynamics covers problems involving a wide range of both length scales and time scales. And yet, they are all governed by a single set of equations: The Navier-Stokes equations.

It is important to quantify the length and time scales in order to narrow down parameters for our problem.

Our general philosophy of problem solving will be as illustrated below:



General problem solving process.

Dimensional Analysis:

Using the dimensions of quantities involved, we may form dimensionless quantities. They serve to:

1) Minimize the number of parameters to consider.

2) Yield basic forms of the solutions (to be further quantified using some other means).

Buckingham Pi's "Theorem":

If a given physical system has m *physical variables* (such as volume, density, frequency, etc) defined in terms of n *physical quantities* (such as mass, length, time, charge), then the system may be uniquely represented by $m - n$ independent dimensionless groups $(\Pi_1, \Pi_2, ...\Pi_{m-n})$.

Important notes:

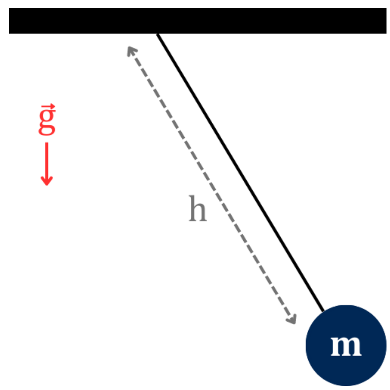
1. Physical laws are independent of the UNITS chosen to represent a quantity.
2. If $m - n = 1$, then the only dimensionless group must be CONSTANT. *This is the most common use of this theorem.*
3. If $m - n = p > 1$, the any dimensionless group may be expressed in terms of the others: $\Pi_p = f(\Pi_1, \Pi_2, ..., \Pi_{p-1})$.

To obtain dimensionless groups,

1. List all relevant variables, with their physical quantities.
2. Form combination where all physical quantities cancel (these are not unique) to reduce the number of parameters.
3. Make sure that every variable appears in at least one group.

Let's see some examples:

1 Pendulum Problem



A classical pendulum system.

Our variables describing the pendulum will be (other choices are possible):

Amplitude = A   [A] = L
Pi_2 = A / h

- Mass m, with units [m] $= M$.
- Length h, with units $[h] = L$.
- Gravitational acceleration $\vec{g}$, with units $g = L/T^2$.
- Period τ, with units $[\tau] = T$.

Note that the physical variables include τ, m, h, and g and the physical quantities include M, L, and T. This means we have $m = 4$, and $n = 3$ (M, L, T).

We can form a single dimensionless group $\Pi_1 = m^a h^b \tau^c g^d$. We must find $a, b, c,$ and d such that the result has no units.

Starting with M, we have $a = 0$.
From L, we find: $b + d = 0$.
From T, we find: $c - 2d = 0$.

This system has infinitely many solutions, but only one degree of freedom. For example, we set $c = 1$ and find $d = 1/2$ and $b = -1/2$.

Since we have a single dimensionless number, it must be some constant C (to be determined by other means):

$$C = \tau \left(\frac{h}{|\vec{g}|} \right)^{1/2}, \quad \text{or} \quad \tau = C \left(\frac{g}{h} \right)^{1/2},$$

Note that if we include an amplitude (A), we would have another dimensionless group $\Pi_2 = A/h \ldots \tau = \left(\frac{g}{h}\right)^{\frac{1}{2}} f(\frac{A}{L})$

In this example, we implicitly assumed that the frequency was independent of the amplitude.

Lecture 2, Dimensional analysis and scaling arguments

Third example: Drag on a sphere translating (not rotating) in a fluid.

- Radius a, with units $[a] = L$.
- Speed U, with units $[U] = L/T$.
- Density ρ, with units $[\rho] = M/L^3$.
- Viscosity (kinematic) ν, with units $[\nu] = L^2/T$.
- Drag D, with units $[D] = ML/T^2$.

Here $m - n = 5 - 3 = 2$, so we will have two dimensionless numbers. We choose:

$$\Pi_1 = \frac{Ua}{\nu} \quad \text{and} \quad \Pi_2 = \frac{D}{\rho U^2 a^2}$$

These are classed Reynolds number for $\Pi_1 = Re$ and Drag coefficient for $\Pi_2 = C_D$. We usually write $C_D = f(Re)$.
In practice, we find from measurements that at high Re, (for $Re > 100$) we have $f(Re)$ ∼Constant so $D \sim \rho U^2 a^2$ and
at low Re, (for $Re < 1$) we have $f(Re) = 1/Re$ so $D \sim \rho U \nu a$.

Dynamic similarity between two systems arises then the systems are described by the same non-dimensional parameters. The dynamics are then identical when proper non-dimensionalization is used. This allows for reduced or enlarged experiments. For example:
1) Wind-tunnels (reduced size, increased speed matches Re).
2) Rotating tables (increased rotation speed reduced viscosity keeps $U/\omega L$ the same as in planetary problems.
3) Low Re tanks (increased viscosity, increased size keeps Re constant).

Scaling Arguments are advanced dimensional analysis. Familiarity with the physical concepts allows one to reduce the number of variables to consider.

Example 3, revisited:

In general, for a round object, we can estimate that the drag is proportional to the **Stress** times the surface area (in general, stress is defined as a force per unit area). At low Reynolds number, when viscous effects are large, we anticipate that the stress will be proportional to the viscosity. The simplest way to get a stress that involves the viscosity is:

$$\text{Viscous stress} \quad \sim \frac{\rho \nu U}{a} = \frac{\mu U}{a}$$

so that the drag in a viscous regime is expected to be

$$D \sim \frac{\mu U a}{a} a^2 = \mu U a$$

Converting in non-dimensional form, we find $C_D = C/Re$.

At high Reynolds number, the stress is due to due to pressure differences, caused by balancing inertia. In that case, viscosity does not appear. We thus have that $\Delta P \sim \rho U^2$, and therefore that $D \sim \rho U^2 a^2$. In dimensionless form, that is:

$$C_D = \quad \text{Constant} \quad .$$

Last example: Row boat speed as a function of n, the number of rowers.

Consider that the boats are shaped the same for any n, and that their size is given by a. Here, the Reynolds number is large, so the drag to overcome is $D \sim \rho U^2 a^2$. The *Power* that can be generated is about proportional to the number of rowers (by observation for group efforts in general; this is not true of the force generated). So we have:

$$P \sim P_0 n \quad \text{and} \quad P = UD \sim U\rho U^2 a^2 = \rho U^3 a^2.$$

where P_0 is the power one rower can generate (rate at which energy is used).

Finally, the size of the boat is chosen so that the volume displaced matches the weight of the rowers. So we have

$$\rho L^3 g n \sim a^3 \rho g \quad \text{and} \quad n \sim (a/L)^3 \quad \text{and} \quad n^{1/3} L \sim a.$$

where L is the size of a rower.

Putting it all together, we get:

$$P \sim P_0 n \sim \rho U^3 a^2 \sim \rho U^3 (n^{1/3} L)^2 \quad \text{so} \quad n^{1/3} \sim \frac{\rho U^3 L^2}{P_0}$$

Finally, if we want to solve for the speed, we find: $U \sim \frac{P_0^{1/3}}{L^{2/3}\,\rho^{1/3}}\, n^{1/9}$.

This is a very weak dependence! But it seems to be confirmed by data (www.sciencebits.com/rowers).

Note: Scaling arguments are used in COMBINATION with data to capture dominant forces at play and to understand mechanisms. They are not sufficient by themselves as they can be misleading...

Very brief introduction to Einstein notation

We will use two notations in this class. One should be familiar, and the other one is Einstein's notation, which is most convenient for calculations. Most importantly, it assumes that a repeated index is summed, from 1 to the dimension used (2 or 3 in this class.

Name	Classic	Einstein
Scalar	a	a
Vector	$\vec{v}$ or v	v_i
Tensor	$\bar{\bar{T}}$	T_{ij}
Matrix Product	$\bar{T}\vec{v}$ or $\bar{T}\cdot\vec{v}$	$T_{ij}v_j$ (result has index i)
Identity	$\bar{\bar{I}}$	δ_{ij}
Dot product	$\vec{v}\cdot\vec{w}$	v_iw_i
Nabla	$\vec{\nabla}$	∇_i
Cross product	$\vec{v}\times\vec{w}$	$v_iw_j\epsilon_{ijk}$ (result has k index)

Here ϵ_{ijk} is the permutation tensor, or Levi-Civita tensor. It is defined as

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are equal} \\ 1 & \text{if } ijk \text{ are a cyclic permutation of 123 (so either 123, 231, or 312)} \\ -1 & \text{if } ijk \text{ are a non-cyclic permutation of 123 (so either 132, 321, or 213)} \end{cases}$$

For example, using Einstein notation, it is easy to compute the product rule for matrix multiplication:

$$\frac{d}{dt}(\bar{\bar{A}}(t)\vec{x}(t)) = \frac{d}{dt}(A_{ij}x_j) = \frac{dA_{ij}}{dt}x_j + A_{ij}\frac{dx_j}{dt} = \frac{d\bar{\bar{A}}}{dt}\vec{x} + A\frac{d\vec{x}}{dt}$$

Lecture 3, The Incompressible Navier-Stokes Equations and Material Derivatives

1) Mass conservation (so-called continuity equation):

$$\vec{\nabla} \cdot \vec{u} = 0$$

2) Momentum conservation

$$\rho \left(\vec{u}_t + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = -\vec{\nabla} P + \rho \nu \vec{\nabla}^2 \vec{u} + \vec{f}$$

Here, our notation is:

- $\vec{u} =$ Fluid velocity vector at a fixed point $\vec{x}$ and at time t.
- $P =$ fluid pressure at a fixed point $\vec{x}$ and at time t.
- $\rho =$ fluid density at a fixed point $\vec{x}$ and at time t. Usually, we will consider ρ constant.
- $\nu =$ kinematic viscosity, usually taken to be constant. Note that $\mu = \rho\nu$ is the dynamic viscosity.
- $\vec{f} =$ body force (typically gravity).

In the momentum equation, the LHS captures INERTIA.
The first term on the RHS is the PRESSURE GRADIENT.
The second term on the RHS is the VISCOUS FORCE, or friction.
The last term on the RHS is the external FORCE PER VOLUME.

With appropriate boundary conditions, this system describes the velocity field and pressure field of an incompressible fluid acted upon by a force $\vec{f}$. Note that we have 4 equations (mass + 3 in the momentum) and 4 unknowns (pressure + 3 components of velocity).

Fundamental notions

1. Continuum hypothesis: We assume that the fluid is a continuous medium so flow lengthscale $>>$ molecular lengthscale.
2. We represent space using a fixed spacial scale $\vec{x}$ and normal time t. This is the Euclidian approach, where coordinates do NOT move with the flow. In the Lagrangian approach, we follow fluid particles.
3. All our field variables are functions of space and time: $\vec{u}(\vec{x}, t)$, $P(\vec{x}, t)$ and potentially $\rho(\vec{x}, t)$.

4. The equations are obtained from mass conservation and Newton's law, (Conservation of linear and angular momentum).

5. The "material derivative" is an important tool. It describes the rate of change of quantities associated to moving deforming elements.

Let $\Phi(\vec{x}, t)$ be a property of fluid at position $\vec{x}_p$ and time t. At time $t + \Delta t$, the same fluid is now at position $\vec{x} + \Delta x$, with property $\Phi(\vec{x}_p + \Delta x, t + \Delta t)$. We define

$$\frac{D\Phi}{Dt} = \lim_{\Delta t \to 0} \frac{\Phi(\vec{x}_p + \Delta x, t + \Delta t) - \Phi(\vec{x}_p, t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Phi(\vec{x}_p + \Delta x, t + \Delta t) - \Phi(\vec{x}_p + \Delta x, t) + \Phi(\vec{x}_p + \Delta x, t) - \Phi(\vec{x}_p, t)}{\Delta t}$$

$$= \frac{\partial \Phi}{\partial t} + \frac{d\vec{x}_p}{dt} \cdot \nabla \Phi$$

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \vec{u} \cdot \nabla \Phi$$

In general, the material derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \tag{1}$$

and it captures changes following a fluid particle:

Example: A leaf in the wind measuring sunlight exposure has material derivative:

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{\text{(A)}} + \vec{u}_{\text{Leaf}} \cdot \underbrace{\frac{\partial T}{\partial z}}_{\text{(B)}} \tag{2}$$

(A) Changes in exposure at a given location.

(B) Changes in exposure due to motion.

Mass Conservation (or budget):
Consider an arbitrary, fixed volume V in space, with boundary S and outward normal $\hat{n}$

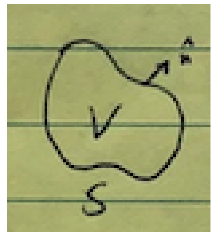


Figure 1: Volume V is enclosed by boundary S with outward normal $\hat{n}$.

The time rate of change of the mass in V is equal to the net flux through S.

$$\frac{d}{dt}\left(\int_V \rho dV\right) = -\int_S \rho\vec{u}\cdot\hat{n}dS \tag{3}$$

V is fixed so

$$\frac{d}{dt}\left(\int_V \rho dV\right) = \int_V \frac{d\rho}{dt}dV \tag{4}$$

Divergence theorem states

$$\int_V \nabla\cdot\vec{F}dV = \int_S \vec{F}\cdot\hat{n}dS \tag{5}$$

So we have

$$-\int_S \rho\vec{u}\cdot\hat{n}dS = -\int_V \nabla\cdot(\rho\vec{u})dV \tag{6}$$

Putting it all together:

$$\int_V \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})dV = 0 \tag{7}$$

Because V is arbitrary, we must have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0. \tag{8}$$

An incompressible fluid is one where density doesn't change as it moves:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho = 0. \tag{9}$$

We can rewrite (8) as:

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0. \tag{10}$$

For an incompressible fluid, this simplifies to $\nabla\cdot\vec{u} = 0$ because $\frac{D\rho}{Dt} = 0$. Note: An incompressible fluid may have a non-constant (in space) density (temperature changes) but the density of a given fluid parcel stays the same.

Surface tension is a measure of how strongly certain fluid molecules prefer to be surrounded by molecules of the same fluid.

For miscible fluids, like water and alcohol, there is no preference and we say that the surface tension is 0 or we simply do not speak of surface tension.

For immiscible fluids, like air and water or oil and water, it is energetically favorable to minimize contacts between molecules of different types. In other words, there is an energy cost to the presence of an interface. For example, a higher energy is associated to a system with a wavy interface that to one with a flat interface.

Physical systems naturally tend to minimize their energy. So in the presence of an interface, they will do so by minimizing the surface area of the interface.

- When there are no constraints, the system prefers to have a flat interface.

- When there is a volume to conserved, the system prefers to have a spherical interface.

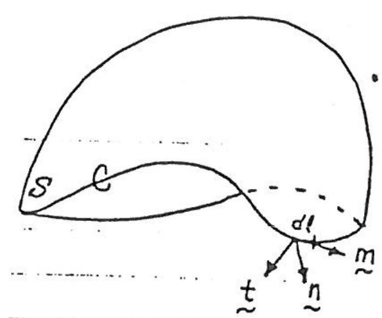
Surface tension, usually denoted as σ (or μ sometimes but not in this class) is the energy cost per area associated to having an interface:

$$\sigma = \frac{\text{Surface Energy}}{\text{Surface Area}}.$$

Its units are therefore:

$$[\sigma] = \frac{ML^2/T^2}{L^2} = \frac{M}{T^2}.$$

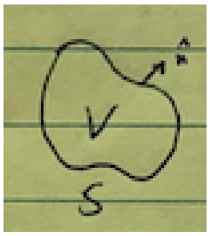
As a result there is a force tangent to a surface trying to flatten any bump in an interface.



For a portion of an interface as shown above, $\vec{n}$ is a unit vector normal to the surface, $\vec{t}$ is tangent vector to the arbitrary boundary C, and therefore also to the surface S, and $\vec{m} = \vec{t} \wedge \vec{n}$ is the *binormal* vector, which is tangent to the surface S and also normal to the curve C.

Lecture 4, The Momentum Equation
We apply Newton's Second Law to a fixed fluid element first, then to a moving one later.

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = \sum \text{All Forces (Internal and External)}$$

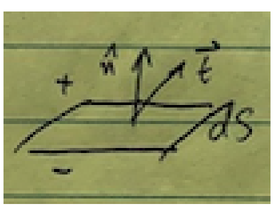


In words:

$$\{\text{Rate of change of momentum in V}\} = \{\text{Body Forces}\} + \{\text{Rate of Inflow of momentum through S}\} + \{\text{Surface Forces Acting on S}\}$$

$$\frac{\partial}{\partial t}\int_V \rho\mathbf{u}\,dV = \underbrace{\int_V \mathbf{f}\,dV}_{\text{I}} + \underbrace{\left(-\int_S \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})\,dS\right)}_{\text{II}} + \underbrace{\int_S \mathbf{t}(\hat{\mathbf{n}})\,dS}_{\text{III}}$$

Where $\mathbf{t}(\hat{\mathbf{n}})$ is the surface stress.



Note on the stress:

- Stress is a force per unit area, and it is thus a vector $\mathbf{t}$;
- It depends on the normal to the surface $\hat{\mathbf{n}}$;
- $\mathbf{t}(\hat{\mathbf{n}})$ is the force exerted from outside (+) on inside (-);
- In general, stress has both a tangential and a normal component.

Again, we keep V fixed so

$$\frac{\partial}{\partial t}\int_V \rho\mathbf{u}dV = \int_V \frac{\partial}{\partial t}(\rho\mathbf{u})dV = \int_V \left(\rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t}\right)dV.$$

We apply the divergence theorem to II:

$$-\int_S \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})dS = -\int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u})dV$$

Now

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) &= \nabla_i(\rho u_i u_j) = u_j u_i \nabla_i\rho + \rho u_j\nabla_i u_i + \rho u_i \nabla_i u_j \\ &= \mathbf{u}(\mathbf{u}\cdot\boldsymbol{\nabla}\rho) + \rho\mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \rho\mathbf{u}\cdot(\boldsymbol{\nabla}\mathbf{u})\end{aligned}$$

Sending II to the left-hand side:

$$\int_V dV\left(\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\underbrace{\left(\frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla}\rho) + \rho(\boldsymbol{\nabla}\cdot\mathbf{u})\right)}_{\text{0 by continuity}}\right)$$

We are then left with:

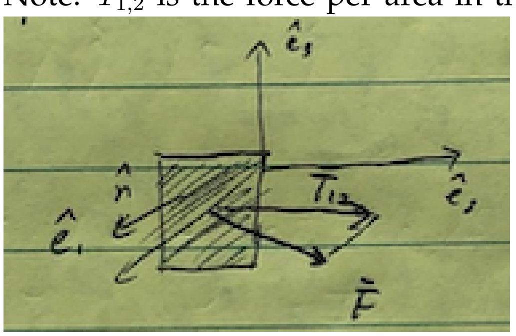
$$\int_V \rho\frac{D\mathbf{u}}{Dt}dV = \int_V \mathbf{f}dV + \int_S \mathbf{t}(\hat{\mathbf{n}})dS.$$

To obtain a PDE, we must express the surface integral as a volume integral. We assume that the stress may be expressed as:

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{T}}}.$$

The <u>stress tensor</u> $\overline{\overline{T}}$ is a second order tensor (two indices), and for now we do not know its components. They will depend on $\mathbf{u}$ and ρ but not on $\hat{\mathbf{n}}$.

Note: $T_{1,2}$ is the force per area in the direction of $\hat{\mathbf{e}}_\mathbf{2}$, on a surface perpendicular to $\hat{\mathbf{e}}_\mathbf{1}$:



The justification for writing $\mathbf{t} = \hat{\mathbf{n}}\cdot\overline{\overline{T}}$ was developed by Cauchy (Handout 3). One may then write III as

$$\int_S \mathbf{t}(\hat{\mathbf{n}})dS = \int_S \hat{\mathbf{n}}\cdot\overline{\overline{T}}dS = \int_V \boldsymbol{\nabla}\cdot\overline{\overline{T}}dV.$$

Putting it all together

$$\int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V \mathbf{f} dV + \int_V \boldsymbol{\nabla} \cdot \overline{\overline{T}} dV,$$

and since V is arbitrary, we have the Cauchy momentum equation

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} + \boldsymbol{\nabla} \cdot \overline{\overline{T}}.$$

Note: this result can also be obtained by following a moving fluid element (HW3).

Reynolds Transport Theorem:

$$\frac{d}{dt}\int_{V(t)} \boldsymbol{\psi}(\mathbf{x}, t) dV = \int_{V(t)} \frac{\partial \boldsymbol{\psi}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{u}\boldsymbol{\psi}) dV,$$

where $V(t)$ is a moving, deforming fluid volume with velocity $\mathbf{u}$.

- This is proven for a scalar ψ in handout #4;
- A more general result can be obtained by setting $\psi = \mathbf{a} \cdot \boldsymbol{\psi}$, for a general $\mathbf{a}$;
- For the special case $\boldsymbol{\psi} = \rho\mathbf{f}$, you will see in HW2 that

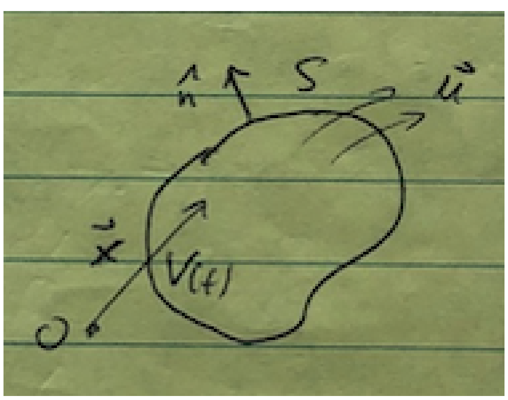
$$\frac{d}{dt}\int_{V(t)} \boldsymbol{\psi} dV = \frac{d}{dt}\int_{V(t)} \rho \mathbf{f} dV = \int_{V(t)} \rho \frac{D\mathbf{f}}{Dt} dV.$$

We proceed by deducing the form of $\overline{\overline{T}}$ by expressing it in terms of $\mathbf{u}$ and ρ. Using conservation of angular momentum, we fist show that $\overline{\overline{T}}$ is symmetric.

Conservation of Angular Momentum

In words:

$$\{\text{Time rate of change of angular momentum of a material volume}\} = \{\text{Sum of all torques}\}$$



$$\underbrace{\frac{d}{dt}\int_{V(t)} \mathbf{x} \times \rho \mathbf{u} dV}_{\text{I}} = \underbrace{\int_{V(t)} ((\mathbf{x} \times \mathbf{f}) + \mathbf{G}) dV}_{\text{II}} + \underbrace{\int_{S(t)} \underbrace{\mathbf{x} \times \overline{t}(\hat{\mathbf{n}})}_{\text{From Surface Shear}} dS}_{\text{III}}.$$

Note:

- $\mathbf{G}$ is a body couple torque, very rarely non-zero
- $\mathbf{G} \neq 0$ in ferro-fluids, particles with dipole moments.

For I: We use Reynolds Transport Theorem (RTT)

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \mathbf{x} \times (\rho \mathbf{u}) dV &= \frac{d}{dt}\int_{V(t)} \rho(\mathbf{x} \times \mathbf{u}) dV = \int_{V(t)} \rho \frac{D}{Dt}(\mathbf{x} \times \mathbf{u}) dV \quad \text{(RTT special case)} \\ &= \int_{V(t)} \rho \left(\frac{D\mathbf{x}}{Dt} \times \mathbf{u} + \mathbf{x} \times \frac{D\mathbf{u}}{Dt} \right) dV = \int_{V(t)} \left(\mathbf{x} \times \rho \frac{D\mathbf{u}}{Dt} \right) dV.\end{aligned}$$

Lecture 5

Conservation of Angular Momentum

To learn something about the stress tensor, namely that it is symmetric, we consider the conservation of angular momentum.

In words:

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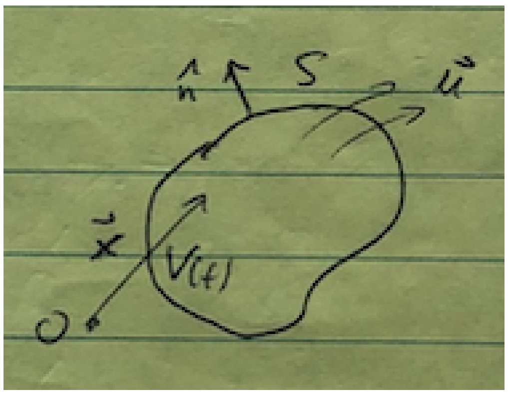

$$\underbrace{\frac{d}{dt}\int_{V(t)} \mathbf{x}\wedge\rho\mathbf{u}dV}_{\text{I}} = \underbrace{\int_{V(t)}((\mathbf{x}\wedge\mathbf{f})+\mathbf{G})dV}_{\text{II}} + \underbrace{\int_{S(t)} \underbrace{\mathbf{x}\wedge\bar{\bar{t}}(\hat{\mathbf{n}})}_{\text{From Surface Shear}} dS}_{\text{III}}.$$

Note:

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For I: We use Reynolds Transport Theorem (RTT)

$$\frac{d}{dt}\int_{V(t)} \mathbf{x}\wedge(\rho\mathbf{u})dV = \frac{D}{Dt}\int_{V(t)}\rho(\mathbf{x}\wedge\mathbf{u})dV = \int_{V(t)}\rho\frac{D}{Dt}(\mathbf{x}\wedge\mathbf{u})dV \text{ (RTT special case)}$$
$$= \int_{V(t)}\rho\left(\frac{D\mathbf{x}}{Dt}\wedge\mathbf{u}+\mathbf{x}\wedge\frac{D\mathbf{u}}{Dt}\right)dV = \int_{V(t)}\left(\mathbf{x}\wedge\rho\frac{D\mathbf{u}}{Dt}\right)dV.$$

For III: $\vec{\boldsymbol{x}}\wedge\bar{\bar{\mathbf{t}}}(\hat{\mathbf{n}}) = \vec{\boldsymbol{x}}\wedge(\hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{T}}}) = -\hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{T}}}\wedge\vec{\boldsymbol{x}}$

So,

$$\int_{\text{S(t)}}\vec{\boldsymbol{x}}\wedge\bar{\bar{\mathbf{t}}}(\hat{\mathbf{n}})dS = -\int_{\text{S(t)}}\hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{T}}}\wedge\vec{\boldsymbol{x}}dS = -\int_{\text{V(t)}}\vec{\boldsymbol{\nabla}}\cdot(\overline{\overline{\mathbf{T}}}\wedge\vec{\boldsymbol{x}})dV \text{ (by Divergenge Theorem)}$$

Aside:

$$\begin{aligned}
\overline{\overline{\mathbf{T}}} \wedge \vec{\boldsymbol{x}} &= T_{kl} x_m \varepsilon_{lmn} \\
\vec{\boldsymbol{\nabla}} \cdot (\overline{\overline{\mathbf{T}}} \wedge \vec{\boldsymbol{x}}) &= \nabla_k T_{kl} x_m \varepsilon_{lmn} \\
&= x_m \nabla_k T_{kl} \varepsilon_{lmn} + T_{kl} (\nabla_k x_m) \varepsilon_{lmn} \\
&= (\vec{\boldsymbol{\nabla}} \cdot \overline{\overline{\mathbf{T}}}) \wedge \vec{\boldsymbol{x}} + \underbrace{T_{ml} : \varepsilon_{lmn}}_{-\overline{\overline{\mathbf{T}}} : \varepsilon}
\end{aligned}$$

where we used that $\nabla_k x_m = \delta_{km}$.

So we have:

$$\int_{\mathrm{V(t)}} \vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{\nabla}} \cdot \overline{\overline{\mathbf{T}}} + \overline{\overline{\mathbf{T}}} : \varepsilon dV$$

All together:

$$\int_{\mathrm{V(t)}} \vec{\boldsymbol{x}} \wedge \underbrace{\left(\rho \frac{D\vec{\boldsymbol{u}}}{Dt} - \vec{\boldsymbol{\nabla}} \cdot \overline{\overline{\mathbf{T}}} - \vec{\boldsymbol{f}} \right)}_{=0 \text{ from linear momentum conservation}} - \overline{\overline{\mathbf{T}}} : \varepsilon dV = 0$$

Since $V(t)$ is arbitrary, we must have $\overline{\overline{\mathbf{T}}} : \varepsilon = 0$.
That is $T_{ij}\varepsilon_{ijk} = 0$ for any k.
Expanding, we find

$$\begin{aligned}
k = 1 \;\; T_{23} - T_{32} = 0 \\
k = 2 \;\; T_{31} - T_{13} = 0 \\
k = 3 \;\; T_{12} - T_{21} = 0
\end{aligned}$$

so $\overline{\overline{\mathbf{T}}}$ is symmetric ($\overline{\overline{\mathbf{T}}} = \overline{\overline{\mathbf{T}}}^T$). We therefore have "only" 6 independent components to determine:
Those on the diagonal (3)
Those off diagonal (3).

<u>Stress in a fluid</u>: we begin with STATIC fluids stresses (at rest).
When $\vec{\boldsymbol{u}} = 0$, the stress tensor is simplified. It is ISOTROPIC (no dependence on orientation). This implies that is has the form

$$\overline{\overline{\mathbf{T}}} = -P\overline{\overline{\mathbf{I}}},$$

where

$$-P\overline{\overline{\mathbf{I}}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

with "-" because pressure is compressive. So our momentum equation is

$$0 = \vec{\boldsymbol{f}} + \vec{\boldsymbol{\nabla}} \cdot \overline{\overline{\mathbf{T}}} = \vec{\boldsymbol{f}} - \vec{\boldsymbol{\nabla}} P.$$

Letting $\vec{\boldsymbol{f}} = \rho \vec{\boldsymbol{g}}$, we find $\rho \vec{\boldsymbol{g}} = \vec{\boldsymbol{\nabla}} P$. For constant ρ, we have $\frac{\partial P}{\partial z} = \rho g$.
A criteria for a static fluid is therefore that:

$$\vec{\boldsymbol{\nabla}} \wedge \rho \vec{\boldsymbol{g}} = 0$$

so

$$\vec{\boldsymbol{\nabla}} \rho \wedge \vec{\boldsymbol{g}} = 0.$$

Thus, if $\vec{\boldsymbol{\nabla}} \rho$ is not parallel to $\vec{\boldsymbol{g}}$, there must be motion!
Also, in a static fluid, $\vec{\boldsymbol{\nabla}} P_{||} \vec{\boldsymbol{\nabla}} \rho$.
So, if $\vec{\boldsymbol{u}} = 0$, we have $\overline{\overline{\mathbf{T}}} = -P\overline{\overline{\mathbf{I}}}$. We write $\overline{\overline{\mathbf{T}}} = -P\overline{\overline{\mathbf{I}}} + \overline{\overline{\boldsymbol{\tau}}}$ where $\overline{\overline{\boldsymbol{\tau}}}$ is the deviatoric stress tensor.
Since $\overline{\overline{\mathbf{T}}}$ is symmetric so is $\overline{\overline{\boldsymbol{\tau}}}$.
Note also that $\overline{\overline{\boldsymbol{\tau}}}$ is independent of $\vec{\boldsymbol{u}}$ itself, to be translationally invariant. So we anticipate that $\overline{\overline{\boldsymbol{\tau}}}$ depends on $\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}}$ (first derivatives of $\vec{\boldsymbol{u}}$). But how?
Note that

$$\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} = \underbrace{\frac{1}{2}(\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} + (\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}})^T)}_{\substack{\text{E, symmetric} \\ \text{Rate of Strain}}} + \underbrace{\frac{1}{2}(\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} - (\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}})^T)}_{\substack{\Omega\text{, antisymmetric} \\ \text{Vorticity (rotation tensor)}}}$$

Taylor Series:

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_0 + \delta \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_0 + (\underbrace{\delta \vec{\boldsymbol{x}}}_{\text{translation}} \cdot \overbrace{E}^{\text{Strain}}) + \delta \vec{\boldsymbol{x}} \cdot \underbrace{\Omega}_{\text{rotation}}.$$

Lecture 6, Newtonian Fluids and Stress Tensor

Assumptions for Newtonian Fluids:

1. Relation between $\bar{\bar{\tau}}$ and $\bar{\bar{E}}$ is local in space and time.
2. Relation to linear: $\bar{\bar{\tau}} = \bar{\bar{\bar{\bar{A}}}} : \bar{\bar{E}}$. (Note: $\bar{\bar{\bar{\bar{A}}}}$ is a 4th order tensor)
3. Fluid is isotropic $\to$ so $\bar{\bar{\bar{\bar{A}}}}$ is isotropic, too.

Note: Non-Newtonian fluids exist! Custard, elastic fluids, polymer, blood, toothpaste, paint, liquid crystals, ketchup, etc.

So we need a symmetric isotropic tensor A.

2nd Rank: $\to$ only δ_{ij}

3rd Rank: $\to$ only ϵ_{ijk}

4th Rank: $\to \delta_{ij}\delta_{kl}, \delta_{ik}\delta_{jl}$, and $\delta_{il}\delta_{jk}$

So we let $A = \lambda_1\delta_{ij}\delta_{kl} + \lambda_2\delta_{ik}\delta_{jl} + \lambda_3\delta_{il}\delta_{jk}$ and $\tau_{ij} = A_{ijkl}E_{kl}$.

For $\bar{\bar{\tau}}$ to be symmetric, we need $A_{ijkl} = A_{jikl}$ so $\lambda_2 = \lambda_3$ since $\bar{\bar{E}}$ is already symmetric.

So

$$\begin{aligned}\tau_{ij} =& \lambda_1\delta_{ij}\delta_{kl}E_{kl} + 2\lambda_2\delta_{ik}\delta_{jl}E_{kl}\\ =& \underbrace{\lambda_1\delta_{ij}E_{kk}}_{*} + 2\lambda_2 E_{ij}\end{aligned}$$

$$* \quad T_r\bar{\bar{E}} = \nabla_i u_i = \nabla\cdot\vec{u}$$

Usually we use $\lambda_2 = \mu = \rho\nu$, the dynamic viscosity. $\lambda_1 = \kappa - \frac{2}{3}\mu$, where $\kappa =$ bulk viscosity and $\bar{\bar{\tau}} = \left(\kappa - \frac{2}{3}\mu\right)\nabla\cdot\vec{u}\bar{\bar{I}} + 2\mu\bar{\bar{E}}$. For incompressible fluids, $\nabla\cdot\vec{u} = 0$ so $\bar{\bar{\tau}} = 2\mu\bar{\bar{E}}$. Navier-Stokes Equation:

$$\rho\frac{D\vec{u}}{Dt} = \vec{f} + \nabla\cdot\bar{\bar{T}} = \vec{f} - \nabla P + 2\nabla\cdot\left(2\mu\bar{\bar{E}}\right) \tag{1}$$

Finally, if μ is constant with flow,

$$\begin{aligned}\nabla\cdot\left(2\mu\bar{\bar{E}}\right) =& 2\mu\nabla\cdot\left(\frac{1}{2}\left(\nabla\cdot(\nabla\vec{u}) + (\nabla\vec{u})^T\right)\right) && (2)\\ =& \frac{2\mu}{2}\left(\nabla_i\nabla_i u_j + \nabla_i\nabla_j u_i\right) && (3)\\ =& \mu\nabla^2\vec{u} + \mu\nabla\cancel{(\nabla\cdot\vec{u})} && (4)\\ =& \mu\nabla^2\vec{u} && (5)\end{aligned}$$

and we finally arrive to:

$$\nabla \cdot \vec{u} = 0 \tag{6}$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u} + \vec{f}. \tag{7}$$

Remarks on the stress tensor $\bar{\bar{T}}$ For an incompressible Newtonian fluid, we had:

$$\bar{\bar{T}} = -P\bar{\bar{I}} + 2\mu\bar{\bar{E}}, \qquad \bar{\bar{E}} = \frac{1}{2}\left((\nabla\vec{u}) + (\nabla\vec{u})\right) \tag{8}$$

What does this look like? In Cartesian:

$$\nabla_i u_j = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{9}$$

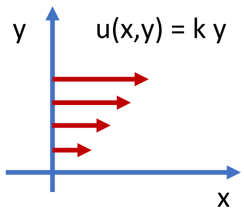
$$\bar{\bar{E}} = \bar{\bar{E}}^T = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{10}$$

$$\bar{\bar{T}} = \bar{\bar{T}}^T = \begin{bmatrix} -P + 2\mu\frac{\partial u_1}{\partial x_1} & \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & -P + 2\mu\frac{\partial u_2}{\partial x_2} & \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & -P + 2\mu\frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{11}$$

Note: $\bar{\bar{T}}$ is a mess in other coordinates. See Batchelor's appendix.

Recall T_{ij} = force per unit area, acting in the direction $\vec{e}_j$ on a face $\perp$ to $\vec{e}_i$.

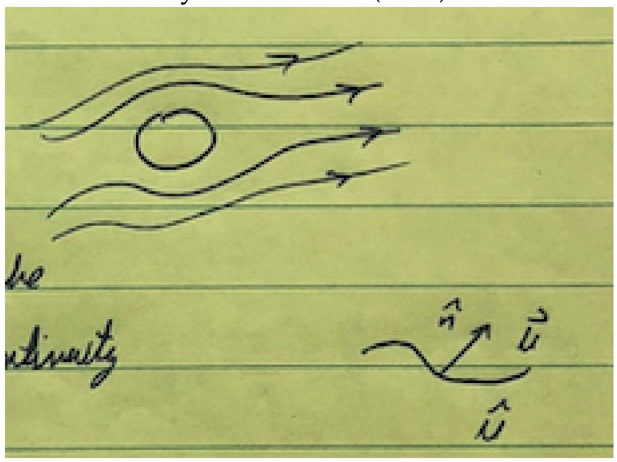
1. Normal stresses (diagonal) involve both pressure and velocity gradients.
2. Tangential stresses depend only on velocity gradients.



Consider for example a basic shear flow, in which the fluid velocity's horizontal component is $u(x, y) = ky$ for a constant k, and all other components are zero, as shown in the figure above.

In that case, the stress tensor only has two non-zero components: $T_{12} = T_{21} = \mu\frac{\partial u}{\partial y} = \mu k$. This corresponds to the horizontal component (in the direction of $\vec{e}^1$) of the force per unit area exerted by the fluid on a horizontal surface (with normal in the direction of $\vec{e}^2$).

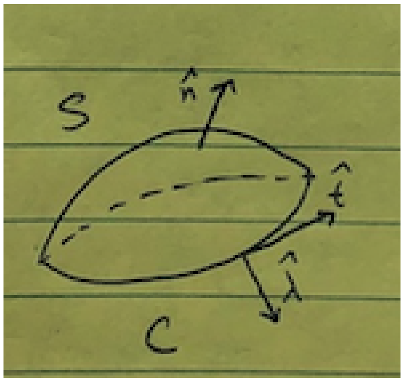
We now have our Navier-Stokes (N-S) equations, with four equations and four unknowns. We need to apply the correct Boundary Conditions (B.C.) to allow us to find solutions.

Type A: B.C. on velocity. 

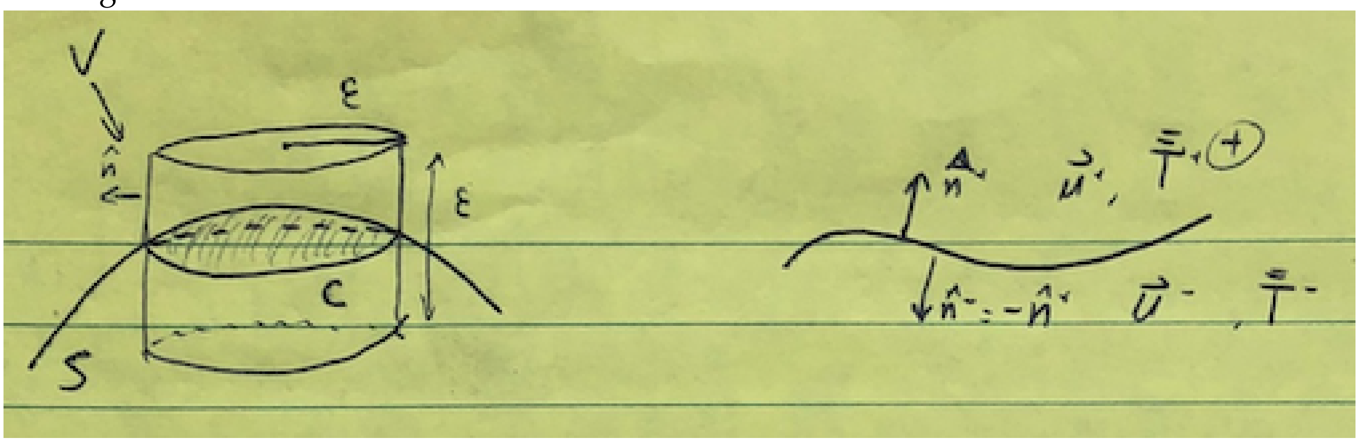
1. Far-field condition (at ∞)
2. Kinematic condition: Condition on the normal component of the velocity require continuity (no penetration) $\vec{u} \cdot \hat{\mathbf{n}} = \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}$
3. Dynamic Condition: Postulate stating that the tangential velocity is also continuity
   (a) Fluid-Solid: If solid has velocity $\vec{v}$, then $\vec{u} = \vec{v}$ at the solid surface (no-slip)
   (b) Fluid-Fluid: $\vec{u} = \hat{\mathbf{u}}$ at the interface $\rightarrow$ In this case, the interface may be deformed, and this is not described by this condition. Stresses need to be considered

Type B: Stress Conditions at a Fluid-Fluid interface.
Background: Because of surface tension (σ), work is needed to deform an interface ($\sigma = \frac{\text{Energy}}{\text{Area}}$ or $\frac{\text{Force}}{\text{length}}$)

- A surface S is bound by a curve C
- $\hat{\mathbf{n}}$ is normal to the surface
- $\hat{\mathbf{t}}$ is tangent to S and C
- $\hat{\boldsymbol{\lambda}}$ is tangent to S and orthogonal to C



Here σ is a force per unit length acting everywhere along C to flatten S, pulling in the direction of $\hat{\boldsymbol{\lambda}}$; σ is also the energy cost per area associated to creating an intercase between two fluids.



We perform a force balance on a volume element V of size ε enclosing a portion on the interface S, with intersection curve C.

$$\int_V \rho \frac{D\vec{\boldsymbol{u}}}{Dt} dV = \int_V \vec{\boldsymbol{f}} dV + \int_S \underbrace{\vec{\boldsymbol{t}}^{(+)}\hat{\mathbf{n}} + \vec{\boldsymbol{t}}^{(-)}}_{\text{surface forces}} dS + \overbrace{\int_{\partial V} \vec{\boldsymbol{t}}(\hat{\mathbf{n}}) dS}^{\text{This cancels out where T is continuous, or scales as volume}} + \int_C \underbrace{\sigma\hat{\boldsymbol{\lambda}}}_{\text{surface tension}} dl, \quad (1)$$

as $\varepsilon \to 0$, the volume forces scale as ε^3, and vanish faster than the surface stress (scaling as ε^2). So surface forces must balance each other:

$$\int_S \hat{\mathbf{n}}^{(+)} \cdot \overline{\overline{\mathbf{T}}}^{(+)} + \hat{\mathbf{n}}^{(-)} \cdot \overline{\overline{\mathbf{T}}}^{(-)} dS + \oint_C \sigma\hat{\boldsymbol{\lambda}} dl = 0 \quad (2)$$

From Handout 5, we may write

$$\oint_C \sigma\hat{\boldsymbol{\lambda}} dl = \int_S \vec{\boldsymbol{\nabla}}\sigma - \sigma\hat{n}(\vec{\boldsymbol{\nabla}} \cdot \hat{\mathbf{n}}) dS, \quad (3)$$

where $\vec{\boldsymbol{\nabla}}\sigma$ is a gradient taken along S only, often denoted $\vec{\boldsymbol{\nabla}}_S\sigma$, and $(\vec{\boldsymbol{\nabla}} \cdot \hat{\mathbf{n}})$ is a divergence along the surface only too. $(\vec{\boldsymbol{\nabla}}_S \cdot \hat{\mathbf{n}})$ is also

$$2 \cdot (\text{mean curvature}) = \left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad (4)$$

where $R_1,\ R_2$ are the principal radii of curvatures. So, we get

$$0 = \int_S^{2\kappa} \hat{\mathbf{n}}^{(+)}(\overline{\overline{T}}^{(+)} - \overline{\overline{T}}^{(-)}) + \vec{\boldsymbol{\nabla}}_S\sigma + \hat{\mathbf{n}}\sigma(\vec{\boldsymbol{\nabla}}_S \cdot \hat{\mathbf{n}})dS \tag{5}$$

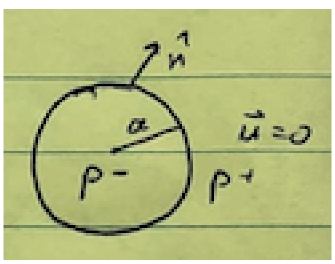
and

$$\hat{\mathbf{n}}^{(+)} \cdot (\overline{\overline{\mathbf{T}}}^{(+)} - \overline{\overline{\mathbf{T}}}^{(-)}) = \hat{\mathbf{n}}^{(+)}\sigma(\vec{\boldsymbol{\nabla}}_S \cdot \hat{\mathbf{n}}^{(+)}) - \vec{\boldsymbol{\nabla}}_S\sigma$$

$$= \underbrace{2\kappa\sigma\hat{\mathbf{n}}^{(+)}}_{\text{normal curvature force}} - \overbrace{\vec{\boldsymbol{\nabla}}_S\sigma}^{\text{Tangential stress associated to gradient of } \sigma} \qquad (*)$$

Normal stress balance gives: $((*) \cdot \vec{\boldsymbol{n}})$

$$\underbrace{\vec{\boldsymbol{n}} \cdot (\overline{\overline{\mathbf{T}}}^{(+)} - \overline{\overline{\mathbf{T}}}^{(-)}) \cdot \vec{\boldsymbol{n}}^{(+)}}_{\textit{Normal stress jump}} = 2\sigma\kappa,\ \textit{because}\ \vec{\boldsymbol{\nabla}}_S\sigma \perp \vec{\boldsymbol{n}} \tag{6}$$



For a drop or bubble at rest:

$$\overline{\overline{\boldsymbol{T}}}^{(+)} = -P^{(+)}\overline{\overline{\mathbf{I}}},\ \overline{\overline{\mathbf{T}}}^{(-)} = -P^{(-)}\overline{\overline{\mathbf{I}}} \tag{7}$$

and we have:

$$P^{(-)} - P^{(+)} = 2\sigma\kappa = \frac{2\sigma}{a} \tag{8}$$

(Note: $\vec{\boldsymbol{\nabla}}_S \cdot \vec{\boldsymbol{n}} = \vec{\boldsymbol{\nabla}}_S \cdot \frac{\vec{\boldsymbol{r}}}{R} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{R}$).
So we have the Laplace pressure jump

$$P^{(-)} - P^{(+)} = \frac{2\sigma}{a},\ \left(\textit{Dimensional Analysis gives } \Delta P \sim \frac{\sigma}{a}\right) \tag{9}$$

Tangential stress balance $((*) \cdot \hat{\boldsymbol{\lambda}})$

$$\vec{\boldsymbol{n}}^{(+)} \cdot (\overline{\overline{\mathbf{T}}}^{(+)} - \overline{\overline{\mathbf{T}}}^{(-)}) \cdot \hat{\boldsymbol{\lambda}} = -\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{\nabla}}_S\sigma$$

$$\textit{Tangential Stress jump} = \underbrace{\textit{Stress jump in tangential direction}}_{\textit{due to surface tension gradients (Marangoni effect)}}$$

Note: There are no pressure terms on the LHS. This implies that any $\vec{\boldsymbol{\nabla}}_S \sigma \neq \vec{\mathbf{0}}$ implies MOTION.

Dynamic vs. regular pressure $(P_d \text{ vs. } P)$

Suppose the fluid is under the influence of gravity so $\vec{\boldsymbol{f}} = \rho \vec{\boldsymbol{g}}$ in NS:

$$\rho \frac{D\vec{\boldsymbol{u}}}{Dt} = -\vec{\boldsymbol{\nabla}} P + \rho \vec{\boldsymbol{g}} + \mu \nabla^2 \vec{\boldsymbol{u}}. \tag{10}$$

Recall that at rest, we found $\vec{\boldsymbol{\nabla}} P_H = \rho \vec{\boldsymbol{g}}$ (Hydrostatic pressure). Define

$$\begin{aligned} P &= \underbrace{P_d}_{\textit{dynamic}} + \overbrace{P_H}^{\textit{hydrostatic}} \\ &= P_d + \rho \vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{x}}, \end{aligned}$$

then we get

$$\rho \frac{D\vec{\boldsymbol{u}}}{Dt} = -\vec{\boldsymbol{\nabla}} P_d + \mu \nabla^2 \vec{\boldsymbol{u}}. \tag{11}$$

By definition, if $\vec{\boldsymbol{\nabla}} P_d \neq 0 \to$ motion.

Note: $\vec{\boldsymbol{g}}$ does not appear in our equation anymore. In the presence of a free surface it would appear in the boundary condition.

Lecture 8, Non-dimensionalization, Vorticity and Strain

1 Non-dimensionalization

We would like to rewrite our equations and Boundary Conditions in dimensionless terms. Consider a flow characterized by a velocity U, length L, and density ρ_0. Change variables as follows:

$$\vec{u}^* = \frac{\vec{u}}{U}, \quad x^* = \frac{x}{L}, \quad t^* = \frac{tU}{L}, \quad \nabla^* = L\nabla, \quad P^* = \frac{P}{\Pi}, \quad \rho^* = \frac{\rho}{\rho_0}$$

where Π is still unknown at this time. We find

$$\rho_0\rho^* \cdot \frac{U^2}{L}\left(\frac{\partial \vec{u}^*}{\partial t^*} + \vec{u}^* \cdot \nabla^* \vec{u}^*\right) = -\frac{\Pi}{L}\nabla^* P^* + \rho_0\rho^*\nu\frac{U}{L^2}\left(\nabla^*\right)^2 \vec{u}^* \tag{1}$$

1.1 Case 1:

Inertial forces $\gg$ Viscous forces.

Pick $\Pi = \rho_0 U^2$. Let $\rho^* = 1$ here leaves

$$\frac{D\vec{u}^*}{Dt^*} = -\nabla^* P^* + \frac{1}{Re}\left(\nabla^*\right)^2 \vec{u}^*, \tag{2}$$

where

$$Re = \frac{UL}{\nu} = \textit{Reynolds number} = \frac{\textit{Inertial Forces}}{\textit{Viscous Forces}} = \frac{U^2/L}{\nu U/L^2}. \tag{3}$$

Note: As $Re \to \infty$, we get $\frac{D\vec{u}^*}{Dt^*} = -\nabla^* P^*$. Euler's equation a singular (highest derivative gone) limit, for inviscid flow.

1.2 Case 2:

Inertial forces $\ll$ Viscous forces.

Pick $\Pi = \rho\nu\frac{U}{L}$ leaves

$$Re\frac{D\vec{u}^*}{Dt^*} = -\nabla^* P^* + \nabla^2\vec{u}^*. \tag{4}$$

In the limit $Re \to 0$, we have

$$\nabla^* P^* = \nabla^2\vec{u}^*, \quad \nabla^* \cdot \vec{u}^* = 0 \quad \texttt{or} \quad \nabla P = \nu\nabla^2\vec{u}, \tag{5}$$

Stoke's equation. This is relevant to flows dominated by viscosity.
Some typical Re:

$$\textit{Bacteria swimming in water}: \quad Re = \frac{10\mu\text{m/s}\cdot 1\mu\text{m}}{10^{-6}\text{m}^2/\text{s}} = \frac{10^{-11}\text{m}^2/\text{s}}{10^{-6}\text{m}^2/\text{s}} = 10^{-5} \quad \textit{small}$$
$$\textit{Airplane in air}: \quad Re = \frac{100\text{m/s}\cdot 10m}{10^{-5}\text{m}^2/\text{s}} = 10^{8} \quad \textit{large}$$

1.3 Case 3:

Flows are defined by a time scale rather than a velocity (periodic forcing). Use $t^* = \omega t$, and keep $\vec{u}^* = \vec{u}/U$.

Consider $Re \gg 1$, so $\Pi = \rho U^2$ to find St.

$$\text{St}\frac{\partial \vec{u}^*}{\partial t^*} + \vec{u}^* \cdot \nabla^* \vec{u}^* = -\nabla^* P^* + \frac{1}{Re}\left(\nabla^*\right)^2 \vec{u}^* \tag{6}$$

$$\text{St} = \frac{\omega L}{U} = \frac{\textit{convective time}}{\textit{forcing time}} = \frac{L/U}{1/\omega} = \textit{Strouhal number} \tag{7}$$

2 Vorticity and Strain

Recall our decomposition:

$$\Delta\vec{u} = \Delta\vec{x}\cdot\nabla\vec{u} = \Delta\vec{x}\cdot\overline{\overline{E}} + \Delta\vec{x}\cdot\overline{\overline{\Omega}} = \frac{D\Delta\vec{x}}{Dt} \tag{8}$$

Or use

$$\vec{r} = \Delta\vec{x}: \quad \frac{D\vec{r}}{Dt} = \vec{r}\cdot\overline{\overline{E}} + \vec{r}\cdot\overline{\overline{\Omega}}. \tag{9}$$

Take $\vec{r}\cdot$ (**??**):

$$\vec{r}\cdot\frac{D\vec{r}}{Dt} = \frac{1}{2}\frac{D|\vec{r}|^2}{Dt} = \vec{r}\cdot\overline{\overline{E}}\cdot\vec{r} + \underbrace{\vec{r}\cdot\overline{\overline{\Omega}}\cdot\vec{r}}_{0\text{ (antisymmetry)}} \tag{10}$$

So $\overline{\overline{E}}$ causes changes in the length of fluid elements.

The vorticity tensor $\overline{\overline{\Omega}}$ may be written as

$$\overline{\overline{\Omega}} = \frac{1}{2}\overline{\overline{\epsilon}}\cdot\vec{\omega} = \frac{1}{2}\overline{\overline{\epsilon}}\cdot(\nabla\times\vec{u}) \tag{11}$$

where $\vec{\omega}$ is the vorticity vector.

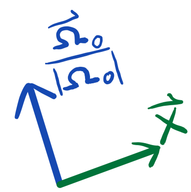
Vorticity is a measure of the local angular velocity:

$$\vec{r}\cdot\overline{\overline{\Omega}} = \frac{1}{2}\vec{r}\cdot\overline{\overline{\epsilon}}\cdot\vec{\omega} = \frac{1}{2}r_i\epsilon_{ijk}\omega_k \tag{12}$$

$$= \frac{1}{2}r_i\omega_k\epsilon_{kij} = \frac{\vec{\omega}}{2}\times\vec{r}. \tag{13}$$

This component of $\Delta\vec{u}$ is a solid body rotation, with angular velocity $\frac{1}{2}\vec{\omega}$. This illustrates that $\overline{\overline{\Omega}}$ does not change lengths, or cause stress.

Example: Rigid body solution $\vec{u} = \vec{\Omega}_0 \times \vec{x}$, $\vec{\omega} = \nabla\times\vec{u} = 2\vec{\Omega}_0$, and uniform everywhere.



We can sometimes gain valuable insight, or numerical convenience, by studying the evolution of vorticity:

Start from N-S, with ρ =constant and $\vec{f} = \rho\vec{g}$.

$$\frac{\partial\vec{u}}{dt} + \vec{u}\cdot\nabla\vec{u} = \frac{-1}{\rho}\nabla P + \nu\nabla^2\vec{u} + \vec{g}, \quad \nabla\cdot\vec{u} = 0. \tag{14}$$

Take the curl of the equation, and note that

$$\nabla\times\vec{u}\cdot\nabla\vec{u} = \vec{u}\cdot\nabla(\nabla\times\vec{u}) - (\nabla\times\vec{u})\cdot\nabla\vec{u} + \cancel{(\nabla\cdot\vec{u})}\nabla\times\vec{u} \tag{15}$$

We get the vorticity equation

$$\underbrace{\frac{D\vec{\omega}}{Dt}}_{\text{Advection of }\vec{\omega}} = \overbrace{\vec{\omega}\cdot\nabla\vec{u}}^{\text{Vortex Stretching}} + \underbrace{\nu\nabla^2\vec{\omega}}_{\text{Dissipation of }\vec{\omega}}. \tag{16}$$

Vortex Stretching: In an inviscid fluid,

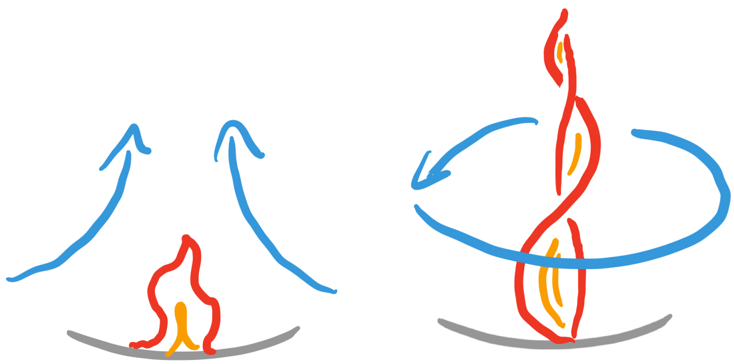
$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{u} \tag{17}$$

Recall $\nabla\vec{u} = \overline{\overline{\Omega}} + \overline{\overline{E}}$ and take (**??**) $\cdot\,\vec{\omega}$. We find

$$\frac{1}{2}\frac{D|\vec{\omega}|^2}{Dt} = \vec{\omega}\cdot\left(\overline{\overline{\Omega}} + \overline{\overline{E}}\right)\cdot\omega = \underbrace{\vec{\omega}\cdot\overline{\overline{\Omega}}\cdot\vec{\omega}}_{0\text{ (antisymmetric)}} + \vec{\omega}\cdot\overline{\overline{E}}\cdot\vec{\omega} \tag{18}$$

$$\frac{1}{2}\frac{D|\vec{\omega}|^2}{Dt} = \vec{\omega}\cdot\overline{\overline{E}}\cdot\vec{\omega} \tag{19}$$

So vorticity can be amplified by local strain. This may be thought of in terms of conservation of angular momentum. For example, a flame vortex:



Toilet, bathtub flushing rotate because of that, too.

Lecture 9

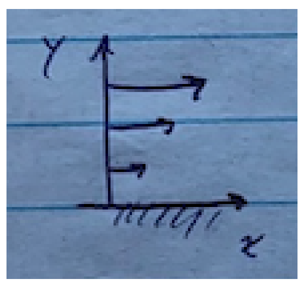
Typically in solving fluid dynamics problems we do:

1. Write N-S Equations and appropriate BCs
2. Non-dimensionalize equations
3. Compute the magnitude of dimensionless # $\rightarrow$ simplify equations as much as possible
4. Solve simplified systen

In our N-S equation

$$\rho(\vec{u}_t + \vec{u}\cdot\vec{\nabla}\vec{u}) = -\vec{\nabla}P + \mu\nabla^2\vec{u} + \vec{f}$$
$$\vec{\nabla}\cdot\vec{u} = 0,$$

the biggest complication is the NON-LINEARITY $\vec{u}\cdot\vec{\nabla}\vec{u}$.
For uni-directional flows, the nonlinearity vanishes, at any Re:$\vec{u}\perp\vec{\nabla}\vec{u}$.



$$\vec{u} = (u, \quad \overbrace{v=0}^{\text{by assumption}} \quad)$$

Because $\vec{\nabla}\cdot\vec{u} = u_x + v_y = 0$, we must have $u_x = 0$, so

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \begin{pmatrix} uu_x + vu_y & = 0 \\ uv_y + vv_y & = 0 \end{pmatrix},$$

For any Re. So we are left with:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$
$$\rho v_t = -P_y + \mu(v_{xx} + v_{yy}) \rightarrow P_y = 0,\ P = P(x)$$

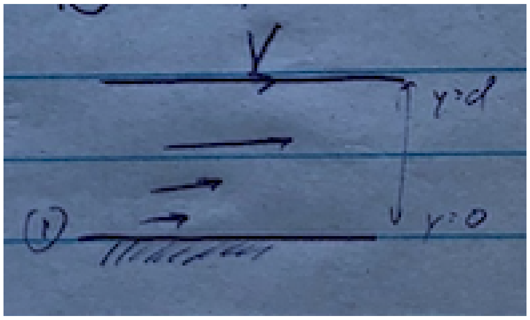
$$\frac{\partial}{\partial x}(1) \to 0 = \frac{\partial^2 P}{\partial x^2}$$

so

$$P(x) = -Gx + C, \tag{2}$$

where C is an unimportant constant.

Steady uni-directional flow:

A Couette flow (simple shear flow) 

1. u(y=0) = 0
2. u(y=d) = V
3. We assume that the pressure is the same and both ends of the container, so $P = C$.

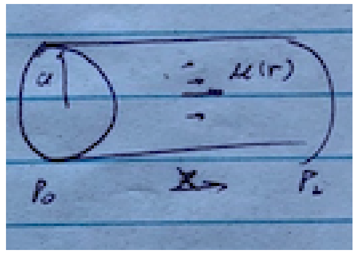
$u_{yy} = 0 \to u = Ay + B = V\frac{y}{d}$.

What is the tangential stress?

$$\begin{aligned} \hat{t} \cdot \vec{\vec{\mathbf{T}}} &= \hat{\boldsymbol{x}} \cdot \vec{\vec{\mathbf{T}}} \cdot \hat{\mathbf{y}} \\ &= T_{xy} \\ &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &= \mu\frac{V}{d} \textit{ (same everywhere)} \end{aligned}$$

This geometry is used to measure μ in viscometers.

B Poiseuille Flow (pronounced Pwazoï) (pipe flow)



Constant pressure gradient:

$$\frac{\partial P}{\partial x} = -G = \left(\frac{P_0 - P_L}{L} \right) \tag{3}$$

In cylindrical coordinates we have only

$$\mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{\partial P}{\partial x} = -G. \tag{4}$$

Integrate once:

$$r \frac{\partial u}{\partial r} = -\frac{r^2 G}{2\mu} + A \tag{5}$$

and we have $\frac{\partial u}{\partial r} = 0$ at $r = 0$ by symmetry, so $A = 0$. Dividing both sides by r and integrating once more yields

$$u = -\frac{r^2 G}{4\mu} + B \tag{6}$$

To find B

$$u(r = a) = 0 = B - \frac{a^2 G}{4\mu}$$

Thus,

$$u(r) = \frac{G}{4\mu}(a^2 - r^2). \tag{7}$$

What is the volumetric flow rate?

$$\begin{aligned} \int \int_A u(r) dA &= 2\pi \int_0^a -\frac{G}{4\mu}(r^2 - a^2) r dr \\ &= -\frac{G\pi}{2\mu} \left(\frac{r^4}{4} - \frac{a^2 r^2}{2} \right) \Big|_0^a \\ &= \frac{G\pi}{8\mu} a^4 \end{aligned}$$

What is the rate of viscous dissipation?

$$\begin{aligned}
\phi_{\text{total}} &= 2\mu \int_V \vec{\tilde{\mathbf{E}}} : \tilde{\tilde{\mathbf{E}}}\mathbf{dV},\ \textit{Bachelor Appendix: } \tilde{\tilde{\mathbf{E}}} : \mathbf{E_{rz}} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\partial \mathbf{u}}{\partial \mathbf{r}},\ \tilde{\tilde{\mathbf{E}}} : \tilde{\tilde{\mathbf{E}}} = \mathbf{2}\left(\frac{\mathbf{1}}{\mathbf{2}}\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right)^{\mathbf{2}} \\
&= 2\mu \int_V 2\left(\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)\right)^2 dV \\
&= 2\mu \int_2 \left(\frac{1}{2}\left(-\frac{2Gr}{4\mu}\right)\right)^2 dV \\
&= \mu \int_0^a dx \cdot 2\pi \int_0^a \frac{G^2 r^2}{4\mu^2} r dr \\
&= L\frac{G^2}{\mu}\frac{\pi}{2}\frac{a^4}{4} \\
&= \frac{G^2\pi a^4}{8\mu}L
\end{aligned}$$

What is the rate of work done by pressure? (Recall: Force$\cdot\vec{u} = \int\int_A \Delta P\ dA\ u$)

$$\begin{aligned}
\frac{dW}{dt} &= A \cdot \Delta P \\
&= \int_V \vec{u}\vec{\nabla} P dV \\
&= Q \cdot L\frac{\partial P}{\partial x} \\
&= Q \cdot \Delta P,
\end{aligned}$$

which is

$$\frac{\pi G a^4}{8\mu} GL,$$

which is the same as the viscous dissipation.

What is the shear on the boundaries?

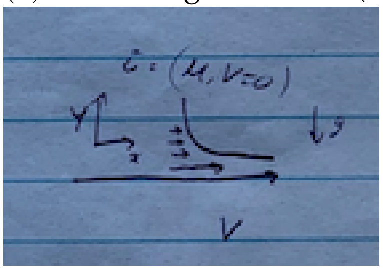
Over a length L, we have

$$\begin{aligned}
L2\pi a\left(\mu \left.\frac{\partial u}{\partial r}\right|_{r=a}\right) &= -2\pi a L\mu\frac{Ga}{2\mu} \\
&= -\pi a^2 LG \\
&= \pi a^2 \Delta P
\end{aligned}$$

So the force on the pipe is (Pipe area)$\cdot\Delta P$.

Unsteady Unidirectional flows:

(1) The Raleigh Problem (Stokes first problem)



A fluid in a semi-infinite container is initially at rest, and started impulsively. Because of the infinite domain, there is no steady-state. From vertical momentum

$$P_y = g\rho$$

From $\frac{\partial}{\partial x}$(horizontal momentum)

$$\frac{\partial^2}{\partial x^2}(P) = 0, \text{ in infinite ambient } P_x = 0$$

So we have only

$$\hat{\mathbf{x}} - \text{momentum} : \rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2},$$

or

$$u_t = \nu u_{xx}. \tag{8}$$

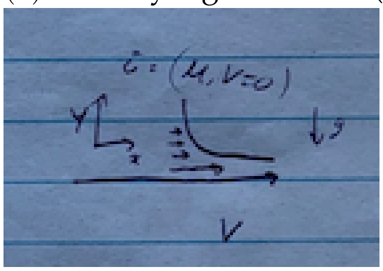
The Heat Equation to solve with:

$$\begin{aligned} \text{I.C.} : \quad & u(x, t = 0) = 0 \\ \text{B.C.} : \quad & u(0, t) = V, \text{ for } t > 0 \\ & u(y, t) < \infty, \text{ as } y \to \infty \end{aligned}$$

Note: There is no characteristic length scale here, and so no good non-dimensionalization. In such cases we often look for a similarity solution.

Lecture 10, The Y of Z

Unsteady Unidirectional flows:
(1) The Rayleigh Problem (Stokes first problem)



A fluid in a semi-infinite container is initially at rest, and started impulsively. Because of the infinite domain, there is no steady-state. From vertical momentum

$$P_y = g\rho$$

From $\frac{\partial}{\partial x}$(horizontal momentum)

$$\frac{\partial^2}{\partial x^2}(P) = 0, \text{ in infinite ambient } P_x = 0$$

So we have only

$$\hat{\mathbf{x}} - \text{momentum}: \rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2},$$

or

$$u_t = \nu u_{yy}. \tag{1}$$

The Heat Equation to solve with:

$$\begin{aligned} \text{I.C.}: \quad & u(y, t = 0) = 0 \\ \text{B.C.}: \quad & u(0, t) = V, \text{ for } t > 0 \\ & u(y, t) < \infty, \text{ as } y \to \infty \end{aligned}$$

Note: There is no characteristic length scale here, and so no good non-dimensionalization. In such cases we often look for a similarity solution.

So we seek a solution $u = f(y, t, \nu, V)$ or by linearity $\frac{u}{V} = f(y, t, \nu)$.

By Buckingham Pi Theorem: $n = 4, \quad m = 2$. So,

$$\Pi_1 = \frac{u}{V}, \quad \Pi_2 = \frac{y}{\sqrt{\nu t c}} = \eta \tag{2}$$

with c to be determined later, and we look for $\Pi_1 = F(\Pi_2)$ so $\frac{u}{V} = F(\eta)$. So we rewrite the heat equation:

$$u_t = V\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = V\frac{y}{\sqrt{\nu t c}}\left(-\frac{1}{2}\right)\frac{1}{t}F_\eta = -V\frac{\eta}{2t}F_\eta$$

$$u_{yy} = V\frac{\partial}{\partial y}(F_\eta \eta_y) = V\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\nu t c}}F_\eta\right) = V\frac{1}{\nu t c}F_{\eta\eta}$$

All together we get

$$-\frac{\eta}{2\not{t}}F_\eta = \frac{\not{\nu}}{\not{\nu}\not{t}c}F_{\eta\eta}$$

We get the ODE

$$-\eta F_\eta = F_{\eta\eta} \tag{3}$$

What about our B.C.? $y = 0 \to \eta = 0, \quad y = \infty \to \eta = \infty, \quad t = 0 \to \eta = \infty,$
so $F(0) = 1, \quad F(\infty) \to 0.$

$$-\eta F_\eta = F_{\eta\eta} \qquad \text{implies} \qquad -\frac{\eta^2}{2} + C_1 = \log F_\eta.$$

$$F_\eta = C_2 e^{-\frac{\eta^2}{2}}$$

Finally,

$$F(\eta) = C_2\int_0^\eta e^{-\frac{s^2}{2}}ds + C_3. \tag{4}$$

$$F(0) = C_3 = 1, \quad F(\infty) = 1 + C_2\frac{\sqrt{\pi}}{2} = 0, \quad \text{so} \quad C_2 = -\frac{2}{\sqrt{\pi}}.$$

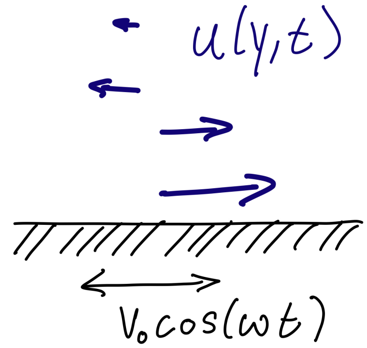
So

$$F(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\frac{s^2}{2}}ds,$$

$$u(y,t) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{y}{\sqrt{2\nu t}}} e^{-\frac{s^2}{2}}ds$$

$$= 1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right).$$

Oscillatory flate plate (Stoke's 2nd problem)



With no applied pressure gradient,

$$u_t = \nu u_{yy}$$

with

$$\begin{aligned} &u \to 0 \quad \text{as} \quad y \to \infty \\ &u(y=0,t) = V_0 \cos(\omega t) \quad \text{(No-slip)} \\ &u(y, t=0) = 0. \end{aligned}$$

We try to separate variables: $u(y,t) = \text{Re}\left(e^{i\omega t} f(y)\right)$

$$\text{To find} \quad u_t = \text{Re}\left(i\omega e^{i\omega t} f(y)\right)$$
$$u_{yy} = \text{Re}\left(e^{i\omega t} f''(y)\right).$$

So $i\omega f = \nu f''$ and $f'' - \frac{i\omega f}{\nu} = 0$.

Note: $\sqrt{i} = \frac{1+i}{\sqrt{2}}$ or $\frac{-1-i}{\sqrt{2}}$. We have

$$f(y) = C_1 \text{exp}\left(\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}y\right) + C_2 \text{exp}\left(\frac{-1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}y\right). \tag{5}$$

Note: $\delta = \sqrt{\frac{2\nu}{\omega}}$, the *decay length*.
$u \to 0$ as $y \to \infty$ implies $C_1 = 0$.
$u(y=0,t) = V_0 \cos(\omega t)$ implies $C_2 = V_0$ and

$$\text{and} \quad u(y,t) = \text{Re}\left(e^{i\omega t} \cdot V_0 e^{\frac{-1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}y}\right) = \text{Re}\left(V_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} e^{i\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)}\right)$$
$$u(y,t) = V_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) = V_0 e^{-\frac{y}{\delta}} \cos\left(\omega t - \frac{y}{\delta}\right)$$

What is the dissipation rate? $\phi = 2\mu \int_V \overline{\overline{E}} : \overline{\overline{E}} dV = \mu \int \left(\frac{\partial u}{\partial y}\right)^2 dV$.

$$u_y = \text{Re}\left(\left(\frac{-1-i}{\sqrt{2}}\right)\left(\frac{\omega}{\nu}\right)^2 V_0 e^{-\frac{y}{\delta}} e^{i\left(\omega t - \frac{y}{\delta}\right)}\right) = \text{Re}\left(\left(\frac{-1-i}{\delta}\right) e^{-\frac{y}{\delta}} e^{i\left(\omega t - \frac{y}{\delta}\right)}\right),$$
$$u_y = \frac{1}{\delta} e^{-\frac{y}{\delta}} V_0 \left(-\cos(\omega t - \frac{y}{\delta}) + \sin\left(\omega t - \frac{y}{\delta}\right)\right),$$
$$u_y^2 = \frac{1}{\delta^2} e^{-\frac{2y}{\delta}} V_0^2 \left(1 - 2\cos\left(\omega t - \frac{y}{\delta}\right)\sin\left(\omega t - \frac{y}{\delta}\right)\right)$$

$$< \phi >= \mu \int_0^{\infty} dy \int_0^{\frac{2\pi}{\omega}} \left(\frac{\omega}{2\pi}\right) dt (u_y)^2 dy = \frac{\mu}{\delta^2}\frac{\delta}{2}V_0^2 = \frac{\mu V_0^2}{2\delta}. \quad (6)$$

Compare this to the rate of work done by the bottom stress:

$$\begin{aligned} u \cdot \mu \frac{\partial u}{\partial y} &= V_0 e^{-\frac{y}{\delta}} \cos\left(\omega t - \frac{y}{\delta}\right) \mu \cdot \frac{1}{\delta} e^{-\frac{y}{\delta}} V_0 \left(-\cos\left(\omega t - \frac{y}{\delta}\right) + \sin\left(\omega t - \frac{y}{\delta}\right)\right) \\ \mu u \frac{\partial u}{\partial y} &= \mu \frac{V_0^2}{\delta}\left(-\cos\left(\omega t - \frac{y}{\delta}\right) + \sin\left(\omega t - \frac{y}{\delta}\right)\right). \end{aligned}$$

Averaged over time:

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left(\mu u \frac{\partial u}{\partial y}\right) dt = \frac{\mu V_0^2}{2\delta} \to \text{The same!}$$

So all the work done at the boundary gets dissipated.

Streak, Path, and Streamlines

There are 3 important types of curves that can help visualize a flow. They often get confused with each other because at steady-state, the are all the same. But in time-dependent flow, they are different.

Streak Lines: Location of all particles having passed through a point.
In experiments, this corresponds to a dye or smoke line. The special point is the source of dye or smoke, and what we can see is where all the dye/smoke gets transported to.

Note that you can write a mathematical expression for a streak line, but it involves tricky notation as you need to be able to invert a function that tracks particles over time.

Path line: Trajectory of a single particle over time.
This is mathematically easier to describe. Is you record a video of an experiment where you can follow a specific particle, this is the path, $x_p(t)$, traced by that particle over time. We then have

$$\frac{dx_p}{dt} = \vec{u}(\vec{x}_p, t)$$

Streamline: This is the most commonly used curve to represent a flow. These are curves that are everywhere tangent to the *instantaneous* velocity field. So you can parametrize such a curve with a parameter s that is different from time. As a certain time t_0, a streamline $\vec{x}_s(s)$ then satisfies

$$\frac{dx_s}{ds} = \vec{u}(\vec{x}_s, t_0)$$

This provides a snapshot of what the flow looks like, though it is tricky to interpret when the flow is not steady.

As mentioned above, if the flow is steady, streamlines, path lines, and streak lines all overlap.

Inviscid Flow and Bernoulli's law

We now turn to inviscid flow, which will be our approximation to High Reynolds number flow (Re $\gg 1$).

In this regime, the Navier-Stokes equations become Euler's equations

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \vec{f}, \qquad \nabla \cdot \vec{u} = 0$$

This is only directly applicable to superfluids, such as liquid Helium at very cold temperatures. But it is a good approximation at high Reynolds numbers away from boundaries.

We consider first a simpler set-up where we have:

1. Constant density ρ.

2. Still inviscid ($\mu = 0$).
3. Steady Flow ($\frac{\partial}{\partial t} = 0$).

Our momentum equation reduces to

$$\rho\, \vec{u} \cdot \nabla \vec{u} = -\nabla P + \vec{f}. \tag{1}$$

We further assume that the body force is conservative, we have $\vec{f} = -\nabla \Psi$. For example, for gravity $\Psi = \vec{g} \cdot \vec{x}$.
We also make use of the identity:

$$\frac{1}{2}(\vec{u} \wedge (\nabla \wedge \vec{u})) = \nabla \left(\frac{|\vec{u}|^2}{2} \right) - \vec{u} \cdot \nabla \vec{u}$$

We may now rewrite equation (1) as

$$\rho(\nabla \left(\frac{|\vec{u}|^2}{2} \right) - \frac{1}{2}(\vec{u} \wedge (\nabla \wedge \vec{u}))) = -\nabla P - \nabla \Psi. \tag{2}$$

Finally, we may take $\vec{u}\cdot$(2) and group terms to find

$$\vec{u} \cdot \nabla \left(\rho \frac{|\vec{u}|^2}{2} + P + \Psi \right) = 0$$

Therefore, *following a streamline* (parallel to $\vec{u}$), we obtain Bernoulli's Law:

$$\frac{\rho |\vec{u}|^2}{2} + P + \Psi = \text{Constant along a streamline} \tag{3}$$

Note that this is a statement of conservation of energy, and can only be applied when viscous dissipation can be ignored (so no turbulence).

Let's see some applications of Bernoulli's law.

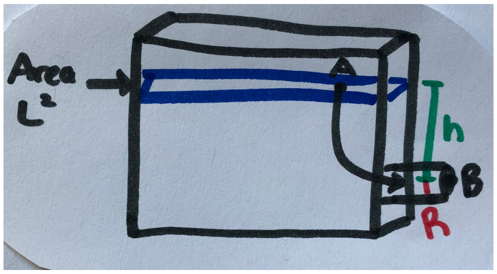
Example 1: Egyptian water clock.

Here the body force is simply gravity, so $\Psi = \rho g z$. Bernoulli's law thus becomes:

$$\frac{\rho U^2}{2} + P + \rho g z = C.$$

where we use $U = |\vec{u}|$. We consider that points A and B are joined by a streamline so that we have

$$\frac{\rho U_A^2}{2} + P_A + \rho g h = \frac{\rho U_B^2}{2} + P_B.$$



Schematics of an Egyptian water clock

Because both A and B are exposed to the atmospheric pressure (neglecting surface tension), we must have that $P_A = P_B$.
To close the problem, we consider mass conservation:

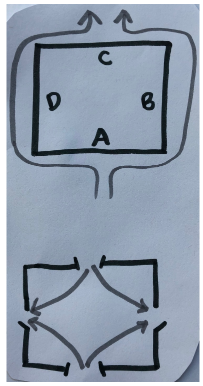
$$L^2 U_A = \pi R^2 U_B \quad \text{or} \quad U_A = \frac{\pi R^2}{L^2} U_B \ll U_B$$

Solving for U_B, we have

$$U_B = \sqrt{\frac{2}{\rho}} \sqrt{\rho g h + \frac{1}{2}\rho U_A^2} \approx \sqrt{2gh}.$$

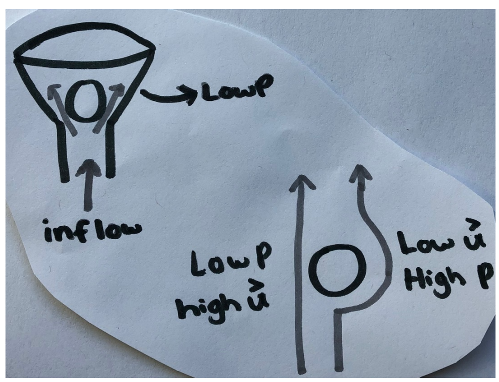
Note that this is not convenient, as the rate of change of the height depends on the water's depth. What shape should the vessel have so that the height changes at a constant rate?

Example 2: Wind past a house with open doors.



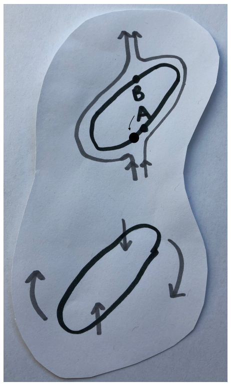
Points A and C are stagnation points, where $|\vec{u}| = 0$. Therefore, P_A and P_C are high. Points B and D see a large wind speed, and thus P_B and P_D are low. This generates flow within the house, and opens doors (which ones depends on how the doors are setup).

Example 3: Ping-pong ball in a funnel and in a jet



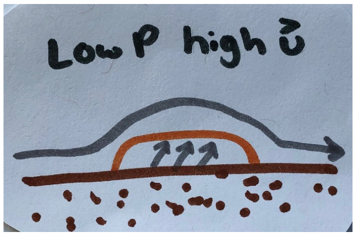
The ball tends to stick to the sides of the funnel and is not blown off
A ping-pong ball in a jet is stable to perturbations

Example 4: Canoe (or other boat) in the wind



Points A and B are stagnation points, where P is maximum. Torque is thus induces on the canoe until it is perpendicular to the wind.

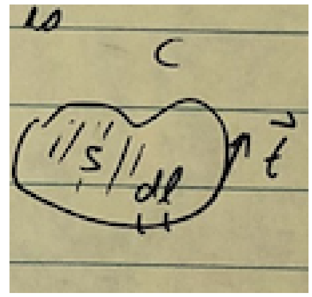
Example 5: Sand-dollars in flow.



Suction is created into a sand-dollar as flow goes over it.

Lecture 11: Kelvin Circulation Theorem

Consider a simple closed curve C that is deforming with the (inviscid) fluid. Recall that circulation is



$$\Gamma = \oint_{C(t)} \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{t}}\, dl = \oint_{C(t)} \vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{dx}} = \int_{S(t)} (\vec{\boldsymbol{\nabla}} \wedge \vec{\boldsymbol{u}}) \cdot \hat{n} dA \tag{1}$$

where S is any smooth surface (with normal $\hat{n}$) that has C as its only boundary. Kelvin's Circulation Theorem states that (if $\mu = 0$), for $\rho =$ constant, $\vec{\boldsymbol{f}} = -\vec{\boldsymbol{\nabla}}\psi$ we have

$$\frac{d\Gamma}{dt} = 0 \rightarrow \Gamma = \text{constant} \tag{2}$$

<u>Proof</u>: We first introduce useful notation. We consider that the curve $C(t)$ is parametrized in terms of a parameter s as $\vec{\boldsymbol{x}}(s,t)$. We also define a differential element for any quantity, scalar or vector, Z along this curve as

$$dZ = \frac{dZ}{ds}\, ds, \quad \text{so notably} \quad \overrightarrow{\boldsymbol{dx}} = \frac{d\vec{\boldsymbol{x}}}{ds}\, ds$$

Also, we consider that the force field applied to the fluid is conservative $\vec{f} = -\vec{\boldsymbol{\nabla}}\Psi$. We also consider a fluid of constant density so that the Euler's equation describing the evolution of the velocity field is

$$\frac{D\vec{\boldsymbol{u}}}{Dt} = \frac{\partial \vec{\boldsymbol{u}}}{\partial t} + \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} = -\vec{\boldsymbol{\nabla}}(\frac{P + \Psi}{\rho})$$

We begin by computing the time derivative of the circulation, using the definition of derivative

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \lim_{\Delta t \to 0} \frac{\Gamma(t + \Delta t) - \Gamma(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\oint_{C(t+\Delta t)} \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s, t+\Delta t), t+\Delta t) \cdot \overrightarrow{\boldsymbol{dx}}(s, t+\Delta t) - \oint_{C(t)} \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t), t+\Delta t) \cdot \overrightarrow{\boldsymbol{dx}}(s,t)}{\Delta t}
\end{aligned}$$

where $\vec{\boldsymbol{x}}(s, t + \Delta t)$ is the parametrisation of $C(t + \Delta t)$.

Our main goal is to rewrite $\Gamma(t + \Delta t)$ as an integral of $C(t)$ so that we can rewrite the derivative of Γ as a single integral.

We begin by approximating $\vec{x}(x, t+\Delta t)$:

$$\vec{\boldsymbol{x}}(s,t+\Delta t) = \vec{\boldsymbol{x}}(s,t) + \Delta t \left.\frac{d\vec{\boldsymbol{x}}}{dt}\right|_{(s,t)} + O((\Delta t)^2) = \vec{\boldsymbol{x}}(s,t) + \Delta t\, \vec{\boldsymbol{u}}(s,t) + O((\Delta t)^2) \tag{3}$$

We can apply a similar expansion to a curve element $\overrightarrow{\boldsymbol{dx}}$

$$\overrightarrow{\boldsymbol{dx}}(s,t+\Delta t) = \overrightarrow{\boldsymbol{dx}}(s,t) + \Delta t\, \overrightarrow{\boldsymbol{du}}(s,t) + O((\Delta t)^2) \tag{4}$$

and to the velocity vector at $\vec{\boldsymbol{x}}(s,t+\Delta t)$ as well

$$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t+\Delta t), t+\Delta t) = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t+\Delta t), t) + \Delta t \left.\frac{\partial \vec{\boldsymbol{u}}}{\partial t}\right|_{(\vec{\boldsymbol{x}}(s,t+\Delta t),t)} + O((\Delta t)^2) \tag{5}$$

We now use equation (3) to expand $\vec{\boldsymbol{u}}$ further

$$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t+\Delta t), t+\Delta t) = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t), t) + \left.\frac{d\vec{\boldsymbol{x}}}{dt} \cdot \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}\right|_{(s,t)} + \Delta t \left.\frac{\partial \vec{\boldsymbol{u}}}{\partial t}\right|_{(\vec{\boldsymbol{x}}(s,t),t)} + O((\Delta t)^2) \tag{6}$$

So the integrand of $\Gamma(t+\Delta t)$ can be rewritten in terms of the parametrization of $C(t)$ as

$$\begin{aligned}\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t+\Delta t), t+\Delta t) \cdot \overrightarrow{\boldsymbol{dx}}(s,t+\Delta t) \quad = \quad & \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t),t) \cdot \overrightarrow{\boldsymbol{dx}}(s,t) + \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}(s,t),t) \cdot \overrightarrow{\boldsymbol{du}}(\vec{\boldsymbol{x}}(s,t)\ \Delta t + \\ & \left.\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}\right|_{(s,t)} + \\ & \left.\frac{\partial \vec{\boldsymbol{u}}}{\partial t}\right|_{(\vec{\boldsymbol{x}}(s,t),t)} \cdot \overrightarrow{\boldsymbol{dx}}(s,t)\ \Delta t + O((\Delta t)^2)\end{aligned}$$

We can therefore write

$$\Gamma(t+\Delta t) = \oint_{C(t)} \vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{dx}} + \vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{du}}\ \Delta t + \left(\frac{\partial \vec{\boldsymbol{u}}}{\partial t} + \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}\right) \cdot \overrightarrow{\boldsymbol{dx}}\ \Delta t + O((\Delta t)^2)$$

Going back to the derivative of Γ, we get some cancellations are and left with

$$\frac{d\Gamma}{dt} = \oint_C \vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{du}} + \frac{D\vec{\boldsymbol{u}}}{Dt} \cdot \overrightarrow{\boldsymbol{dx}}$$

Now, because we consider an inviscid fluid, we have

$$\frac{D\vec{\boldsymbol{u}}}{Dt} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}(P+\Psi)$$

and we may also write that $\vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{du}} = \frac{1}{2}\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}) \cdot \overrightarrow{\boldsymbol{dx}}$. This leaves

$$\frac{d\Gamma}{dt} = \oint_C \vec{\boldsymbol{\nabla}} \left(\frac{1}{2}\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} - \frac{1}{\rho}(P+\Psi)\right) \cdot \overrightarrow{\boldsymbol{dx}} = 0$$

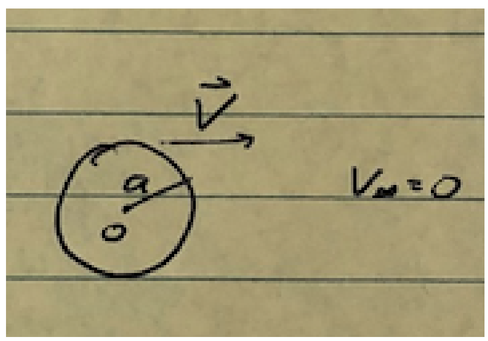
Where the last equaloty comes from Stokes theorem applied to a gradient field.

Irrotational Flows

(Helmoltz decomposition: if $\vec{\boldsymbol{u}} \in C^2$, then $\vec{\boldsymbol{u}} = \vec{\boldsymbol{\nabla}}\phi + \vec{\boldsymbol{\nabla}} \wedge \vec{\boldsymbol{A}}$ is always possible).

Special class of flows where $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\nabla}} \wedge \vec{\boldsymbol{u}} = \vec{\boldsymbol{0}}$. Key point: Because of Kelvin's Circulation Theorem, if a fluid is inviscid and irrotational, then it will remain irrotational forever.
Velocity potential: All irrotational velocity fields may be written as the gradient of a potential: $\vec{\boldsymbol{u}} = \vec{\boldsymbol{\nabla}}\phi$. Because $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}} = 0$, we have $\nabla^2\phi = 0$, so ϕ is an harmonic function.
The boundary conditions on ϕ are that of no penetration: $\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}}\phi = 0$. One usually solves for ϕ via:

1. Separation of variables
2. Complex potentials
3. Fourier Transforms
4. Numerics

Ex: Irrotational flow past a sphere 

$$\nabla^2\phi = 0, with \vec{\boldsymbol{\nabla}}\phi \to 0 \text{ as } \hat{\mathbf{r}} \to \infty$$
$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}}\phi = \vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}} \text{ on } r = a$$

In spherical coordinates:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\phi\right) = 0 \text{ (assume } \phi = \phi(r) \text{ only)} \tag{7}$$

Solving directly, we may find a scalar solution: $\phi_{-1} = \frac{C}{r} \to \frac{1}{r}$.
Note: $\nabla^2\nabla_i\phi = \nabla_i\nabla^2\phi,\ \nabla^2\nabla_i\nabla_j\phi = \nabla_i\nabla_j\nabla^2\phi$.
So we can use this to find vector, tensor solutions to $\nabla^2\phi = 0$.

$$\phi_{-2} = \vec{\boldsymbol{\nabla}}\phi_{-1} = \frac{\vec{\boldsymbol{x}}}{r^3} \text{ is a vector solution}$$
$$\phi_{-3} = \vec{\boldsymbol{\nabla}}\phi_{-2} = \frac{\bar{\bar{I}}}{r^3} - \frac{3\vec{\boldsymbol{x}}\vec{\boldsymbol{x}}}{r^5} \text{ is a tensor solution.}$$

Here, we need a scalar solution, linear in $\vec{\boldsymbol{V}}$. Try:

$$\phi = \vec{\boldsymbol{V}} \cdot \phi_{-2} = C\frac{\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}}}{r^3} \tag{8}$$

This implies that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{\nabla}}\phi = C\vec{\boldsymbol{V}} \cdot \left(\frac{\bar{\bar{I}}}{r^3} - \frac{3\vec{\boldsymbol{x}}\vec{\boldsymbol{x}}}{r^5}\right)$
Our B.C. says that

$$\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}|_{r=a} = C\left(\frac{\vec{\boldsymbol{V}}}{r^3} \cdot \hat{\boldsymbol{n}} - \frac{3\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{x}}(\vec{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}})}{r^5}\right) = \vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}$$

On the boundary $\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{x}}}{a}$, so

$$\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}|_{r=a} = C\left(\frac{\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}}{a^3} - \frac{3\vec{\boldsymbol{V}} \cdot a\hat{\boldsymbol{n}}(a\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}})}{a^5}\right) = \underbrace{C\frac{\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}}{a^3}(-2)}_{\to C = -\frac{a^3}{2}} = \vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}$$

Lecture 13: Unsteady Invisid Flows, Irrotational

Assume we have an inviscid limit

$$\mu = 0$$

and irrotational flow such that

$$\vec{\boldsymbol{\omega}} = \nabla \times \vec{\boldsymbol{u}} = 0$$
$$\vec{\boldsymbol{u}} = \nabla\overline{\phi}.$$

The the Navier-Stokes equations reduce to

$$\vec{\boldsymbol{\nabla}}^2\overline{\phi} = 0 \tag{1}$$

$$\rho\frac{\partial\overline{\phi}}{\partial t} + \frac{\rho}{2}\nabla\overline{\phi}\cdot\nabla\overline{\phi} + P + \psi = f(t) \tag{2}$$

if $\vec{\boldsymbol{f}} = -\nabla\psi$. For us, $\psi = \rho g h$ where g is gravity and h is height. Note also that $f(t)$ is in time only.

To simplify the problem, we can absorb $f(t)$ into $\phi(\vec{\boldsymbol{x}}, t)$ as seen below.

$$\phi(\vec{\boldsymbol{x}}, t) = \overline{\phi}(\vec{\boldsymbol{x}}, t) + \int_0^t f(s)ds$$

Using this, Eqs. (1)-(2), now becomes Eqs. (3)-(4).

$$\vec{\boldsymbol{\nabla}}^2\phi = 0 \tag{3}$$

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}\nabla\phi\cdot\nabla\phi + P + \psi = 0 \tag{4}$$

Let's apply this to water waves as seen in Fig. 1.

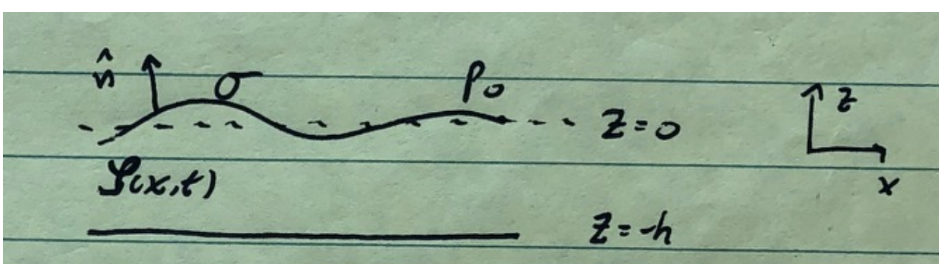


Figure 1: Water Waves set-up

To a good approximation, small amplitude waves are irrotational and inviscid.

Previously, we derived a stress jump condition for constant surface tension σ

$$\hat{\boldsymbol{n}}\cdot(\overline{\overline{T_a}} - \overline{\overline{T_w}}) = \hat{\boldsymbol{n}}\sigma(\nabla_S\cdot\hat{\boldsymbol{n}})$$

where $\overline{\overline{T_a}} = -P_0\overline{\overline{I}}$ and $\overline{\overline{T_w}} = -P\overline{\overline{I}}$. We know P_0 and want to find P. So,

$$P = P_0 + \sigma(\nabla_S \cdot \hat{\boldsymbol{n}}).$$

We need a normal vector:

$$\hat{\boldsymbol{n}} = \frac{(-\zeta_x, 1)}{(1+\zeta_x^2)^{-1/2}}.$$

And we need to compute the curvature:

$$\begin{aligned}\nabla_S \cdot \hat{\boldsymbol{n}} &= \frac{\partial}{\partial x}\hat{\boldsymbol{n}}_x \\ &= \frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}.\end{aligned}$$

So,

$$P = P_0 - \frac{\sigma\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}. \tag{5}$$

We can plug Eq. (5) into Eqs. (3)-(4) to find the final system of equations that we are solving.

What are the relevant Boundary Conditions?

1. at $z = -h$, $v = \frac{\partial \phi}{\partial z}$
2. As $x \to \pm\infty$, ϕ, h are finite
3. At $z = \zeta$:
$$\hat{\boldsymbol{n}} \cdot \nabla\phi = \hat{\boldsymbol{n}}\frac{\partial \zeta}{\partial t} \cdot \hat{k}$$
$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial x}$$
   - This is also known as the kinematic condition
4. $\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}\nabla \cdot \phi\nabla \cdot \nabla\phi + P_0 + \sigma(\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}) + \rho g h = 0$
   - This is also known as the dynamic condition

We now need to make some simplifications:

1. Assume small waves, where $\zeta^2 << \zeta$, $\phi^2 << \phi$, $\phi\zeta < \zeta$
2. Assume that boundary conditions at $z = \zeta$ can be applied at $z = 0$

We then have:

$$\nabla^2\phi = 0$$
$$\frac{\partial\phi}{\partial z} = 0 \text{ at z =-h}$$
$$\frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial z} \text{ at z = 0}$$

Linearizing the dynamic condition:

$$\rho\frac{\partial\phi}{\partial t} + P_0 - \sigma\zeta_{xx} + \rho g\zeta = 0 \text{ at the interface.}$$

And linearizing the kinematic condition:

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial y} \text{ at the interface.}$$

To describe waves, we seek solutions of the form:

$$\zeta = \hat{\zeta}e^{ik(x-ct)}$$
$$\phi = \hat{\phi}e^{ik(x-ct)}$$

Where $k = \frac{2\pi}{\lambda}$ is the wave number, λ is the wave length, and $c = \frac{\omega}{k}$ is the speed at which the wave travels.

From $\nabla^2\phi$, we find

$$\hat{\phi}'' - k^2\hat{\phi} = 0$$

so

$$\hat{\phi}(z) = c_1\cosh zk + c_2\sinh zk.$$

Using $\frac{\partial\hat{\phi}}{\partial z}|_{z=-h} = 0$, we get $\hat{\phi}(z) = c_1\cosh k(z+h)$.

Using $\frac{\partial\hat{\phi}}{\partial z}|_{z=0} = \frac{\partial\phi}{\partial z}$, we get $-ikc\hat{\zeta} = c_1 k\sinh kh$.

Finally from the dynamic condition, we find $-\rho ikcc_1\cosh(kh) + \rho g\hat{\zeta} + \sigma k^2\hat{\zeta} = 0$.

Combining:

$$(-k\rho c^2\tanh^{-1}(kh) + \rho g + \sigma k^2)\hat{\zeta} = 0$$
$$c^2 = \frac{\sigma k^2 + \rho g}{k\rho}\tanh kh$$
$$\omega^2 = \left(gk + \frac{\sigma k^3}{\rho}\right)\tanh kh$$

Physical Interpretation:

Consider the Bond number:

$$Bo = \frac{\rho g}{\sigma k^2} = \frac{\text{gravity}}{\text{surface tension}} = \frac{\rho g}{k^2 \sigma}$$

We can use this to re-write ω^2

$$\omega^2 = gk\left(1 + \frac{1}{Bo}\right)\tanh(kh)$$

For air-water, if Bo 1, $\lambda_c \approx 1.7cm \rightarrow$ capillary wave.

For $Bo >> 1$ (or $\lambda >> \lambda_c$, surface tension is negligible $\rightarrow$ gravity wave

For gravity waves: $Bo >> 1$, $c^2 = \frac{g}{k} \tanh kh$

a In shallow water, $kh << 1$, $\tanh kh \approx kh$
  So , $c = \sqrt{gh}$, $\omega = \sqrt{gh}k$, meaning all waves travel at the same speed
  One can only reliable surf in shallow water!

b In deep water, $kh >> 1$, $\tanh kh \approx 1$
  So , $c = \sqrt{\frac{g}{k}}$, $\omega = \sqrt{gk}$, meaning long waves travel faster

For capillary waves: $Bo << 1$, $c^2 = \frac{\sigma k}{\rho} \tanh kh$

a In shallow water, $kh << 1$, $\tanh kh \approx kh$
  So , $c = \sqrt{\frac{\sigma h}{\rho}}k$, $\omega = \sqrt{\frac{\sigma h}{\rho}}k^2$, meaning short wavelengths travel faster

b In deep water, $kh >> 1$, $\tanh kh \approx 1$
  So , $c = \sqrt{\frac{\sigma}{\rho}}k^{1/2}$, $\omega = \sqrt{\frac{\sigma}{\rho}}k^{3/2}$, meaning short wavelengths travel faster

This relationship is plotted in Fig. 2

A note on dispersive systems:

Whenever c depends on k, the system is called dispersive and the various modes (Fourier components) separate (disperse). In these systems, the ENERGY on the waves travel at a speed different than the phase speed $c = \frac{\omega}{k}$. Instead they travel at the group velocity $c_g = \frac{d\omega}{dk}$. This is the velocity at which a wave-packet travels.

For each of the regimes above, the group velocities are:

- Shallow gravity waves: $c_g = c$

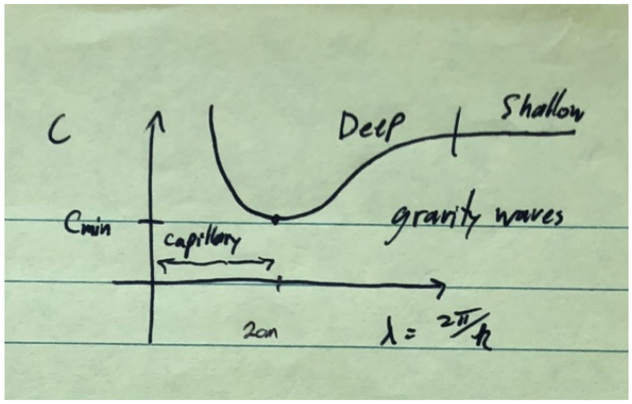


Figure 2: Plotting c vs λ

- Deep gravity waves: $c_g = \frac{1}{2}c$
- Shallow capillary waves: $c_g = 2c$
- Deep Capillary waves $c_g = \frac{3}{2}c$

Consider flow past a submerged obstacle, such as a rock in a river, as depicted in Fig. 3.

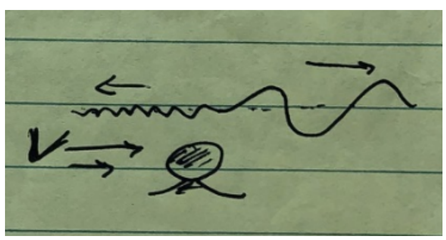


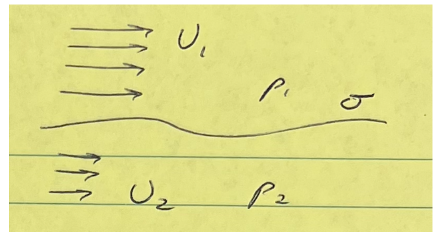
Figure 3: Flow past a submerged obstacle.

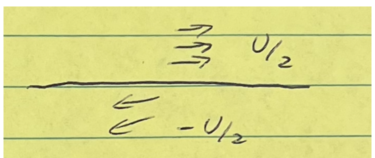
If $v < c_{min}$, no steady waves are generated.

If $v > c_{min}$, there are 2 k values for which $c = v$ and the waves appear steady. In this case, the smaller k (bigger λ) is a gravity wave. It will have $c_g < c = v$, meaning the energy will be swept downstream. The larger k (smaller λ) is a capillary wave. It will have $c_g > c = v$, meaning the energy will travel upstream.

Lecture 13: Kelvin-Helmholtz Instability

Introduction: The question is whether the perturbation will grow or decay over time. Consider inviscid layers moving with velocity U_1 and U_2 parallel to their interface (shown on right). We will use inviscid limit, and look at small perturbation of an interface moving with velocity $(U_1 + U_2)/2$ as shown below.





With interface moving at $U_s = (U_1 + U_2)/2$, let $U = U_1 - U_2$ and $U_s = 0$, we get the setup as shown on the left. From this point on, we will denote the velocity of top layer as $U_1 = U/2$ and velocity of bottem layer as $U_2 = -U/2$ (obtained by plugging in U_1 and U_2).

Setup: First consider an interface $y = \eta(x, z, t)$. When interface is unperturbed, $\eta = 0$.

Assume irrotational and inviscid flow tells us that with Helmhold decomposition, we get

$$\vec{\boldsymbol{u}}_i = \nabla \Phi_i \text{ with } i = 1, 2 \tag{1}$$
$$\nabla^2 \Phi_i = 0 \text{ with } i = 1, 2. \tag{2}$$

We can write the velocity potential that describe the perturbed flow above and below the interface as

$$\Phi_i = \Phi_{iB} + \Phi_{io} \text{ with } i = 1, 2$$

where Φ_B is the base-state and Φ_o is the perturbation.

If we are at base-state (when surface is unperturbed) we have

$$\eta = 0, \tag{3}$$
$$\Phi_{1B} = \frac{Ux}{2}, \tag{4}$$
$$\Phi_{2B} = \frac{-Ux}{2}. \tag{5}$$

Boundary Conditions: We consider following boundary condition

$$\Phi_1 \to \frac{Ux}{2} \text{ as } y \to \infty, \Phi_2 \to \frac{-Ux}{2} \text{ as } y \to -\infty, \tag{6}$$
$$\hat{\boldsymbol{n}} \cdot \nabla \Phi_1 = \hat{\boldsymbol{n}} \cdot \nabla \Phi_2 = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{u}}_s \text{ Kinematic B.C. on } y = \eta, \tag{7}$$
$$P_1 - P_2 = \sigma \cdot \text{curvature} \approx -\sigma \nabla^2 \eta \text{ Dynamic B.C..} \tag{8}$$

where σ is the surface tension, $\hat{\boldsymbol{n}}$ is the normal of the interface, and $\vec{\boldsymbol{u}}_s$ is the surface velocity. The surface is $y = \eta(x, z, t)$ or $f(x, z, y, t) = y - \eta = 0$. Since we assume small perturbation, we can assume velocity of the surface is purely vertical

$$\vec{\boldsymbol{u}}_s = \frac{\partial \eta}{\partial t}\boldsymbol{e}_y. \tag{9}$$

Also, normal vector $\hat{\boldsymbol{n}}$ is

$$\hat{\boldsymbol{n}} = \frac{\nabla f}{|\nabla f|} = \frac{-(\partial\eta/\partial x)\boldsymbol{e}_x - (\partial\eta/\partial z)\boldsymbol{e}_z + \boldsymbol{e}_y}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial z)^2}} \tag{10}$$

Bernoulli's relation: Here we still consider the Bernoulli's relation, which gives us

$$\rho_1\frac{\partial \Phi_1}{\partial t} + \rho_1\frac{|\nabla\Phi_1|^2}{2} + P_1 + \rho_1 g y = f_1(t), \tag{11}$$
$$\rho_2\frac{\partial \Phi_2}{\partial t} + \rho_2\frac{|\nabla\Phi_2|^2}{2} + P_2 + \rho_2 g y = f_2(t). \tag{12}$$

At base-state, we have

$$0 + \rho_1\frac{U^2}{8} + P_1 + 0 = f_1(t), \tag{13}$$
$$0 + \rho_2\frac{U^2}{8} + P_2 + 0 = f_2(t). \tag{14}$$

At base-state, $P_1 = P_2$ so subtrating the two equations at base-state gives us

$$f_1 - f_2 = (\rho_1 - \rho_2)\frac{U^2}{8}. \tag{15}$$

We apply the dynamic boundary condition to Eq. 11 and Eq. 12 and get

$$\begin{aligned} P_1 - P_2 = -\sigma\nabla^2\eta = (f_1 - f_2) + \rho_2\frac{\partial\Phi_2}{\partial t} - \rho_1\frac{\partial\Phi_1}{\partial t} + (\rho_2 - \rho_1)g\eta \\ + \frac{\rho_2}{2}|\nabla\Phi_2|^2 - \frac{\rho_1}{2}|\nabla\Phi_1|^2. \end{aligned} \tag{16}$$

Before we plug in the base-state function, linearize the following by droopping quadratic terms

$$\begin{aligned} \Phi &= \Phi_B + \Phi_o \\ \nabla\Phi &= \nabla\Phi_B + \nabla\Phi_o \\ |\nabla\Phi|^2 &= |\nabla\Phi_B|^2 + 2\nabla\Phi_B \cdot \nabla\Phi_o + |\nabla\Phi_o|^2 \\ &= \frac{U^2}{4} + U\frac{\partial\Phi_o}{\partial x} + \text{ small.} \end{aligned}$$

After plugging in, we get

$$P_1 - P_2 = -\sigma\nabla^2\eta = \cancel{(\rho_1 - \rho_2)\frac{U^2}{8}} + \rho_2\frac{\partial\Phi_2}{\partial t} - \rho_1\frac{\partial\Phi_1}{\partial t} + (\rho_2-\rho_1)g\eta + \cancel{(\rho_2 - \rho_1)\frac{U^2}{8}} \tag{17}$$

$$+ \rho_2\frac{U}{2}\frac{\partial\Phi_{2o}}{\partial x} + \rho_1\frac{U}{2}\frac{\partial\Phi_{1o}}{\partial x} \tag{18}$$

$$= \rho_2\frac{\partial\Phi_{2o}}{\partial t} - \rho_1\frac{\partial\Phi_{1o}}{\partial t} + (\rho_2-\rho_1)g\eta + \rho_2\frac{U}{2}\frac{\partial\Phi_{2o}}{\partial x} + \rho_1\frac{U}{2}\frac{\partial\Phi_{1o}}{\partial x}$$

Note that this is the equation perturbed equation Φ_{io} has to satisfy since by plugging in base state, we are basically subtraction the requirement of the base-state out of the equation.

<u>Method of Normal Modes:</u> Here we look for oscillatory solution for perturbed equation Φ_{1o} and Φ_{2o}

$$\Phi_1 = \frac{Ux}{2} + \Phi_{1o}e^{i\alpha x + i\beta z + \omega t}e^{-ky} \tag{19}$$

$$\Phi_2 = \frac{-Ux}{2} + \Phi_{2o}e^{i\alpha x + i\beta z + \omega t}e^{-ky} \tag{20}$$

$$\eta = 0 + \eta_0 e^{i\alpha x + i\beta z + \omega t}. \tag{21}$$

Here by using Laplace's equation $\nabla^2\Phi_1 = 0$, we can deduce that $\alpha^2 + \beta^2 = k^2$.

Next, we apply the **Kinematic B.C.**:

$$\frac{\partial\eta}{\partial t}\boldsymbol{e}_y \cdot \hat{\boldsymbol{n}} = \nabla\Phi_1 \cdot \hat{\boldsymbol{n}}$$

$$\omega\eta_0 = -\frac{i\alpha U}{2}\eta_0 + \Phi_{1o}\alpha^2\eta_0 + \Phi_{1o}\beta^2\eta_0 - k\Phi_{1o}$$

Ignoring the product of quadratic terms since small perturbations

$$\omega\eta_0 = -\frac{U}{2}\eta_0 - k\Phi_{1o}$$

$$\Phi_{1o} = \frac{-1}{k}\left(\omega\eta_0 + \frac{i\alpha U}{2}\eta_0\right).$$

Doing the same steps, we get

$$\Phi_{2o} = \frac{1}{k}\left(\omega\eta_0 - \frac{i\alpha U}{2}\eta_0\right).$$

From balancing the stress:

$$+\sigma(\eta_{xx}+\eta_{zz}) = \rho_2\Phi_{2o}\omega - \rho_1\Phi_{1o}\omega + (\rho_2-\rho_1)g\eta_0 + \frac{\rho_2}{2}\frac{U^2}{4} + \frac{\rho_2}{2}(-2\frac{U}{2}i\alpha\Phi_{2o})$$
$$-\frac{\rho_1}{2}\frac{U^2}{8} - \frac{\rho_1}{2}(2\frac{U}{2}i\alpha\Phi_{1o}) + \frac{U^2}{8}(\rho_1-\rho_2)$$
$$-\sigma\nabla^2\eta_0 = (\rho_2-\rho_1)g\eta_0 + \omega(\rho_2\Phi_{2o}-\rho_1\Phi_{1o}) + \frac{Ui\alpha}{2}(-\rho_2\Phi_{2o}-\rho_1\Phi_{1o})$$

so we get

$$0 = \eta_0\Bigg[\sigma\nabla^2 + (\rho_2-\rho_1)g + \frac{\omega}{k}(\rho_2\left(\omega - \frac{i\alpha U}{2}\right) + \rho_1\left(\omega + \frac{i\alpha U}{2}\right)) + \frac{Ui\alpha}{2k}(-\rho_2\left(\omega - \frac{i\alpha U}{2}\right) + \rho_1\left(\omega + \frac{i\alpha U}{2}\right))\Bigg] \tag{22}$$

and, finally, (Phew...)

$$0 = \omega^2 + \omega\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)i\alpha U + \frac{\sigma k^2}{\rho_1+\rho_2} + \frac{\rho_1-\rho_2}{\rho_1+\rho_2}gk - \frac{U^2\alpha^2}{4}. \tag{23}$$

If we solve for ω, we get

$$\omega = \left[\frac{i\alpha U(\rho_2-\rho_1)}{2(\rho_1+\rho_2)} \pm \left(\frac{\alpha^2U^2\rho_1\rho_2}{(\rho_1+\rho_2)^2} - \frac{\sigma k^3 + (\rho_2-\rho_1)gk}{\rho_1+\rho_2}\right)^{1/2}\right]. \tag{24}$$

The stability of the system is determined by the

$$sgn\left(\frac{\alpha^2U^2\rho_1\rho_2}{(\rho_1+\rho_2)^2} - \frac{\sigma k^3 + (\rho_2-\rho_1)gk}{\rho_1+\rho_2}\right). \tag{25}$$

Note that β does not appear here. The most unstable modes will be those where $\beta = 0$, α is maximize, $\alpha = k$

There are modes parallel to shear U

We have instability if $Re(\omega) > 0$, so $\nabla > 0$ so

$$\left(\frac{\alpha^2U^2\rho_1\rho_2}{(\rho_1+\rho_2)^2} > \frac{\sigma k^3 + (\rho_2-\rho_1)gk}{\rho_1+\rho_2}\right) \tag{26}$$

so if

$$U > \frac{(\rho_1+\rho_2)}{\rho_1\rho_2}(\sigma k + (\rho_2-\rho_1)\frac{g}{k})^{1/2}. \tag{27}$$

In a Special case: At rest $U = 0$ (neglecting diffusion), we have instability if $\sigma k^2 + (\rho_2 - \rho_1)g < 0$ or $\sigma k^2 < (\rho_1 - \rho_2)g$. So if the upper layer is heavier $(\rho_1 - \rho_2) > 0$, then surface tension stabilizes large k (small wavelengths) If the horizontal extent is too small ($\lambda = 2\pi/k$)

$$k > \left(\frac{(\rho_1 - \rho_2)g}{\sigma}\right)^{1/2}, \lambda < 2\pi\left(\frac{\sigma}{(\rho_1 - \rho_2)g}\right)^{1/2} \tag{28}$$

Stokes Flow

We now turn to the opposite limit where $Re \ll 1$, and inertial effects become negligible.

The limit case where $Re = 0$ is known as Stokes Flow. Note that this is not a singular limit. Recall that

$$Re = \frac{\text{Diffusive time scale}}{\text{Intertial time scale}}$$

and we can also introduce a Strouhal number, St

$$St = \frac{\text{Inertial time scale}}{\text{Forcing time scale}}$$

The non-dimensional momentum equation the becomes

$$Re\ St\,\frac{\partial \vec{u}}{\partial t} + Re\,\vec{u}\cdot\nabla\vec{u} = -\vec{\nabla}P + \nabla^2\vec{u} + \vec{f}$$

If the forcing is sufficiently fast, the time dependent term may have to be kept even at small Reynolds number. We will focus here on situations where that is not the case and consider the Stokes equations

$$0 = -\vec{\nabla}P + \nabla^2\vec{u} + \vec{f} \quad \text{and} \quad \vec{\nabla}\cdot\vec{u} = 0$$

or in dimensional form

$$0 = -\vec{\nabla}P + \mu\nabla^2\vec{u} + \vec{F} \tag{1}$$

Note that even if there are no time derivatives, unsteady flows may arise due to time-dependent boundary conditions.

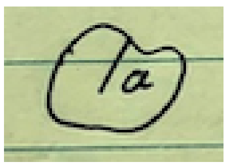
A few things to consider for Stokes flow:

1. Taking the divergence of (1) yields $\nabla^2 P = 0$.
2. Taking the curl of (1) yields $\nabla^2\vec{\omega} = 0$. This can be thought of as $\frac{\partial\vec{\omega}}{\partial t} = D\nabla^2\omega$ in the limit of D going to infinity. So the vorticity diffuses instantaneously in Stokes flow.
3. Taking the curl of the curl of (1) yields $\nabla^2\nabla^2\vec{u} = 0$. This can be used to solve for Stokes flow, especially when combined with a streamfunction $\vec{\Psi}$ that we define such that $\vec{u} = \vec{\nabla}\wedge\vec{\Psi}$ with $\vec{\nabla}\cdot\vec{\Psi} = 0$. In that case, we also have the biharmonic equation for Ψ: $\nabla^2\nabla^2\Psi = 0$.

There are a few important **General Properties of Stokes Flow**:

1. No inertia
2. Quasi-Steady
3. Linearity
4. Time reversibility

1. No inertia: Because inertia plays no role, the flow is completely determined by the instantaneous forces (no memory/history).
Consider the translation of an object due to a force $\vec{F}$



Hydrodynamic Drag at low Re

$$\begin{aligned} D &= 6\mu\pi UaC \text{ (For a sphere } C = 1\text{, proof to come)} \\ &= \text{Force in steady motion} \end{aligned}$$

If we now set $F = 0$, how long does it take for the body to stop? Roughly:

$$m\frac{du}{dt} = -6\mu Ua\pi, \quad (2)$$

which is readily solved

$$\begin{aligned} U(t) &= U_0 e^{-6\mu a\pi t/m} \\ &= U_0 e^{-t/\tau}, \end{aligned}$$

where $\tau = \frac{m}{6\pi\mu a} = \frac{2a^2}{\nu}$.
Compare τ to the time taken to travel one particle size:

$$t_a = \frac{a}{U},$$

so

$$\frac{\tau}{t_a} = \frac{2a^2}{\nu}\frac{U}{a} = 2Re. \quad (3)$$

For example, for an organism of $10\mu m$, with $U \sim 10^{-3}cm/s,\ \nu = 10^{-2}cm^2/s$, we get

$$\tau = \frac{2 \cdot 10^{-10}m^2}{10^{-6}m^2/s} = 2 \cdot 10^{-4}s,$$

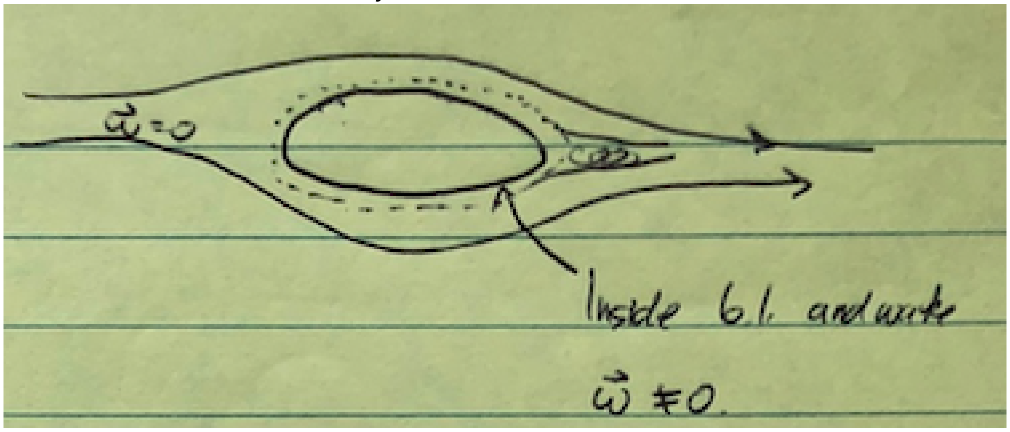
and a distance travelled of

$$d = U\tau \approx 10^{-5} m/s \cdot 2 \cdot 10^{-4} s = 2 \cdot 10^{-9} m = 2nm \leftarrow \text{very tiny}$$

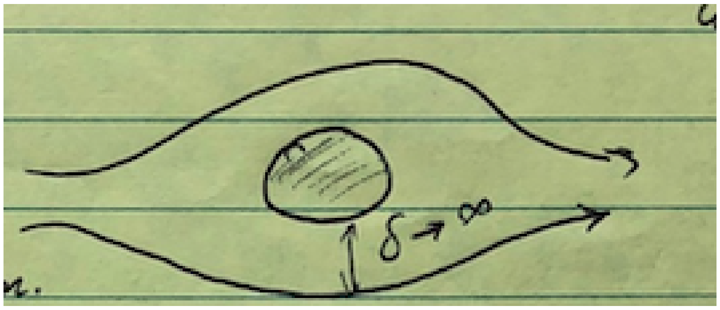
Do not be confused: The absence of inertia does not mean that acceleration does take place. It only says that those accelerations are not dynamically significant.

2. Quasi-steady: There are no time derivatives in the Stokes equations. However, the B.C. may be time-dependent.
   Because of the instantaneous diffusion of momentum and vorticity, the velocity is completely prescribed by the force and B.C. at any given time.
   You can think of boundaries as a source of vorticity.

Ex.: For a streamlined body

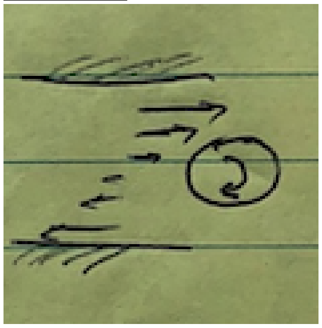


For a low Re flow the whole domain is a viscous boundary layer.



3. Linearity: The Stokes equation are linear. So if we have solutions $u^{(1)}$, $P^{(1)}$ and $u^{(2)}$, $P^{(2)}$, then $c_1u^{(1)}+c_2u^{(2)},\ c_1P^{(1)}+c_2P^{(2)}$ are solutions too.

Example: Sphere in a linear shear flow $\to$ is there lift



B.C.:

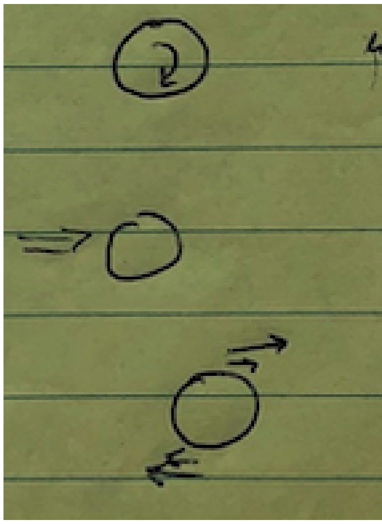
1. $u\to u_\infty=\vec{\boldsymbol{u}}+\underbrace{\Gamma\cdot\vec{\boldsymbol{x}}}_{\text{Linear shear}}$ as $x\to\infty$;
2. $u=\Omega\wedge\vec{\boldsymbol{x}}$ on S.

Solve

$$\mu \nabla^2 \vec{\boldsymbol{u}} = \vec{\boldsymbol{\nabla}} P, \;\; \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}} = 0.$$

Break it down

- Sphere rotating in a stagnant fluid: $u_\infty \to 0, \; \vec{\boldsymbol{u}} = \Omega \wedge \vec{\boldsymbol{x}}$ on S. So Lift $= 0$ by symmetry;
- Sphere in uniform flow: $u_\infty = \vec{\boldsymbol{u}}$. So Lift $= 0$ by symmetry;

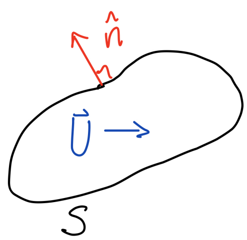


- Non-rotating flow centered on a particle Lift $= 0$ by symmetry.

So adding it up, we must have Lift $= 0$ in our problem.

Lecture 14

Further consequence of linearity:



Consider the motion $\vec{U}(t)$ of an arbitrary rigid particle. The fluid response $\vec{u}(\vec{x},t), P(\vec{x},t)$ are forced by the motion of the boundary $\vec{U}(t)$.

$\vec{u}, P$ are lineary related to $\vec{U}$. The stress tensor $\overline{\overline{T}} = -P\overline{\overline{I}} + \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T\right)$ is also linearly related to $\vec{U}$.

So $\vec{F}_H = \int_S \hat{n} \cdot \overline{\overline{T}} = -P\overline{\overline{T}} dS$ is also linearly related to $\vec{U}$, so may write $\vec{F}_H = \mu\overline{\overline{A}} \cdot \vec{U}$, where $\overline{\overline{A}}$ depends on the particle shape. Drag $\propto \vec{U}$.

If the particle also rotates with angular velocity Ω, we can write $\vec{F}_H = \mu\overline{\overline{A}} \cdot \vec{U}$ and similarly for the torque.

(4) Time reversibility (symmetry in time):

We saw that $\vec{u} \propto$ *applied forces*. So if we reversed forces, we would reverse the velocities $\rightarrow$ time reversed.

Note: For regular Navier-Stokes, this is not true because of inertia.

Using time reversibility and spacial symmetries, it is possible to rule out certain flows.

Ex: 1) Sphere falling near a wall:

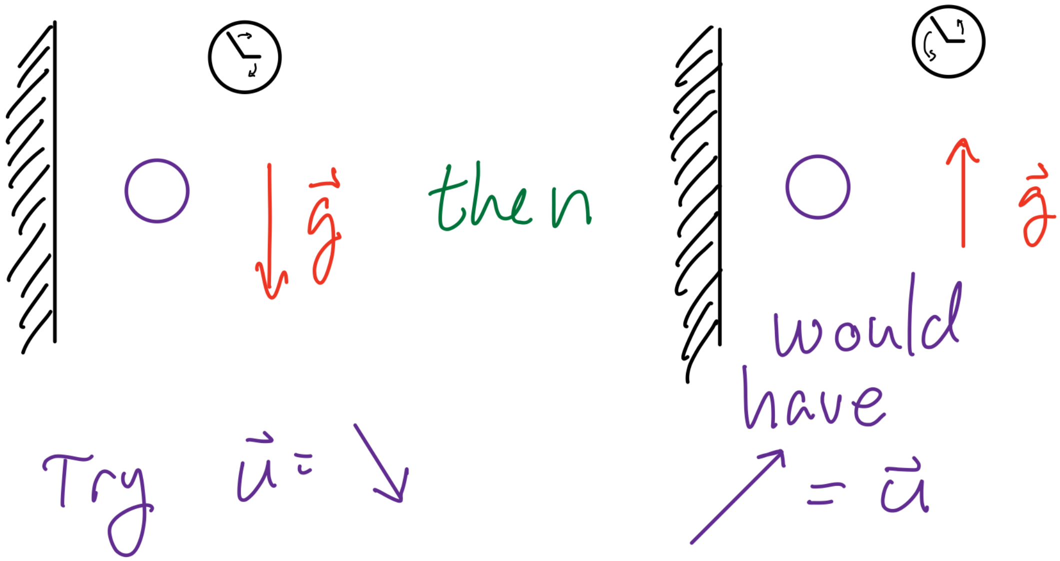


Figure 1: So the sphere must fall vertically.

Ex: 2) Ellipsoid falling under gravity, does it rotate?

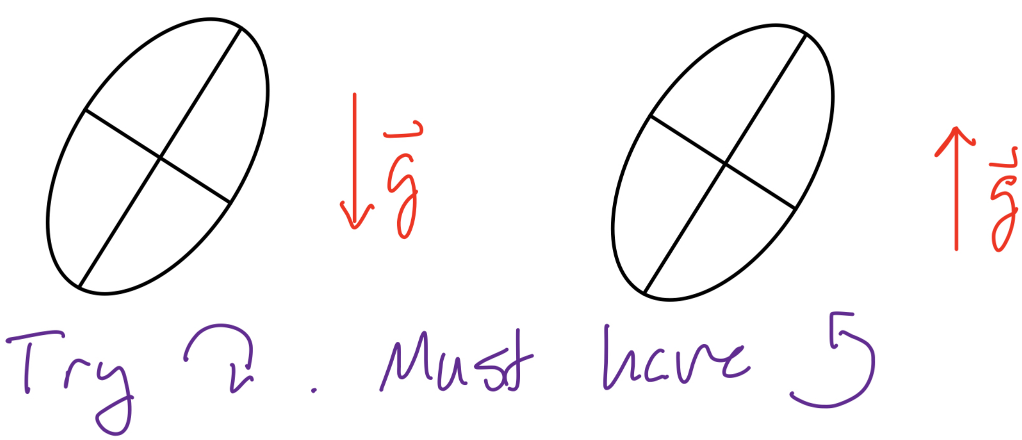


Figure 2: Try clockwise $\vec{u}$, we must have counterclockwise $\vec{u}$ for reversed gravity, but this is the same system rotated, so no rotation is possible.

Vector methods for solving Stokes flow problems

Ref. Disorder and Mixing, Guyon, Chap 3, Hinch's Method.

Preliminaries: We distinguish between "true" and "pseudo" vectors.

Ex: True vectors are $\vec{u}, \vec{x}, \vec{F}, \nabla$.

Pseudo-vectors are obtained via a cross-product, which involves a "right-hand" rule, which has an arbitrary direction. So angular velocity $\vec{\omega}$, torque $\vec{L}$ are pseudo-vectors.

Relation between *true vectors* ^{T} and *pseudo-vectors* ^{P}:

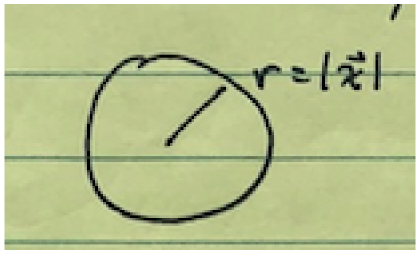
(i) $T \times T = P$; $\qquad \vec{x} \times \vec{F} = \vec{L}, \qquad \nabla \times \vec{u} = \vec{\omega}$

(ii) $P \times T = T$, and $T \times P = T$; $\qquad \vec{\Omega} \times \vec{x} = \vec{u}, \qquad \nabla \times \vec{\omega} = \nabla^2 \vec{u}$

(iii)
- $T \cdot P =$ pseudo-scalar, because the sign is arbitrary.
- $T \cdot T =$ true scalar
- $P \cdot P =$ true scalar

(iv) $P \times P = P$ of arbitrary sign. (Count "signs", with $\times$ counting as one, $\cdot$ not counting as one.)

Vector methods for solving Stokes flow problems, continued

Consider a sphere 

and look for spherically symmetric solutions.
Validity region:

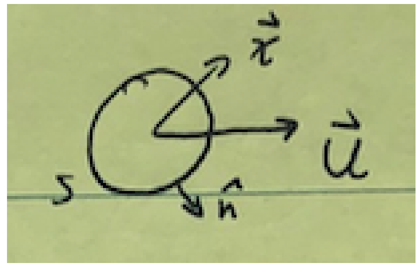
Exterior	Interior
$(r \neq 0, r \to \infty)$	$(r < R_0,\ r = 0)$
$\phi_{-1} = 1/r$	$1 = \phi_0 = \nabla \cdot \vec{\phi}_1$
$\vec{\phi}_{-2} = \nabla \vec{\phi}_{-1} = \vec{x}/r^3$	$\vec{x} = \vec{\phi}_1$
$\overline{\overline{\phi}}_{-3} = \frac{\overline{\overline{I}}}{r^3} - \frac{3\vec{x}\vec{x}}{r^5}$	$r^2\overline{\overline{I}} - 3\vec{x}\vec{x} = \overline{\overline{\phi}}_2$

We are now ready to use the Hinch Method for Stokes flow.
We are solving
$$\mu \nabla^2 \vec{u} = \vec{\nabla} P, \quad \vec{\nabla} \cdot \vec{u} = 0.$$
We postulate the following forms of the velocity and pressure:

$$\vec{u}(\vec{x}) = \vec{\nabla}\phi + \vec{x} \wedge \vec{\nabla}\psi + \vec{\nabla}(\vec{x} \cdot \vec{A}) - 2\vec{A},$$
$$P(\vec{x}) = 2\mu \vec{\nabla} \cdot \vec{A},$$

where $\nabla^2\phi = \nabla^2\psi = \nabla^2\vec{A} = 0$ (you will verify the two equations above in your homework).
Note that here ψ is a pseudo-scalar, ϕ is a true scalar and $\vec{A}$ is true vector.
We want to exploit the linearity of the Stokes equations to solve for ϕ, ψ and $\vec{A}$. We can then compute $\vec{u}$ and P from there. We will separate the homogeneous and the inhomogeneous parts:

$$\vec{u} = \vec{u}_H + \vec{u}_i \qquad \mu\nabla^2\vec{u}_H = 0 \qquad \mu\nabla^2\vec{u}_i = \vec{\nabla}P$$
$$\vec{\nabla} \cdot \vec{u}_H = 0 \qquad \vec{\nabla} \cdot \vec{u}_P = 0$$

Example: Sphere translating in a fluid 

In non-dimensional form:

$$\nabla^2\vec{u} = \vec{\nabla}P \ \text{ B.C } \ \vec{u} = \vec{U} \text{on S}$$
$$\vec{\nabla}\cdot\vec{u} = 0 \ \ \vec{u} \to 0 \text{ as } r \to \infty$$

We look for $\phi,\ \vec{A},\ \psi$ that are linear in $\vec{U}$ and the exterior harmonics $\phi_{(n+1)}$. We must choose:

$$\phi = \alpha\vec{U}\cdot\vec{\phi}_{-2} = \alpha\frac{\vec{U\cdot}\vec{x}}{r^3}$$
$$\psi = 0 \text{ (Nothing else works)}$$
$$\vec{A} = \beta\vec{U}\phi_{-1} + \gamma\vec{U}\cdot\overline{\overline{\phi}}_{-3} = \beta\frac{\vec{U}}{r} + \gamma\vec{U}\cdot\overline{\overline{\phi}}_{-3}.$$

Note:

$$\vec{U}\cdot\overline{\overline{\phi}}_{-3} = \vec{U}\cdot\vec{\nabla}\vec{\nabla}\phi_{-1} = \vec{U}\cdot\vec{\nabla}\vec{\nabla}\left(\frac{1}{r}\right)$$

and

$$\vec{\nabla}\cdot(\vec{U}\cdot\overline{\overline{\phi}}_{-3}) = \vec{U}\cdot\vec{\nabla}\left(\nabla^2\left(\frac{1}{r}\right)\right) = 0,$$

so this does not change P. In general, any divergece-free contribution to $\vec{A}$ can be absorbed into terms involving $\vec{\psi}$ or ϕ, so they may be neglected. So

$$\phi = \alpha\frac{\vec{U}\cdot\vec{x}}{r^3},\ \ \vec{A} = \beta\vec{U}/r$$
$$\vec{u} = \vec{\nabla}\phi + \vec{\nabla}(\vec{x}\cdot\vec{A}) - 2\vec{A}$$

Note:

$$\vec{\nabla}(\vec{x}\cdot\vec{A}) - 2\vec{A} = \overline{\overline{I}}\cdot\vec{A} + \vec{x}\cdot\vec{\nabla}\vec{A} - 2\vec{A} = \vec{x}\cdot\vec{\nabla}\vec{A} - \vec{A},$$

which gives:

$$\beta\vec{U}\cdot\left(-\frac{\vec{x}\vec{x}}{r^3} - \frac{\overline{\overline{I}}}{r}\right) \text{ and } \vec{\nabla}\phi = \alpha\vec{U}\cdot\left(\frac{\overline{\overline{I}}}{r^3} - 3\frac{\vec{x}\vec{x}}{r^5}\right)$$

So
$$\vec{u} = \vec{U} \cdot \left(\alpha \left(\frac{\overline{\overline{I}}}{r^3} - 3\frac{\vec{x}\vec{x}}{r^5} \right) - \beta \left(\frac{\overline{\overline{I}}}{r} + \frac{\vec{x}\vec{x}}{r^3} \right) \right) \tag{1}$$
At $r = 1$, we have $\vec{u} = \vec{U}$, which gives
$$(\alpha - \beta)\overline{\overline{I}} + (-3\alpha - \beta)\vec{x}\vec{x} = \overline{\overline{I}}$$
So,
$$\alpha - \beta = 1, \quad -3\alpha - \beta = 0$$
we find $\alpha = 1/4, \;\; \beta = -3/4$. So overall, in dimensional form:
$$P(\vec{x}) = 2\mu\vec{\nabla} \cdot \left(\beta\frac{\vec{U}}{r} \right) = \frac{3}{2}\mu\vec{U}a \cdot \frac{\vec{x}}{r^3}$$
$$\vec{u}(\vec{x}) = \vec{U} \cdot \left(\frac{3a}{4} \left(\frac{\overline{\overline{I}}}{r} + \frac{\vec{x}\vec{x}}{r^3} \right) + \frac{a^3}{4} \left(\frac{\overline{\overline{I}}}{r^3} - \frac{3\vec{x}\vec{x}}{r^5} \right) \right)$$
<u>Note</u>: As $r \to \infty$, this decays only as a/r (very slow).
For irrotational flow, we had
$$\vec{u} = \vec{\nabla}\phi = \frac{a^3}{2} \left(\vec{U} \cdot \left(\frac{\vec{x}\vec{x}}{r^5} - \frac{\overline{\overline{I}}}{r^3} \right) \right),$$
which decays as a^3/r^3. Now
$$\vec{u} \sim U\vec{a} \cdot \left(\frac{3}{4} \left(\frac{\overline{\overline{I}}}{r} + \frac{\vec{x}\vec{x}}{r^3} \right) \right) \sim \frac{Ua}{r}$$

Stokes Drag

Importantly, we can now calculate the hydrodynamic drag on a sphere
$$\vec{F}_H = \int_S \hat{n} \cdot \overline{\overline{T}} dS = \int_S \hat{n} \cdot (-P\overline{\overline{I}} + \mu(\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T))dS$$
On the surface, $\hat{n} = \vec{x}/a$ and one finds
$$\hat{n} \cdot \overline{\overline{T}}|_{r=a} = -\frac{3}{2}\frac{\vec{U}\mu}{a},$$
so
$$\vec{F}_H = -\frac{3}{2}\frac{\vec{U}\mu}{a}\int_S dS = -\frac{3}{2}\frac{\vec{U}\mu}{a} \cdot 4\pi a^2 = -6\pi\mu\vec{U}a$$

For a sphere sedimentary under gravity:

$$\vec{F}_H + \text{Bouyancy} + \text{Gravity} = 0 \;\; (\text{No acceleration if } Re = 0)$$
$$-6\pi\mu\vec{U}a - \frac{4\pi}{3}a^3\vec{g}\rho_w + \frac{4\pi}{3}a^3\vec{g}\rho_S = 0$$

So

$$\vec{U} = \frac{1}{6\pi\mu a}\frac{4\pi}{3}a^3\vec{g}(\rho_S - \rho_w) = \frac{2}{9}\frac{a^2\vec{g}(\rho_S - \rho_w)}{\mu} \leftarrow \text{Stokes Settling speed}$$

and the flow around it is

$$\vec{u}(\vec{x}) = \frac{3a}{4}\vec{U}\cdot\left(\frac{\overline{\overline{I}}}{r} + \frac{\vec{x}\vec{x}}{r^3}\right) + \frac{a^3}{4}\vec{U}\cdot\left(\frac{\overline{\overline{I}}}{r^3} - \frac{3\vec{x}\vec{x}}{r^5}\right)$$

$$\vec{F}_H = -6\pi\mu U\vec{a}$$

If we let $\vec{F}$ fix, but let $a \to 0$, we find the response to a POINT FORCE

$$\vec{u} = -\frac{\vec{F}_H}{8\pi\mu}\left(\frac{\overline{\overline{I}}}{r} + \frac{\vec{x}\vec{x}}{r^3}\right)$$

this is called a STOKESLET.

Second Example: Rotating sphere in our infinite fluid

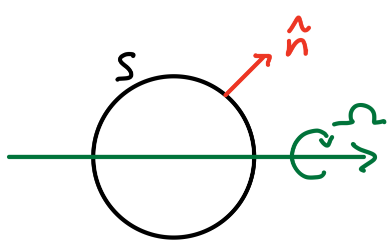


Figure 1: *

Solve:

$$\nabla^2\vec{u} = \nabla P, \qquad \nabla\cdot\vec{u} = 0, \tag{2}$$

with boundary conditions:

$$\vec{u} = \vec{\Omega}\times\vec{x} \quad on \quad S, \quad (r = 1, \hat{n} = \vec{x}) \tag{3}$$

$$\vec{u} \to 0 \quad as \quad |\vec{x}| \to \infty. \tag{4}$$

We seek $\phi, \psi, \vec{A}$ linear in the pseudo-vector $\vec{\Omega}$ and ϕ_{-n}.

$$\phi = 0, \qquad \phi = \alpha \frac{\vec{\Omega} \cdot \vec{x}}{r^3}, \qquad \vec{A} = \beta \frac{\vec{\Omega} \times \vec{x}}{r^3} + \underbrace{\left(\gamma \frac{\vec{\Omega}}{r}\right)}_{\text{pseudo, so no.}} \tag{5}$$

But

$$\nabla \cdot \vec{A} = \beta \nabla_i \left(\frac{\Omega_j x_k}{r^3} \varepsilon_{jki} \right) = \beta \Omega_j \varepsilon_{jki} \left(\frac{\delta_{ik}}{r^3} - \frac{3 x_j x_k}{r^5} \right) = 0. \tag{6}$$

So $\vec{A}$ will not contribute and we have:

$$\begin{aligned} \vec{u}(\vec{x}) = \vec{x} \times \nabla \psi = \alpha \left(\vec{x} \times \nabla \frac{\vec{\Omega} \cdot \vec{x}}{r^3} \right) &= \alpha x_i \nabla_j \frac{(\Omega_k x_k)}{r^3} \varepsilon_{ijm} \\ &= \alpha x_i \Omega_k \left(\frac{\delta_{jk}}{r^3} - \frac{3 x_k x_j}{r^5} \right) \varepsilon_{ijm} = \alpha \left(\frac{x_i \Omega_k}{r^5} \varepsilon_{ikm} - \underbrace{\frac{3 \Omega_k x_i x_j x_k}{r^5} \varepsilon_{ijm}}_{\to 0} \right) \\ &= \alpha \frac{\vec{x}}{r^3} \times \vec{\Omega}. \end{aligned}$$

So at $r = 1$, $\quad \vec{u}(\vec{x}) = -\alpha (\vec{\Omega} \times \vec{x}) \quad$ and $\alpha = -1$ (or dimensionally $-a^3$).

So we find a uniform pressure ($P = 0$) and a (dimensional) velocity of

$$\vec{u}(\vec{x}) = \frac{\vec{\Omega} \times \vec{x}}{r^3} a^3 \tag{7}$$

Note: This decays as $\frac{1}{r^2}$ as $r \to \infty$.

What is the torque?

$$\vec{L}_H = \int_S \vec{x} \times \hat{n} \cdot \overline{\overline{T}} dS, \qquad \vec{A} = 0 \quad \text{so} \quad P = 0. \tag{8}$$

$$u_k = \Omega_i \frac{x_j}{r^3} \varepsilon_{ijk}, \quad \nabla \vec{u} = \nabla_m u_k = \Omega_i \left(\frac{\delta_{mj}}{r^3} - \frac{3 x_j x_m}{r^5} \right) \varepsilon_{ijk}$$

and

$$(\nabla \vec{u})^T = \Omega_i \left(\frac{\delta_{jk}}{r^3} - \frac{3 x_j x_k}{r^5} \right) \varepsilon_{ijm}.$$

So

$$T_{mk} = \Omega_i \underbrace{\left(\frac{\varepsilon_{ikm}}{r^3} + \frac{\varepsilon_{imk}}{r^3} \right)}_{\to 0} - 3 \frac{\Omega_i}{r^5} \left(x_j x_m \varepsilon_{ijk} + x_j x_k \varepsilon_{ijm} \right)$$

and

$$\begin{aligned} \hat{n} \cdot \bar{\bar{T}} \Big|_{r=1} &= x_m \left(-3\Omega_i \left(x_j x_m \varepsilon_{ijm} + x_j x_k \varepsilon_{ijm}\right)\right) \\ &= -3\Omega_i \left(x_j \varepsilon_{ijk} + \cancelto{0}{x_j x_k x_m \varepsilon_{ijm}}\right) \\ &= -3\left(\vec{\Omega} \times \vec{x}\right) \end{aligned}$$

So

$$\begin{aligned} \vec{L} &= \int_S \hat{n} \times \left(-3\Omega \times \hat{n}\right) dS = 3 \int_S \hat{n} \left(\hat{n} \cdot \vec{\Omega}\right) - \vec{\Omega} dS \\ &= 3 \int_S \left(\hat{n}\hat{n} - \bar{\bar{I}}\right) \cdot \vec{\Omega} dS = 3 \left(\int_V \nabla \vec{x} dV - \int_S \bar{\bar{I}} dS\right) \cdot \vec{\Omega} \\ &= 3 \left(\frac{4\pi}{3} \bar{\bar{I}} - 4\pi \bar{\bar{I}}\right) \cdot \vec{\Omega} = -8\pi \vec{\Omega}. \end{aligned}$$

In dimensional form, $\vec{L}^H = -8\pi\mu a^3 \vec{\Omega}$. (opposes rotation)
So

$$\vec{u}(\vec{x}) = -\frac{\vec{L}^H}{8\pi\mu} \times \underbrace{\frac{\vec{x}}{r^3}}_{\textit{rotlet}} \tag{9}$$

Introduction to stability analyses (linear)

We will use the method of normal modes to study the stability of a PDE or system of PDEs.

The approach is:

1) Find an equilibrium solution (often it is the trivial solution). It must satisfy PDE and BCs
2) Introduce perturbations: $e^{\sigma t} e^{i\alpha x}$ or $e^{\sigma t} f(x)$. They too must satisfy BCs, but not PDE
3) Use PDE to determine in what conditions $\sigma > 0$ leading to perturbation growth, and instability. Usually this will depend on parameters in the equation. It can also depend on aspects of the perturbation, like their wavelength.

Ex: Consider a periodic (= B.C.) domain of size L and the PDE

$$u_t = D u_{xx} + E u_{xxxx}$$

$u_e = 0$ is a solution. Is it stable?

perturbations are: $p(x,t) = e^{\sigma t} e^{i\frac{2\pi n}{L}x}$ ← General form of periodic modes with period L.

Plug it in: $u = u_e + p(x,t) \rightarrow \sigma p = -D\left(\frac{2\pi n}{L}\right)^2 p + E\left(\frac{2\pi n}{L}\right)^4 p$

The critical value will be when the real part of σ changes sign. Here everything is real (but that is not always the case)

Unstable if $\sigma_c \geq 0 \rightarrow D\left(\frac{2\pi n}{L}\right)^2 \leq E\left(\frac{2n\pi}{L}\right)^4 \rightarrow D \leq E\left(\frac{2n\pi}{L}\right)^2$

~~$\frac{D}{E} \leq \left(\frac{2\pi n}{L}\right)^2$~~

See next page

~~If $D>0$, for $E>0$ $\exists n$-unstable; for $E<0$ always stable; If $D<0$; $E>0$ all unstable, $E<0$ most unstable is n=1~~

~~$+D$ ≥ $\left(\frac{2m\pi}{L}\right)^2$, $+E$~~ ~~$\frac{D}{E} \leq \left(\frac{2\pi}{L}\right)^2$~~ → stable

Most unstable is n=1

Analysis of $D \leq E \left(\frac{2n\pi}{L}\right)^2$ ①

If D=0, E>0 → unstable. D=0, E<0 → stable. E=0, D>0 stable, E=0, D<0 → unstable

If D>0 and E>0

There is always some n for which ① is met

so the system is unstable, with larger n being the most unstable

If D>0 and E<0

① is not satisfied for any n → stable

If D<0 and E>0

① is always satisfied → unstable for any n

If D<0 and E<0

① becomes $|D| \geq |E| \left(\frac{2n\pi}{L}\right)^2$ ② is the condition for instability.

For large n, ② will not be satisfied.

The smallest possible n is n=1.

So if $|D| \geq |E| \left(\frac{2\pi}{L}\right)^2$ the system is unstable

The most unstable mode is n=1.

Rayleigh-Taylor Linear Stability Analysis

We now turn to study the instabilities generally called convection, which arise (literally) when a fluid is heated from below. This is a large class of problems and the observed flows are typically far from the onset of instability and thus non-linear. We focus here on the linear regime that captures the boundary between stable and unstable systems.

We first assume that the density of the fluid is linearly related to the temperature

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

where T_0 is a reference temperature at which the density is ρ_0. Here $\alpha = \frac{1}{\rho_0}\frac{d\rho}{dT}$ is the *thermal expansion coefficient*. For example in water at room temperature, $\alpha \approx 10^{-4}/^\circ K$.

We consider a two-dimensional system (it can be shown that the instability is triggered by two-dimensional disturbances, first. We denote the horizontal coordinate as x and the vertical as z, increasing upward. We assume a layer of thickness h bound by two horizontal boundaries. The equilibrium temperature profile will be chosen to vary linearly

$$T_e(x, z) = T_0 - \Gamma z$$

where Γ is a positive constant.

The fluid is thus hot and light at the bottom and cold and heavy at the top, an unstable configuration under the influence of gravity (the system wants to reduce its potential energy). However, diffusive and viscous effects will act to stabilize the system. We will begin with a scaling analysis to quantify these effects.

In a layer of thickness h, the relative density difference is at most

$$\frac{\Delta\rho}{\rho_0} = h\Gamma\alpha.$$

A fluid blob of size h with this density difference but resisted by viscous effects would rise at speed (as we found when studying Stokes flow)

$$U_s \sim \frac{\Delta\rho}{\rho_0}\frac{gh^2}{\nu} = \frac{g\alpha h\Gamma h^2}{\nu}$$

To travel a distance h would therefore require a time (an advective time)

$$t_a = \frac{h}{U_s} = \frac{h\nu}{g\alpha\Gamma h^3} = \frac{\nu}{g\alpha\Gamma h^2}$$

On the other hand, diffusive effects may dissipate the density difference. Denoting the temperature diffusivity as κ, the time associated to the heat dissipation over a thickness h is

$$t_d = \frac{h^2}{\kappa}.$$

The ratio of these time scales should dictate if an instability can develop. We define the **Rayleigh number** as

$$Ra = \frac{t_d}{t_a} = \frac{g\alpha\Gamma h^4}{\kappa\nu}.$$

We expect instability if this ratio is large, meaning that the fluid can move before losing its density difference. So we anticipate that the system will be **unstable if** $Ra > Ra_{cr}$, for some number Ra_{cr}.

The exact value of Ra_{cr} depends on the boundary conditions but it is always of order about 1000.

Governing Equations

We now apply the method of normal modes to determine the stability of the system. Our governing equations are:

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) &= -\nabla P_d + \rho\nu\nabla^2\vec{u} - g\rho\hat{k} \\ \frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T &= \kappa\nabla^2 T \\ \rho &= \rho_0(1-\alpha(T-T_0)) \end{aligned}$$

Here the third equation is the advection-diffusion equation, which describes the behavior of a scalar quantity moving by a velocity field $\vec{u}$ and while undergoing diffusion.

We will first use the *Boussinesq approximation*, which states that density variations only have a non-negligible effect when multiplied by the gravitational acceleration. In all other instances where the density appears, we will use $\rho = \rho_0$. This approximation is typically valid provided $\Delta\rho/\rho \leq 5\%$ to 10%.

We will also introduce the dynamic pressure, which we define as the difference between the pressure and the static pressure

$$P_d = P - \int \rho_e g dz = P - \rho_0 g(z - \alpha\Gamma\frac{z^2}{2}).$$

where ρ_e is the density associated with the equilibrium temperature distribution T_e. We also introduce the perturbed temperature $T_p = T - (T_0 - \Gamma z)$. Our equations then become

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \rho_0\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) &= -\nabla P_d + \rho_0\nu\nabla^2\vec{u} + g\rho_0\alpha T_p\hat{k} \\ \frac{\partial T_p}{\partial t} + \vec{u}\cdot(\nabla T_p - \Gamma\hat{k}) &= \kappa\nabla^2 T_p. \end{aligned}$$

Our base (equilibrium) state is a static one where $\vec{u}_e = \vec{0}$, $P_{d,e} = 0$, and $T_{p,e} = 0$. Note that these are all zero because of our choices of dynamic pressure and temperature perturbations. Other choices would have been fine as well, and they would instead have led to non-zero equilibrium values.

In general, we would now write our variables as the sum of equilibrium values and perturbations:

$$\vec{u} = \vec{u}_e + \vec{u}' \ , \quad P_d = P_{d,e} + P', \quad \text{and } T_p = T_{p,e} + T'.$$

However, since all equilibrium quantities are zero here, we do not need to introduce the primed quantities.

Our goal is now to linearize the governing equations. In practice, we will neglect all the products of the variables we are solving for because they are all assumed to be small (because they are perturbations of the equilibrium value). We are then left with

$$\begin{aligned}
\nabla \cdot \vec{u} &= 0 \\
\rho_0 \frac{\partial \vec{u}}{\partial t} &= -\nabla P_d + \rho_0 \nu \nabla^2 \vec{u} + g\rho_0 \alpha T_p \hat{k} \\
\frac{\partial T_p}{\partial t} - \Gamma \vec{u} \cdot \hat{k} &= \kappa \nabla^2 T_p.
\end{aligned}$$

We will focus on the vertical component of the momentum equation

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial P_d}{\partial z} + \rho_0 \nu \nabla^2 w + g\rho_0 \alpha T_p \tag{1}$$

where $w = \vec{u} \cdot \hat{k}$.

We would now like to eliminate the pressure term, which we can do at the expense of an increase in the order of the equations. To do so, we first take the Laplacian of equation (1)

$$\rho_0 \frac{\partial \nabla^2 w}{\partial t} = -\frac{\partial \nabla^2 P_d}{\partial z} + \rho_0 \nu \nabla^2 \nabla^2 w + g\rho_0 \alpha \nabla^2 T_p \tag{2}$$

We now seek another way to produce $\frac{\partial \nabla^2 P_d}{\partial z}$. To get it, we first take the divergence of the vector form of the momentum equation and take advantage of $\nabla \cdot \vec{u} = 0$. We are left with

$$\begin{aligned}
\rho_0 \frac{\partial \nabla \cdot \vec{u}}{\partial t} &= -\nabla^2 P + \rho_0 \nu \nabla^2 (\nabla \cdot \vec{u}) + g\rho_0 \alpha \frac{\partial T_p}{\partial z} \hat{k} \\
0 &= -\nabla^2 P_d + g\rho_0 \alpha \frac{\partial T_p}{\partial z} \hat{k}
\end{aligned}$$

and we differentiate the result with respect to z to find

$$\frac{\partial \nabla^2 P_d}{\partial z} = g\rho_0 \alpha \frac{\partial^2 T_p}{\partial z^2} \hat{k}.$$

We can now replace the pressure term in equation (2) to find

$$\rho_0 \frac{\partial \nabla^2 w}{\partial t} = -g\rho_0\alpha \frac{\partial^2 T_p}{\partial z^2}\hat{k} + \rho_0 \nu \nabla^2\nabla^2 w + g\rho_0\alpha\nabla^2 T_p$$

which we rewrite as

$$\frac{\partial \nabla^2 w}{\partial t} = \nu\nabla^2\nabla^2 w + g\alpha\nabla_H^2 T_p$$

where we introduced the horizontal Laplacian $\nabla_H^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$.

Our system of equation now only has two unknowns: w and T_p:

$$\begin{aligned} \frac{\partial \nabla^2 w}{\partial t} &= \nu\nabla^2\nabla^2 w + g\alpha\nabla_H^2 T_p & (3)\\ \frac{\partial T_p}{\partial t} - \kappa\nabla^2 T_p &= \Gamma w & (4) \end{aligned}$$

Finally, we apply the operator $\frac{\partial}{\partial t} - \kappa\nabla^2$ to equation (3) to allow us to substitute for T_p using equation (4). We get

$$\begin{aligned} \frac{\partial(\frac{\partial}{\partial t} - \kappa\nabla^2)\nabla^2 w}{\partial t} &= \nu(\frac{\partial}{\partial t} - \kappa\nabla^2)\nabla^2\nabla^2 w + g\alpha\nabla_H^2(\frac{\partial}{\partial t} - \kappa\nabla^2)T_p \\ \frac{\partial(\frac{\partial}{\partial t} - \kappa\nabla^2)\nabla^2 w}{\partial t} &= \nu(\frac{\partial}{\partial t} - \kappa\nabla^2)\nabla^2\nabla^2 w + g\alpha\Gamma\nabla_H^2 w & (5) \end{aligned}$$

Cleaning things up a little, we get a single, sixth order equation, in the unknown w:

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2 w\right] - \nu\left(\frac{\partial}{\partial t} + \kappa\nabla^2\right)\nabla^2\nabla^2 w - g\alpha\Gamma\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) w = 0. \quad (6)$$

Method of normal modes

We are now ready to use the method of normal modes by considering an expansion of the form

$$w = f(z)e^{ikx+\sigma t}$$

We will determine $f(z)$ and impose constraints on k based on the boundary conditions chosen. We are interested in the stability boundary of the system, where $Re(\sigma)$ changes sign.

In general, this corresponds to solving a a 6×6 linear system. However, we will focus here on a particularly convenient set of boundary conditions that allow us to obtain an exact solution.

We consider the case of *stress-free*, no penetration, fixed temperature boundary conditions. Both are very reasonable, but they don't often come together. Stress-free makes sense at a free air-liquid surface, while fixed-temperature requires some control that would typically involve a solid surface.

The fixed-temperature BC implies that our perturbations T_p must be zero at the boundaries $z = 0$ and $z = h$. In addition, the no-penetration condition implies that $w(z = 0) = w(z = h) = 0$.

The stress-free condition implies that $\rho\nu\frac{\partial u}{\partial z} = 0$ at the top and bottom. We will combine this with the divergence-free equation

$$\frac{\partial(\nabla \cdot \vec{u})}{\partial z} = \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} = 0.$$

We remark that at the boundaries, where $\frac{\partial u}{\partial z} = 0$, this implies that $\frac{\partial^2 w}{\partial z^2} = 0$. This is in addition to the boundary condition $w = 0$.

Lastly, because $T_p = 0$, and $w = 0$ and $\frac{\partial^2 w}{\partial z^2} = 0$, at the boundaries, the equation

$$\frac{\partial \nabla^2 w}{\partial t} = \nu\nabla^2(\nabla^2 w) + g\alpha\nabla_H^2 T_p$$

implies that $\frac{\partial^4 w}{\partial z^4} = 0$ at the boundaries as well. So we have 6 boundaries conditions on w:

$$\text{At } z = 0 \text{ and } z = h: \quad w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0, \text{ and } \frac{\partial^4 w}{\partial z^4} = 0.$$

As you can see, these boundary conditions are very convenient as a simple function of z can satisfy them all. We will use $f(z) = \sin(n\pi z/h)$ for $n = 1, 2, ...$

Before we go further, we will non-dimensionalize our equation. We could have done this from the beginning, the result would have been the same. Note that in equation (6), the only physical quantities that appear are time and length. We select as a typical length the height of the layer, $L^* = h$, and we use a diffuse time scale $t^* = h^2/\kappa$. We then define dimensionless quantities, denoted with a prime:

$$\vec{x}' = \frac{\vec{x}}{h}, \quad \nabla' = h\nabla, \quad t' = \frac{\kappa t}{h^2}, \quad \frac{\partial}{\partial t'} = \frac{h^2}{\kappa}\frac{\partial}{\partial t}, \quad w' = \frac{h}{\kappa}w$$

and

$$\vec{x} = h\vec{x}', \quad \nabla = \frac{\nabla'}{h}, \quad t = \frac{h^2 t'}{\kappa}, \quad \frac{\partial}{\partial t} = \frac{\kappa}{h^2}\frac{\partial}{\partial t'}, \quad w = \frac{\kappa}{h}w', \quad \nabla^2 = \frac{\nabla'^2}{h^2}$$

We then rewrite equation (6)

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2 w\right] - \nu\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2\nabla^2 w - g\alpha\Gamma\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)w = 0.$$

as

$$\frac{\kappa}{h^2}\frac{\partial}{\partial t'}\left[\left(\frac{\kappa}{h^2}\frac{\partial}{\partial t'} - \kappa\frac{\nabla'^2}{h^2}\right)\frac{\nabla'^2}{h^2}\frac{\kappa}{h}w'\right] - \nu\left(\frac{\kappa}{h^2}\frac{\partial}{\partial t'} - \kappa\frac{\nabla'^2}{h^2}\right)\frac{\nabla'^2}{h^2}\frac{\nabla'^2}{h^2}\frac{\kappa}{h}w' - g\alpha\Gamma\left(\frac{\nabla'^2}{h^2} - \frac{1}{h^2}\frac{\partial^2}{\partial z'^2}\right)\frac{\kappa}{h}w' = 0.$$

Simplifying and grouping terms, we get

$$\frac{\kappa^3}{h^7}\frac{\partial}{\partial t'}\left[\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\right]w'-\frac{\nu\kappa^2}{h^7}\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\nabla'^2 w'-\frac{g\alpha\Gamma\kappa}{h^3}\left(\nabla'^2-\frac{\partial^2}{\partial z'^2}\right)w'=0.$$

Multiplying by $\frac{h^7}{\nu\kappa^2}$, we find

$$\frac{\kappa}{\nu}\frac{\partial}{\partial t'}\left[\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\right]w'-\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\nabla'^2 w'-\frac{g\alpha\Gamma\kappa h^4}{\kappa\nu}\left(\nabla'^2-\frac{\partial^2}{\partial z'^2}\right)w'=0.$$

This equation involves two non-dimensional numbers. One is the Prandtl number, $Pr = \frac{\nu}{\kappa}$ and the other is the Rayleigh number we encountered earlier

$$Ra = \frac{g\alpha\Gamma\kappa h^4}{\kappa\nu}.$$

This is a sign that we are on the right track!

We are now ready to substitute for w according to our assumed form $w' = \sin(n\pi z)e^{ikx+\sigma t}$ into our non-dimensional equation

$$\frac{1}{Pr}\frac{\partial}{\partial t'}\left[\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\right]w'-\left(\frac{\partial}{\partial t'}-\nabla'^2\right)\nabla'^2\nabla'^2 w'-Ra\left(\nabla'^2-\frac{\partial^2}{\partial z'^2}\right)w'=0.$$

We note that

1. $\frac{\partial}{\partial t'} \longrightarrow \sigma$
2. $\frac{\partial^2}{\partial z'^2} \longrightarrow -n^2\pi^2$
3. $\frac{\partial^2}{\partial x'^2} \longrightarrow -k^2$
4. $\nabla'^2 \longrightarrow -(n^2\pi^2+k^2)$

We thus obtain

$$e^{ikx+\sigma t}\left[\frac{1}{Pr}\sigma\left[\left(\sigma+n^2\pi^2+k^2\right)\right]-\left(\sigma+(n^2\pi^2+k^2)\right)(n^2\pi^2+k^2)^2+Ra\left(k^2\right)\right]=0.$$

In general, σ may be complex. However, here it is possible to show (see Kundu's textbook, for example) that σ is real at the onset of instability. We are therefore interested in the critical value that will arise when $\sigma = 0$. This simplifies the equation considerably and, in particular, makes the Prandtl number unimportant.

$$-\left((n^2\pi^2+k^2)\right)(n^2\pi^2+k^2)^2+Ra\left(k^2\right)=0.$$

and we find

$$Ra = \frac{(n^2\pi^2 + k^2)^3}{k^2}.$$

The critical value of the Rayleigh number will be the smallest possible value for which this equality is satisfied. We therefore need to look for the minimum of this expression as a function of n and k. Clearly, smaller values of n decrease the value of Ra and we thus select n to be as small as possible, namely $n = 1$.

Now the wavenumber k can take any real value as we have no conditions on the horizontal extent of the flow. We therefore need to find the minimum of

$$Ra(k^2) = \frac{(\pi^2 + k^2)^3}{k^2}.$$

We find the derivative to be

$$\frac{dRa}{dk^2} = \frac{3(\pi^2 + k^2)^2 k^2 - (k^2 + \pi^2)^3}{k^4}$$

This is zero if $3k^2 = k^2 + \pi^2$ or $k^2 = \pi^2/2$. The means that **the most unstable wavenumber** is

$$k_m = \frac{\pi}{\sqrt{2}}.$$

The corresponding **critical Rayleigh number** is

$$Ra_{cr} = \frac{(\pi^2 + k_m^2)^3}{k_m^2} = \frac{(3\pi^2/2)^3}{\pi^2/2} = \frac{3^3\pi^4}{4} \approx 657$$

We conclude by computing som Rayleigh numbers, to get a sense of when a system may get unstable. Our definition of the Rayleigh number can be rewritten

$$Ra = \frac{g\alpha\Gamma\kappa h^4}{\kappa\nu} = \frac{\Delta\rho}{\rho}\frac{gh^3}{\kappa\nu}$$

For water, a temperature difference of 1°C corresponds to $\frac{\Delta\rho}{\rho} \sim 10^{-4}$. Using $\kappa \approx 10^{-7}$m^2/s and $\nu \approx 10^{-6}$m^2/s, and $g = 10$m/s^2 we find

$$Ra \approx \frac{10^{-4}\ 10d^3}{10^{-6}\ 10^{-7}} m^{-3} = 10^{10} d^3/m^3$$

To get a Rayleigh number greater than 10^3, and therefore instability,
we need $d > 10^{-7/3}$m ≈ 4.6mm.

For magma, the viscosity can rise to about $\nu = 1$m^2/s, giving

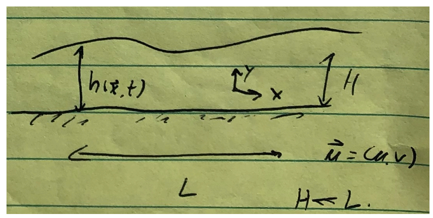
$$Ra \approx \frac{10^{-4}\ 10d^3}{1\ 10^{-7}} m^{-3} = 10^4 d^3/m^3$$

To get a Rayleigh number greater than 10^3, and therefore instability,
we need $d > 10^{-1/3}$m ≈ 46cm.

Lubrication Theory

Here we consider systems where the flow is *nearly* unidirectional because the geometry has a very large aspect ratio. Most typically, this will apply to flow where a free surface describes the top of a nearly flat region, such as a spreading liquid sheet, in a regime where viscous effects will be important.

For example: Lava flow, blood flow in capillaries, thin films, bearings.



Because of the domain's aspect ratio, where $H \ll L$, we will non-dimensionalize differently in x and in y:

$$x' = \frac{x}{L} \quad \text{and} \quad y' = \frac{y}{H}$$

Suppose we have a velocity scale U for the horizontal velocity, then for the vector, $\vec{u} = (u, v)$, we use

$$u' = \frac{u}{U} \quad \text{and} \quad v' = \frac{v}{V}$$

where we want to determine V.

From the divergence-free condition, we then have

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{U}{L}\frac{\partial u'}{\partial x'} + \frac{V}{H}\frac{\partial v'}{\partial y'}$$

So we expect that

$$V \sim U\frac{H}{L} \ll U$$

and therefore the flow is almost unidirectional, as the vertical flow is much smaller than the horizontal one.

We now look at the momentum equations. Beginning in the $\hat{x}$ direction

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

which non-dimensionalizes to

$$\rho \frac{U^2}{L}\frac{Du'}{Dt'} = -\frac{P_0}{L}\frac{\partial P'}{\partial x'} + \mu\left(\frac{U}{L^2}\frac{\partial^2 u'}{\partial x'^2} + \frac{U}{H^2}\frac{\partial^2 u'}{\partial y'^2}\right)$$

which can be rewritten as

$$\left(\frac{\rho U H}{\mu}\right)\frac{H}{L}\frac{Du'}{Dt'} = -\left(\frac{P_0 H^2}{\mu U L}\right)\frac{\partial P'}{\partial x'} + \left(\frac{H^2}{L^2}\right)\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}$$

We now define $P_0 = \frac{\mu U L}{H^2}$ for convenience.

So if we define $Re = \frac{\rho U H}{\mu}$ and assume that $Re\ H/L \ll 1$ because $H/L \ll 1$, we are left with the dominant terms

$$0 = -\frac{\partial P'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}$$

This is the same equation that we obtained in unidirectional flow. However, here it is possible that the pressure gradient is not constant.

Turning now to the $\hat{y}$ direction

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

Recalling that $V \sim U\frac{H}{L}$, this non-dimensionalizes to

$$\rho\frac{UV}{L}\ \frac{Dv'}{Dt'} = -\frac{P_0}{H}\frac{\partial P'}{\partial y'} + \mu\left(\frac{V}{L^2}\frac{\partial^2 v'}{\partial x'^2} + \frac{V}{H^2}\frac{\partial^2 v'}{\partial y'^2}\right)$$

that we rewrite as

$$\rho\frac{U^2 H}{L^2}\ \frac{Dv'}{Dt'} = -\frac{\mu U L}{H^3}\frac{\partial P'}{\partial y'} + \mu\left(\frac{UH}{L^3}\frac{\partial^2 v'}{\partial x'^2} + \frac{U}{HL}\frac{\partial^2 v'}{\partial y'^2}\right)$$

which we again rewrite as

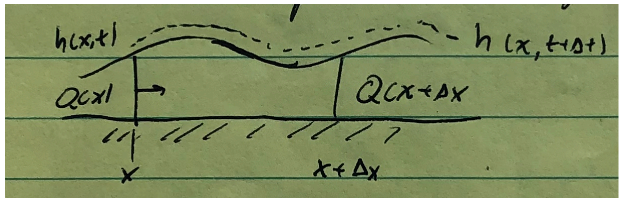
$$Re\left(\frac{H}{L}\right)^3\ \frac{Dv'}{Dt'} = -\frac{\partial P'}{\partial y'} + \left(\frac{H}{L}\right)^4\frac{\partial^2 v'}{\partial x'^2} + \left(\frac{H}{L}\right)^2\frac{\partial^2 v'}{\partial y'^2}$$

Note that all terms in the equation above are of order H^2/L^2 or smaller, EXCEPT the pressure gradient. This implies that

$$0 = \frac{\partial P'}{\partial y'}$$

and thus we have that P' can be written as a function of x' and t' only: $P'(x', t')$. We can therefore integrate the $\hat{x}$-momentum equation twice with respect to y'. We find, in dimensional form

$$u(x,y,t) = \frac{1}{2\mu}\frac{\partial P}{\partial x}\ y^2 + C_1(x,t)\ y. + C_2(x,t)$$



So at any location x, the flow is parabolic in y. It will depend on the the horizontal pressure derivative, and on the boundary conditions.

We can compute the volume flux (per depth unit) and use it to describe how the height of the current changes over time. Define

$$Q(x,t) = \int_0^{h(x,t)} u(x,y,t)\, dy$$

We can compute its x derivative

$$\frac{\partial Q}{\partial x} = u(x,h,t)\frac{\partial h}{\partial x} + \int_0^{h(x,t)} \frac{\partial u(x,y,t)}{\partial x}\, dy$$

From the divergence-free equation, we know that $\frac{\partial u}{\partial x} = -\frac{\partial v)}{\partial y}$ so we have

$$\frac{\partial Q}{\partial x} = u(x,h,t)\frac{\partial h}{\partial x} - \int_0^{h(x,t)} \frac{\partial v(x,y,t)}{\partial y}\, dy$$

$$\frac{\partial Q}{\partial x} = u(x,h,t)\frac{\partial h}{\partial x} - v(x,h,t)$$

where we used that $v(x,0,t) = 0$, as even in Lubrication theory there is no-penetration at a solid boundary.

Finally, we note that $v(x,h,t) = \frac{\partial h}{\partial t}$ so we get

$$\frac{\partial Q}{\partial x} = u(x,h(x,t),t)\frac{\partial h}{\partial x} - \frac{\partial h}{\partial t}$$

If we assume slow variations, consistent with the large aspect ratio, we have $\frac{\partial h}{\partial x} \ll 1$. However, recall that $U \gg V = \frac{\partial h}{\partial t}$, so we must take the limit carefully. So if

$$U\frac{\partial h}{\partial x} \ll V\; U\frac{H}{L} \quad \text{so} \quad \frac{\partial h}{\partial x} \ll \frac{H}{L}$$

we end up with

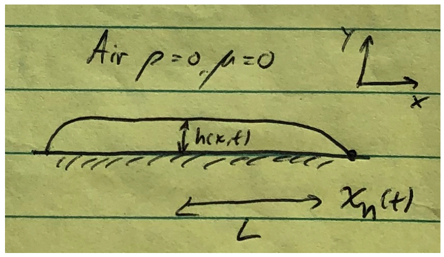
$$\frac{\partial Q}{\partial x} + \frac{\partial h}{\partial t} = 0$$

This means that any flux change in space displaces the interface up or down (if you pile up material in one place, the height changes).

Viscous Gravity Current

How does a finite amount of liquid spread? How about one with a given influx?

2D Geometry



We will first use a scaling argument:

In $\hat{y}$, we have

$$\frac{\partial P}{\partial y} = \rho g \longrightarrow P \sim \rho g h$$

In $\hat{x}$, we have

$$\frac{\partial P}{\partial x} = \nu \nabla^2 u \longrightarrow \frac{\rho g H}{L} \sim \frac{\nu U}{H^2}$$

We can also estimate that $U \sim \frac{L}{T}$.

For the case of a finite area spreading in 2D, we have that **H L = A** is constant. We therefore find

$$\frac{\rho g A}{L^2} \sim \frac{\nu L^3}{A^2 T}$$

which can be solved to find

$$L^5 \sim \frac{\rho g A^3 T}{\nu} \quad \text{and} \quad H^5 \sim \frac{\nu A^2}{\rho g T}$$

Now let's see if we can find a more accurate result by solving the equations of Lubrication Theory. In $\hat{y}$, we have that when $y = h(x,t)$, $P = P_0$, the atmospheric pressure. Therefore,

$$\frac{\partial P}{\partial y} = -\rho g \longrightarrow P \sim P_0 - \rho g(y - h(x,t))$$

We thus see that

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial h}{\partial x}.$$

Our $\hat{x}$-momentum equation then becomes

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x} = \rho g \frac{\partial h}{\partial x}.$$

This yields

$$u(x,y,t) = \frac{\rho g}{2\mu}\frac{\partial h}{\partial x}y^2 + C_1(x,t)y + C_2(x,t)$$

We have as boundary conditions that:

$$\text{No-slip at the bottom } u(x,0,t) = 0 \longrightarrow C_2(x,t) = 0$$
$$\text{No tangential stress at the top } \left.\frac{\partial u}{\partial y}\right|_{y=h} = 0 \longrightarrow u(x,y,t) = \frac{\rho g}{2\mu}\frac{\partial h}{\partial x}y(y-2h)$$

We may now compute the horizontal volume flux

$$Q = \int_0^h u\,dy = \frac{\rho g}{2\mu}\frac{\partial h}{\partial x}\left(\frac{y^3}{3} - hy^2\right) = -\frac{h^3\rho g}{3\mu}\frac{\partial h}{\partial x}.$$

Combining this with our previously obtained relation between the flux and the time derivative of the height, we obtain a single equation in terms of the height

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \longrightarrow \frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^3\rho g}{3\mu}\frac{\partial h}{\partial x}\right) = 0. \tag{1}$$

If we introduce the notation that the front of the current is at $x = x_N(t)$, the boundary conditions for this PDE are

$$h(x_N,t) = 0 \quad \text{and} \quad \int_0^{x_N(t)} h(x,t)\,dx = \frac{A}{2}.$$

where A is the constant area of the whole current.

Note that this is a difficult system to solve. It is a second order, non-linear PDE, with one regular BC and one global BC. Even numerically, this is tricky. So we will call on a similarity solution, making use of our previous scaling analysis. This means that we will not be able to satisfy a general initial condition, but we will get the long-time behavior right.

Recall our previous scaling

$$L \sim \left(\frac{\rho g A^3 t}{\nu}\right)^{1/5} \quad \text{and} \quad H \sim \left(\frac{\nu A^2}{\rho g t}\right)^{1/5}.$$

We will define a dimensionless length $\eta = \frac{x}{L(t)}$ and assume that the current total length is $x_N(t) = \beta L(t)$, for a constant β. We then assume that the height is of the form

$$h(x,t) = \left(\frac{\nu A^2}{g}\right)^{1/5} t^{1/5} f(\eta) = Ct^{-1/5}f(\eta).$$

For reference, we also have that

$$\eta = \frac{C}{A}\frac{x}{t^{1/5}}.$$

We will now be looking for $f(\eta)$ and hope that we can rewrite the entire problem in terms of η and f only.

First, we note that

$$\frac{\partial \eta}{\partial x} = \frac{C}{A}\frac{1}{t^{1/5}} = \frac{\eta}{x} \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \frac{-1}{5t}\frac{C}{A}\frac{x}{t^{1/5}} = \frac{-\eta}{5t}$$

We can now compute that

$$\frac{\partial h}{\partial t} = \frac{-C}{5}t^{-6/5}f(\eta) + Ct^{-1/5}f'(\eta)\frac{-\eta}{5t} = \frac{-C}{5}t^{-6/5}(f + \eta f')$$

and

$$h^3\frac{\partial h}{\partial x} = \frac{C^5}{A}t^{-1}f'(\eta)f^3$$

so that

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) = C\frac{C^5}{A^2}t^{-6/5}\frac{d}{d\eta}\left(f'(\eta)\,f^3\right).$$

We can finally rewrite equation (1) as

$$\frac{-C}{5}t^{-6/5}(f + \eta f') - \frac{g}{3\nu}\,C\frac{C^5}{A^2}t^{-6/5}\frac{d}{d\eta}\left(f'(\eta)\,f^3\right) = 0.$$

Canceling a $-C$ and $t^{-6/5}$, and noting that

$$\frac{g}{3\nu}\frac{C^5}{A^2} = \frac{g}{3\nu}\frac{\nu A^2}{g}\frac{1}{A^2} = \frac{1}{3}$$

we obtain the much-coveted ODE

$$\frac{1}{5}(f + \eta f') + \frac{1}{3}\frac{d}{d\eta}\left(f'\,f^3\right) = \frac{1}{5}(\eta f)' + \frac{1}{3}\left(f'\,f^3\right)' = 0.$$

The converted conditions are that

$$f(\beta) = 0$$

and

$$\int_0^{x_N} h\,dx = \frac{A}{2} \longrightarrow \int_0^{x_N}\frac{C}{A}t^{-1/5}f(\eta)dx = \int_0^{\beta} f\,d\eta = \frac{1}{2}.$$

We may now integrate our ODE once

$$\frac{1}{5}(\eta f) + \frac{1}{3}\left(f'\,f^3\right) = K_1$$

and find that $K_1 = 0$ because of the nose condition $f(\beta) = 0$. Dividing the result by f

$$\frac{1}{5}(\eta) + \frac{1}{3}\left(f'\, f^2\right) = 0$$

we may integrate once more

$$\frac{\eta^2}{10} + \frac{f^3}{9} = K_2.$$

This same condition implies that $K_2 = \frac{\beta^2}{10}$ so that

$$f(\eta) = \left[\frac{9}{10}(\beta^2 - \eta^2)\right]^{1/3} = \left[\frac{9\beta^2}{10}\right]^{1/3}\left(1 - \left(\frac{\eta}{\beta}\right)^2\right)^{1/3}$$

Finally, we use the integral condition to determine β

$$\begin{aligned} \int_0^\beta f(\eta)\, d\eta &= \left[\frac{9\beta^2}{10}\right]^{1/3}\int_0^\beta\left(1 - \left(\frac{\eta}{\beta}\right)^2\right)^{1/3}\, d\eta \\ &= \left[\frac{9\beta^5}{10}\right]^{1/3}\int_0^1\left(1 - z^2\right)^{1/3}\, dz \\ &= \left[\frac{9\beta^5}{10}\right]^{1/3}\frac{\sqrt{\pi}\Gamma(1/3)}{5\Gamma(5/6)} = \frac{1}{2} \end{aligned}$$

We finally find that

$$\beta = \left[\frac{5\Gamma(5/6)\; 10^{1/3}}{2\sqrt{\pi}\Gamma(1/3)\; 9^{1/3}}\right]^{3/5} \approx 0.7474$$

so that

$$h(x,t) = \left(\frac{\nu A^2}{gt}\right)^{1/5}\left(\frac{9}{10}0.74^2\right)^{1/3}\left(1 - \left(\frac{C\,x}{0.74At^{1/5}}\right)^2\right)^{1/3}$$

which really isn't as bad as you thought, right?