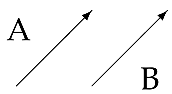
Calculus of Vectors, Dyadics, and Tensors

A: Introduction and Review

1. Scalars And Vectors

   Scalar = magnitude only (eg. mass, temp, etc.)
   Vector: characterized by magnitude and direction; represented geometrically as an arrow. $\nearrow$
   $\Rightarrow$ 2 vectors are equal if they have the same magnitude and direction; "parallel transport vectors"

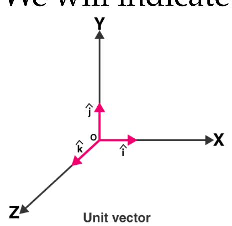
   $\Rightarrow A = B$

   (Nevertheless, it is important to keep in mind that the effect of a given vector may depend upon its location)

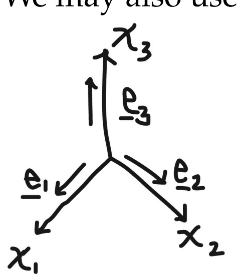
$\boxed{\text{Notation}}$: I will typically indicate a vector quantity by an underline, eg. $\underline{a}$ or $\underline{b}$.
Another common method is to use arrows $\vec{a}\ \vec{b}$

2. Cartesian Coordinate System

   (a) We will indicate the unit or base vectors as:

   We may also use $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$. ($\underline{e}_1 = i$, $\underline{e}_2 = j$, and $\underline{e}_3 = k$).

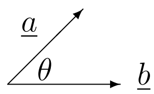
   

   $\underline{i} = (0, 0, 1)$, $\underline{j} = (0, 1, 0)$, and $\underline{k} = (1, 0, 0)$

   (b) In order to describe a vector you must give both the components and the base vectors.
   e.g., $\underline{a} = a_x\underline{i} + a_y\underline{j} = a_z\underline{k}$

3. Recall the definition of the SCALAR PRODUCT (also called the dot or inner product) of two vectors:

   (a) $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

   Where $|\underline{a}|$, $|\underline{b}|$ are the magnitudes of $\underline{a}$ and $\underline{b}$
   Also, since $\underline{i} \cdot \underline{i} = 1$, $\underline{i} \cdot \underline{j} = 0$, $\underline{i} \cdot \underline{k} = 0$ etc.,
   then $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$.
   NOTE: if $\underline{a} \cdot \underline{b} = 0$ and $|\underline{a}| \neq 0$, $|\underline{b}| \neq 0$, then $\underline{a} \perp \underline{b}$

   (b) Clearly, we also have
   $[\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}]$
   and
   $[\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}]$
   and
   $[|\underline{a}|^2 = \underline{a} \cdot \underline{a} = a^2]$

4. Vector Product (also called cross product)

   (a) The vector product of 2 vectors a, b is define as

   $$\underline{a} \wedge^{[1]} \underline{b} = |\underline{a}|\ |\underline{b}| \sin\theta\ \underline{e}$$

   where e is a unit vector in the direction perpendicular to the plane formed by a and b, as given by the RIGHT-HAND RULE.

   (b) From the definition: $\underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a}$ and $\underline{a} \wedge (\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$. It also follows that $\underline{i} \wedge \underline{j} = \underline{k},\ \underline{i} \wedge \underline{k} = -\underline{j},\ \underline{j} \wedge \underline{k} = \underline{i},\ \underline{i} \wedge \underline{i} = 0$, etc.

   (c) You may remember writing something like

   $$\underline{a} \wedge \underline{b} = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \underline{i}(a_y b_z - a_z b_y) + \underline{j}(a_z b_x - a_x b_z) - \underline{k}(a_x b_y - a_y b_x)$$

   $\Rightarrow$ much of the above is cumbersome and frightfully lengthy to write. We now introduce a special notation which will simplify many manipulations.

B: Einstein Index Notation and the Summation Concentration

1. Let us reconsider the some of the above. From now on keep in mind that we are representing vectors in a three-dimensional world. So, we will now label (x, y, z) coordinates by $(1, 2, 3)$.

[1] NOTICE: My notation for this operator is $\wedge$; many others write $\times$

Let $\underline{a}$ have components a_i, base vectors e_i. Then,

$$\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3 = \sum_{i=1}^{3} a_i\underline{e}_i \equiv a_i\underline{e}_i \quad (= a_j\underline{e}_j)^2$$

This idea must be clear in your mind before you move on.

⇒ From now on, we will **not** write the summation symbol. Instead we will invoke the summation convention - if an index appears twice, we will know that we should do a summation $i = 1, 2, 3$.

2. Scalar product revisited Consider two vectors $\underline{a} = a_i\underline{e}_i$ and $\underline{b} = b_j\underline{e}_j$ (use a different index for each vector)
Then,

$$\longrightarrow \underline{a} \cdot \underline{b} = \Sigma_{i=1}^3 a_i\underline{e}_i \cdot \Sigma_{j=1}^3 b_j\underline{e}_j = \Sigma_{i=1}^3 a_i \cdot b_i = a_i \cdot b_i$$
$$(= a_1b_1 + a_2b_2 + a_3b_3)$$

base vectors are orthogonal: $\underline{e}_i \cdot \underline{e}_j = 1$ when $i = j$, $\underline{e}_i \cdot \underline{e}_j = 0$ otherwise.

Note: We dropped the sigma from our expression since we invoke the summation convention

3. Kronecker delta δ_{ij} (i=1,2,3 j=1,2,3)

   a. Definition:

$$\delta_{ij} = 0 \qquad i \neq j \tag{1}$$
$$\delta_{ij} = 1 \qquad i = j \tag{2}$$

   - clearly $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$
   - Note: You can think about δ_{ij} as the components of the identity matrix

$$\text{identity matrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   b. With this shorthand we write

$$\underline{a} \cdot \underline{b} = a_i\underline{e}_i \cdot b_j\underline{e}_j = a_ib_j \underbrace{(\underline{e}_i \cdot \underline{e}_j)}_{\delta_{ij}} = a_ib_j\delta_{ij} = a_ib_i$$

$$\therefore \boxed{\underline{a} \cdot \underline{b} = a_ib_i = a_jb_j}$$

[2] dummy index

- in the final expression, i and j are considered summation indicies
- $a_i b_j \delta_{ij}$ implies the double sum $\Sigma_{i=1}^{3}\Sigma_{j=1}^{3} a_i b_j \delta_{ij}$
- For $a_i \underline{e}_i \cdot b_j \underline{e}_j$, the vector operation only acts on the base vectors, not the components

and we again remark that a different dummy index was used for each vector ($a_i \underline{e}_i$, $b_j \underline{e}_j$)
Note: NEVER write $a_i \underline{e}_i \cdot b_i \underline{e}_i$

c. Remarks:

i. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33}$

ii. $\delta_{ij} \to$ the REPLACEMENT OPERATOR: $\boxed{\delta_{ij} c_j = c_i}$

iii. Very often, one will not write the unit vectors e_i, and will write A_i where it is understood that i may be either 1, 2, or 3. In this case i would be called a free index since it is free to take on the values 1, 2, or 3. Similarly, the vector eqn $a = b$ may be written

$$a_i \underline{e}_i = b_i \underline{e}_i \text{ or } a_i = b_i$$

and since i only appears once on each side of the eqn, it is free to take on the value 1,2, or 3 so this stands for 3 separate equalities: $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

Another example:
$(\underline{a} \cdot \underline{b})\underline{c} = a_i b_i \underline{c} = a_i b_i c_j \underline{e}_j$ or $a_i b_i c_j$

- i appears twice in $a_i b_i$ so we sum $i = 1 \to 3$
- j is free on c_j so it can take on values 1, 2, or 3

4. Permutation Symbol:

This symbol will be useful whenever vector products arise: ϵ_{ijk}, $\quad i = 1,2,3 \quad j = 1,2,3 \quad k = 1,2,3$

(a) Definition:

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ or } -1, \text{ if } i,j,k \text{ are all different} \\ 0 \text{ if any two indices are the same} \end{cases}$$

In particular,

$$\varepsilon_{ijk} = +1 \text{ if } i,j,k \text{ are an EVEN permutation of } 1,2,3$$
$$\implies \varepsilon_{123} = 1 \quad \varepsilon_{312} = 1 \quad \varepsilon_{231} = 1$$
$$\varepsilon_{ijk} = -1 \text{ if } i,j,k \text{ are an ODD permutation of } 1,2,3$$
$$\implies \varepsilon_{213} = -1 \quad \varepsilon_{132} = -1 \quad \varepsilon_{321} = -1$$

Note: By even permutation we mean that an even number of interchanges of the indices must occur to get back to the order 123; analogous for meaning of odd permutation.

(b) This definition has the following **cyclic** and **interchange** property.

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

and if two indices are simply interchanged, the sign changes,

$$\varepsilon_{ijk} = -\varepsilon_{ikj} \quad \text{or} \quad \varepsilon_{ijk} = -\varepsilon_{jik}$$

Also, since i, j, k can each independently take on the values $1, 2, 3$ then ε_{ijk} represents 27 quantities.

(c) We also have $\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\underline{e}_k$ and by referring to the figure, we can verify everything is ok: Note: The cross-product of base vectors (or any 2 vectors)

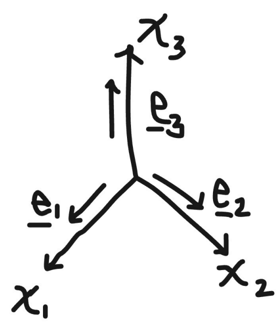


Figure 1: $\underline{e}_1 \wedge \underline{e}_2 = +\underline{e}_3 = \varepsilon_{123}\underline{e}_3 = +1 \cdot \underline{e}_3$

will always involve the permutation symbol ε_{ijk}.

(d) We now have an effective shorthand notation for representing the vector product.

$$\begin{aligned}
&\text{Let} \quad \underline{c} = \underline{a} \wedge \underline{b}; \text{ write } \underline{a} = a_i\underline{e}_i, \quad \underline{b} = b_j\underline{e}_j \\
&\implies = a_i\underline{e}_i \wedge b_j\underline{e}_j = a_i b_j(\underline{e}_i \wedge \underline{e}_j) \\
&\underline{a} \wedge \underline{b} = a_i b_j \varepsilon_{ijk}\underline{e}_k \quad \text{NOTE CAREFULLY THE ORDER OF INDICES} \\
&\text{or with } \underline{c} = c_k\underline{e}_k, \text{ we have } c_k = a_i b_j \varepsilon_{ijk} \text{ (use summation convention on } i, j)
\end{aligned}$$

EXERCISE: Verify that this is in agreement with the "matrix" definition on pg. 57.

(e) triple scalar product: $\underline{a} \cdot (\underline{b} \times \underline{c})$
Again we are careful to use different dummy indices for each vector so

$$\begin{aligned}
\underline{a} \cdot (\underline{b} \times \underline{c}) &= a_i \underline{e}_i \cdot (b_j \underline{e}_j \times c_k \underline{e}_k) \\
&= a_i \underline{e}_i \cdot (b_j c_k \varepsilon_{jkl} \underline{e}_l) \\
&= a_i b_j c_k \varepsilon_{jkl} \underbrace{(\underline{e_i} \cdot \underline{e_l})}_{\delta_{il}} \\
&= \varepsilon_{jki} a_i b_j c_k \\
&= \varepsilon_{ijk} a_i b_j c_k \\
&= \underbrace{(\underline{a} \times \underline{b}) \cdot c = (\underline{c} \times \underline{a}) \cdot b}_{\text{by using cyclic property of } \varepsilon_{ijk}}
\end{aligned}$$

Exercise: convince yourself that these last 2 identities follow from index expression.
Recall also that

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{\varepsilon_{ijk} a_i b_j c_k}_{\text{index representation of the } 3 \times 3 \text{ determinant}}$$

5. Useful identities involving ε and δ

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

i, j, l, m can each independently take on value 1, 2, 3. Hence, this can corresponds to 81 quantities.
$\boxed{\text{Proof}}$:
Verify by brute force for each of the 81 equations. However, it is best to make your life easier by noticing that both sides change sign if either i and j or l and m are interchanged. Also, both sides vanishes if $i = j$ or $l = m$. Then, consider remaining terms like:

$$\varepsilon_{12k}\varepsilon_{k12} = \underbrace{\varepsilon_{121}\varepsilon_{112}}_{0} + \underbrace{\varepsilon_{122} + \varepsilon_{212}}_{0} + \varepsilon_{123}\varepsilon_{312} = 1.$$

and

$$\delta_{11}\delta_{22} - \delta_{12}\delta 12 = 1$$

Likewise

$$\varepsilon_{12k}\varepsilon_{k13} = \varepsilon_{121}\varepsilon_{113} + \varepsilon_{122}\varepsilon_{213} + \varepsilon_{123}\varepsilon_{313} = 0$$

also

$$\delta_{11}\delta_{23} - \delta_{13}\delta_{23} = 0$$

$\boxed{\text{Example 1}}$: Show that $\underline{e_i} = \frac{1}{2}\varepsilon_{mni}\underline{e_m} \times \underline{e_n}$
Well,

$$\begin{aligned}\varepsilon_{mni}\underline{e_m} \times \underline{e_n} = \varepsilon_{mni}\varepsilon_{mnj}\underline{e_j} &= \overbrace{\varepsilon_{nim}\varepsilon_{mnj}}^{\delta_{nn}\delta_{ij}-\delta_{nj}\delta_{ni}}\ \underline{e_j} \\ &= (3\delta_{ij} - \delta_{ij})\underline{e_j} = 2\underline{e_i}\end{aligned}$$

$\boxed{\text{Example 2}}$: Show that $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$

$$\begin{aligned}\underline{a} \times (\underline{b} \times \underline{c}) &= a_i\underline{e_i} \times (b_j\underline{e_j} \times c_k\underline{e_k}) = a_i\underline{e_i} \times (b_jc_k\varepsilon_{jkl}\underline{e_l}) \\ &= a_ib_jc_k\varepsilon_{jkl}(\underline{e_i} \times \underline{e_l}) = a_ib_jc_k\varepsilon_{jkl}\varepsilon_{ilm}\underline{e_m} \\ &= a_ib_jc_k\varepsilon_{jkl}\varepsilon_{lmi}\underline{e_m} = a_ib_jc_k(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km})\underline{e_m} \\ &= a_ib_mc_i\underline{e_m} - a_ib_ic_m\underline{e_m} \\ &= (a_ic_i)b_m\underline{e_m} - (a_ib_i)c_m\underline{e_m} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}\end{aligned}$$

6. Some additional examples of the use of index notation

   (i) $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

   (ii) $\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\underline{e}_k$,
   $$\varepsilon_{ijk} = \begin{cases} 1, & i,j,k \text{ is an even permutation of 1,2,3} \\ -1, & i,j,k \text{ is an odd permutation of 1,2,3} \\ 0, & \text{any two indices are the same} \end{cases}$$

   (iii) **Summation convention**: Whenever a subscript appears twice, a summation from $1 \to 3$ is implied.

   Examples:

   (i) $\delta_{ik}\delta_{jk} = \delta_{ij}$.
      - Since δ_{jk} is only nonzero when $j = k$, the k in δ_{ik} may be replaced with with j.

   (ii) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.
      - **Note:** Since i is a dummy index, $\delta_{ii} = \delta_{jj} = \delta_{kk}$, etc.

   (iii) $\delta_{ij}\varepsilon_{ijk} = 0$, since $\delta_{ij} = 0$ if $i \neq j$, and $\varepsilon_{ijk} = \varepsilon_{iik} = 0$ if $i = j$.

(iv) $\varepsilon_{ijk}\varepsilon_{njk} = \varepsilon_{ijk}\varepsilon_{knj}$, by first rotating the indices on the second ε.
Next, use the identity $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ to get

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{knj} &= \delta_{in}\underbrace{\delta_{jj}}_{3} - \delta_{ij}\delta_{nj}\\ &= 3\delta_{in} - \delta_{in}\\ &= 2\delta_{in}\end{aligned}$$

(v) $a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} =?$
Notice that m and n appear twice, so a summation is implied. But, m and n are simply dummy variable, so we could just use another letter. Looking at the second term,

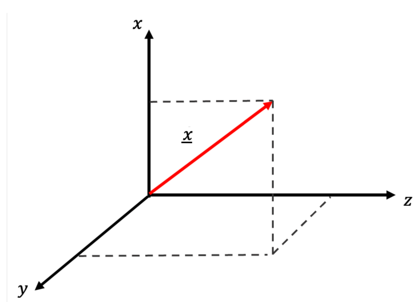
$$\begin{aligned}a_n b_m \varepsilon_{mnq} &= -a_n b_m \varepsilon_{nmq}\\ &= -a_j b_k \varepsilon_{jkq}\\ &\text{(letting } j=n,\ m=k\text{, this expression is the same as above.)}\\ &= -a_m b_n \varepsilon_{nmq} \qquad \text{(letting } j=m,\ k=n\text{)}\end{aligned}$$

So, we see that $a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = 2a_m b_n \varepsilon_{mnq}$.

- **Note:** $a_m b_n \varepsilon_{mnq} = (\underline{a} \wedge \underline{b})_q$ or the qth component of $\underline{a} \wedge \underline{b}$

C: Some vector calculus (taking derivatives of vector function)

1. Notation: We will use the vector $\underline{x}$ to denote the vector location of a point in a space



One can discuss scalar fields: $\phi(\underline{x}) = \phi(x_1, x_2, x_3) \quad \rightarrow \quad$ (Note: the value of ϕ depends on the location in space)

And

One can discuss vector fields $\underline{a}(\underline{x}) = a_1(x_1, x_2, x_3)\underline{e}_1 + a_2(x_1, x_2, x_3)\underline{e}_2 + a_3(x_1, x_2, x_3)\underline{e}_3$ $\rightarrow$ (Note: each component of the vector $\underline{a}$ depends on the location on space and it could simply write $a_j(x_i)$)

2. Differentiation of vectors

Suppose $\underline{a} = \underline{a}(t) = a_i(t)\underline{e}_i$
Then $\frac{d\underline{a}}{dt} = \frac{da_i(t)}{dt}\underline{e}_i$ since its Cartesian base vectors i.e. constant vectors

We will now consider spatial derivatives of vectors, e.g.,
$\frac{\partial}{\partial x}\underline{b}(\underline{x})$ or $\frac{\partial}{\partial y}\underline{b}(\underline{x})$

3. Gradient operator: Section 9.3 Greenberg

Let $\phi(\underline{x})$ be a scalar function which vanishes with position x, y, z in spaces.
The rate of variation of ϕ in the x-direction is $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x_1}$, in the y-direction is $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x_2}$, in the z-direction is $\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x_3}$
We introduce the vector,

$grad\ \phi \equiv \underline{\nabla}\phi = \frac{\partial \phi}{\partial z} = \underline{e}_1\frac{\partial \phi}{\partial x_1} + \underline{e}_2\frac{\partial \phi}{\partial x_2} + \underline{e}_3\frac{\partial \phi}{\partial x_3} = \underline{e}_i\phi_{,i} \rightarrow$ "comma" notation to indicate differentiation with respect to x_i

$\Rightarrow$ Relation between the gradient and the **directional derivative**

Consider a small displacement $\underline{dr}$, where $|\underline{dr}| = ds$. Note: $\underline{dr} = dx_1\underline{e}_1 + dx_2\underline{e}_2 + dx_3\underline{e}_3$. The unit tangent vector $\underline{t}$ in the direction of $\underline{dr}$ is $\underline{t} = \frac{\underline{dr}}{ds}$. Then, the rate-of-change of ϕ in the direction of $\underline{t}$ is

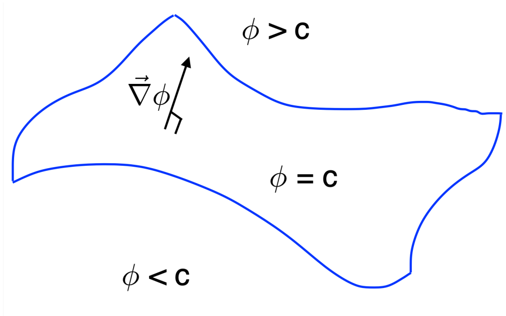
$$\underline{t} \cdot \underline{\nabla}\phi = t_i\underline{e}_i \cdot \underline{e}_j\frac{\partial \phi}{\partial x_j} = t_i\frac{\partial \phi}{\partial x_i} = t_1\frac{\partial \phi}{\partial x_1} + t_2\frac{\partial \phi}{\partial x_2} + t_3\frac{\partial \phi}{\partial x_3}$$

$$\underline{t} \cdot \underline{\nabla}\phi = \frac{dx_1}{ds}\frac{\partial \phi}{\partial x_1} + \frac{dx_2}{ds}\frac{\partial \phi}{\partial x_2} + \frac{dx_3}{ds}\frac{\partial \phi}{\partial x_3} = \frac{d\phi}{ds}$$

$$\therefore \boxed{\frac{d\phi}{ds} = \underline{t} \cdot \underline{\nabla}\phi} \quad \boxed{\text{Directional derivative of } \phi \text{ in the } \underline{t} \text{ direction}}$$

Now, consider a surface $\phi(\underline{x}) = c$, a constant.

Clearly $\Delta\phi = 0$ for any displacement **along** the surface, then since $\underline{t}$ is a tangent vector to the surface, it follows that $\underline{t} \perp \underline{\nabla}\phi$, i.e., $\underline{\nabla}\phi$ is a vector perpendicular to the surface $\phi =$ constant.



$\Rightarrow \underline{\nabla}\phi$ is a vector **normal** to the surface $\phi =$ constant.

4. Divergence of a vector field: $\underline{\nabla} \cdot \underline{f}$ or div$\underline{f}$

   (a) Simply compute using standard ideas.

$$\underline{\nabla} \cdot \underline{f} = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \cdot (f_j \underline{e}_j) = \underline{e}_i \cdot \frac{\partial f_j}{\partial x_i} \underline{e}_j + f_j \underline{e}_i \cdot \frac{\partial \underline{e}_j}{\partial x_i} \quad \text{(using the product rule)}$$

   Note $\frac{\partial \underline{e}_j}{\partial x_i} = 0$ since the $\underline{e}_j' s$ are unit vectors which do not vary with position in space.

$$\underline{\nabla} \cdot \underline{f} = \delta_{ij} \frac{\partial f_j}{\partial x_i}$$
$$= \frac{\partial f_j}{\partial x_j} \left(= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) = f_{j,j} \quad \text{(comma notation sometimes used)}$$

   Note: Now that you have gone through this, make your life easier. The $\underline{e}_j$ are **constant** vectors with respect to differentiation so we know it is ok to simply write

$$\underline{\nabla} \cdot \underline{f} = \underline{e}_i \frac{\partial}{\partial x_i} \cdot (f_j \underline{e}_j) = \underline{e}_i \cdot \underline{e}_j \frac{\partial f_j}{\partial x_i} = \frac{\partial f_j}{\partial x_j}.$$

   Also, whenever you see a term like $\frac{\partial f_k}{\partial x_k}$, you now know $\frac{\partial f_k}{\partial x_k} = \underline{\nabla} \cdot \underline{f}$ (where the k's are the same index).

   (b) An identity using index notation

   Let $\Phi(x)$ be a scalar field.

$$\begin{aligned}
\underline{\nabla} \cdot (\Phi \underline{f}) &= \underline{e}_i \frac{\partial}{\partial x_i} \cdot (\Phi f_j \, \underline{e}_j) && \text{by using the product rule} \\
&= (\underline{e}_i \cdot \underline{e}_j) \frac{\partial}{\partial x_i} (\Phi f_j) && \underline{e}_j \text{ are constant vectors} \\
&= \delta_{ij} \left[\frac{\partial \Phi}{\partial x_i} f_j + \Phi \frac{\partial f_j}{\partial x_i} \right] \\
&= \frac{\partial \Phi}{\partial x_j} f_j + \Phi \frac{\partial f_j}{\partial x_j} = (\underline{\nabla} \Phi) \cdot \underline{f} + \Phi \, \underline{\nabla} \cdot \underline{f}.
\end{aligned}$$

The inner product $(\cdot)$ only operates on vectors, <u>not</u> the scalar components Φf_j.

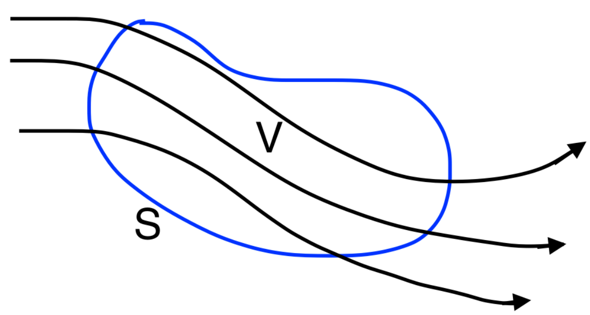
$$\therefore \quad \underline{\nabla} \cdot (\Phi \underline{f}) = (\underline{\nabla} \Phi) \cdot \underline{f} + \Phi \, \underline{\nabla} \cdot \underline{f}.$$

Notice how similar this is to the normal product rule of differentiation.

(c) Interpretation of the Divergence of a vector field

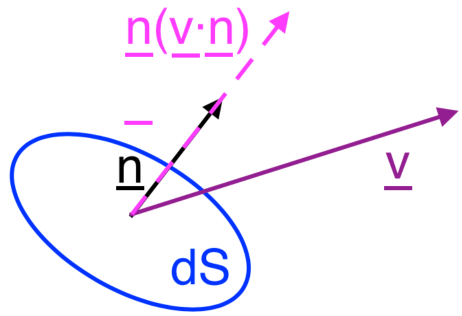
Recall the divergence theorem which relates certain volume integrals to integrals over a bounding surface - similar to electric or magnetic field lines in a medium:

$$\int_V \underline{\nabla} \cdot \underline{f} \, \mathrm{d}V = \int_S \underline{f} \cdot \underline{n} \, \mathrm{d}S.$$



In the field of fluid dynamics we find a very nice physical interpretation of the divergence of a vector field.
Consider the flow of a fluid of <u>constant density</u> (e.g. water). Such a flow is called *incompressible*.

Let $\underline{v}(\underline{x})$ be the velocity of the fluid at a point $\underline{x}$. Let S be some fixed boundary drawn in the fluid, surrounding the fixed volume V. The net flow rate through a surface with differential area $\mathrm{d}S$ is $(\underline{v} \cdot \underline{n}) \; \mathrm{d}S$.



$$\int_V \underline{\nabla} \cdot \underline{f}\, \mathrm{d}V = \int_S \underline{f} \cdot \underline{n}\, \mathrm{d}S.$$

The total flow through the surface is found by integrating over S:

$\int_S (\underline{v} \cdot \underline{n})\, \mathrm{d}S = 0$ since for a fluid of constant density : $\{\text{in-flow}\} - \{\text{out-flow}\} = 0$

$\int_V \underline{\nabla} \cdot \underline{v}\, \mathrm{d}V = 0$ by the Divergence theorem.

You may remember this from Math 21. If not, we will discuss it shortly.
And since this must be true for any choice of the volume element V, we conclude :

$$\underline{\nabla} \cdot \underline{v} = 0 \quad \text{for all } \underline{x}.$$

For an incompressible fluid, the vanishing of the divergence of the velocity field is associated with conservation of mass.

5. Curl of a vector field $\underline{\nabla} \wedge \underline{f}$ or curl$\underline{f}$

   (a) Again, simply compute using standard ideas

$$\begin{aligned}\underline{\nabla} \wedge \underline{f} &= \underline{e}_i \frac{\partial}{\partial x_i} \wedge (f_j \underline{e}_j) \\ &= (\underline{e}_i \wedge \underline{e}_j) \frac{\partial f_j}{\partial x_i} \\ &= \varepsilon_{ijk} \frac{\partial f_j}{\partial x_i} \underline{e}_k\end{aligned}$$

Note:

- As before, the $\underline{e}_j$ are constant vectors and the curl ($\wedge$) operation only affects vectors.
- Sometimes people will write this as $(\underline{\nabla} \wedge \underline{f})_k = \varepsilon_{ijk} \frac{\partial f_j}{\partial x_i}$, where the subscript k indicates the kth component of the vector $\underline{\nabla} \wedge \underline{f}$.

(b) Alternatively, let's just go through and show that the above agrees with what you have seen in earlier vector calculus courses. First,

$$\underline{\nabla} \wedge \underline{f} = (\underline{e}_i \wedge \underline{e}_j)\,\frac{\partial f_j}{\partial x_i}$$

and since the summation convention has been assumed and the variables i, j appear twice, we must sum $i = 1 \to 3$ and $j = 1 \to 3$ as follows

$$\begin{aligned}
\underline{\nabla} \wedge \underline{f} &= \underbrace{(\underline{e}_1 \wedge \underline{e}_1)}_{0}\frac{\partial f_1}{\partial x_1} + \underbrace{(\underline{e}_1 \wedge \underline{e}_2)}_{\underline{e}_3}\frac{\partial f_2}{\partial x_1} + \underbrace{(\underline{e}_1 \wedge \underline{e}_3)}_{-\underline{e}_2}\frac{\partial f_3}{\partial x_1} + \underbrace{(\underline{e}_2 \wedge \underline{e}_1)}_{-\underline{e}_3}\frac{\partial f_1}{\partial x_2} + \underbrace{(\underline{e}_2 \wedge \underline{e}_2)}_{0}\frac{\partial f_1}{\partial x_2} + \ldots \\
&= \underline{e}_1\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_3}{\partial x_3}\right) + \underline{e}_2\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) + \underline{e}_3\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) \\
&= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}
\end{aligned}$$

which is how you probably saw it represented previously.

(c) Another identity:

$$\underline{\nabla} \wedge \underline{\nabla}\phi = \underline{e}_i\frac{\partial}{\partial x_i} \wedge \left(\underline{e}_j\frac{\partial \phi}{\partial x_i}\right) = \underline{e}_i \wedge \underline{e}_j\,\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \varepsilon_{ijk}\frac{\partial^2 \phi}{\partial x_i \partial x_j}\underline{e}_k$$

But notice that by using properties of ε_{ijk},

$$\varepsilon_{ijk}\frac{\partial^2 \phi}{\partial x_i \partial x_j} = -\varepsilon_{jik}\frac{\partial^2 \phi}{\partial x_i \partial x_j} = -\varepsilon_{ijk}\frac{\partial^2 \phi}{\partial x_j \partial x_i} = 0 \text{ (by comparing with the first term)}$$

Therefore,

$$\underline{\nabla} \wedge \underline{\nabla}\phi = 0 \quad \text{(for any scalar function } \phi) .$$

Note: In the second equality we interchanged $i \leftrightarrow j$ and assumed twice differentiability.

(d) Evaluate $\underline{\nabla} \cdot (\underline{a} \wedge \underline{b})$.
We have:

$$\underline{\nabla} \cdot (\underline{a} \wedge \underline{b}) = \underline{e}_i\frac{\partial}{\partial x_i} \cdot (a_j\underline{e}_j \wedge b_k\underline{e}_k) \quad = \quad \underline{e}_i\frac{\partial}{\partial x_i} \cdot (a_j b_k)\epsilon_{jkl}\underline{e}_l$$

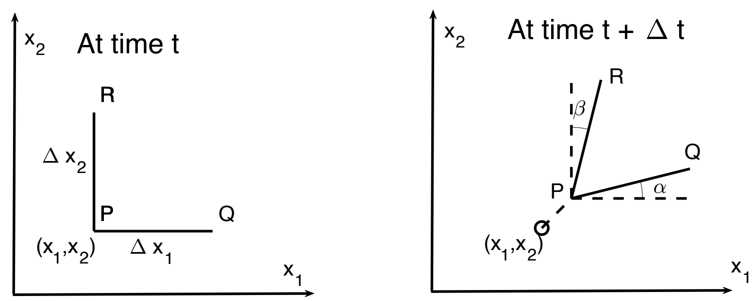
Note that here ϵ_{jkl} is constant and can therefore be taken out of the parenthesis of the derivative operator.

$$\begin{aligned}
(\underline{e}_i \cdot \underline{e}_l)\frac{\partial(a_j b_k)}{\partial x_i}\epsilon_{jkl} = \delta_{il}\frac{\partial(a_j b_k)}{\partial x_i}\epsilon_{jkl} &= \frac{\partial(a_j b_k)}{\partial x_i}\epsilon_{jki} = \\
\frac{\partial a_j}{\partial x_i}\epsilon_{jki}b_k + \frac{\partial b_k}{\partial x_i}\epsilon_{jki}a_j &= \frac{\partial a_j}{\partial x_i}\epsilon_{ijk}b_k - \frac{\partial b_k}{\partial x_i}\epsilon_{ikj}a_j = \\
(\underline{\nabla} \wedge \underline{a})_k b_k - (\underline{\nabla} \wedge \underline{b})_j a_j &= (\underline{\nabla} \wedge \underline{a}) \cdot \underline{b} - (\underline{\nabla} \wedge \underline{b}) \cdot \underline{a}
\end{aligned}$$

6. Interpretation of the curl of a vector field.

   Again, we use the velocity field of a fluid flow, denoted as $\underline{v}(\underline{x})$, as an example. We will now see that $\underline{\omega} = \underline{\nabla} \wedge \underline{v}$ provides a measure of the local angular velocity.

   Consider 2 line segments, $\overline{PR}$ and $\overline{PQ}$ in the flow; examine planar motions for simplicity.



   For small Δt, the rotation of the segments will be small and we therefore have $\alpha \sim \tan\alpha$ and $\beta \sim \tan\beta$. We estimate these angles as

$$\alpha \sim \tan\alpha = \Delta t \frac{v_2(x_1 + \Delta x_1) - v_2(x_1)}{\Delta x_1} \sim \Delta t \frac{\partial v_2}{\partial x_1}$$

   and

$$\beta \sim \tan\beta = \Delta t \frac{v_1(x_2 + \Delta x_2) - v_1(x_2)}{\Delta x_2} \sim \Delta t \frac{\partial v_1}{\partial x_2}$$

   The average rate of counterclockwise rotation of a fluid particle about the x_3 axis is thus

$$\frac{1}{2}\left(\Delta t \frac{\partial v_2}{\partial x_1} - \Delta t \frac{\partial v_1}{\partial x_2}\right) = \frac{1}{2}(\underline{\nabla} \wedge \underline{v})_3 = \frac{1}{2}\underline{\omega}_3$$

   and, in general, the average rate of rotation of a fluid particle about the x_i axis is

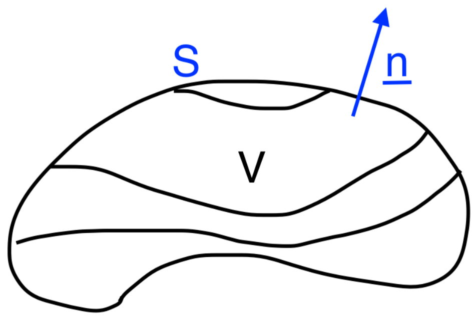
$$\frac{1}{2}(\underline{\nabla} \wedge \underline{v})_i = \frac{1}{2}\underline{\omega}_i$$

   where $\underline{\omega}$ is the vorticity vector.

Handout 2: Integral theorems in Vector Calculus

1 Divergence Theorem (or Gauss' Theorem)

This theorem relates integrals over volumes to integrals over their bounding surface(s).



The theorem states that given a continuous vector function $\vec{f}$ with continuous partial derivatives, then

$$\int_V \vec{\nabla} \cdot \vec{f}\, dV = \int_S \vec{n} \cdot \vec{f}\, dS$$

where $\vec{n}$ is the unit outward normal to S, the surface bounding the volume V. *Note that it is a good habit to write $\vec{n}$ on the left, as a replacement of $\vec{\nabla}$.*

You may find a proof of this theorem in most vector calculus textbooks. It relies on computing the outward flux on a small volume element and taking the limit as this elements shrinks to a point.

Using index notation, we can write this theorem as

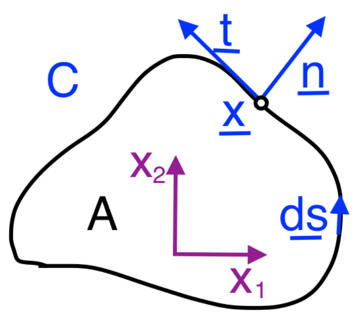
$$\int_V \nabla_i f_i\, dV = \int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S n_i f_i\, dS$$

and written out in 3D, this becomes

$$\int_V \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dV = \int_S n_1 f_1 + n_2 f_2 + n_3 f_3\, dS$$

2 Planar Versions of the Divergence Theorem

Consider some area A in the plane bounded by a curve C. Let $\vec{n}$ and $\vec{t}$ be the unit outward normal and tangent vectors along the boundary, respectively. We then have, for a point $\vec{x}$



on the boundary and using s to denote the arclength along the boundary

$$\vec{t} = \frac{d\vec{x}}{ds} \quad \text{and so} \quad \vec{t}ds = dx_1\vec{e}_1 + dx_2\vec{e}_2.$$

Moreover, $ds = |d\vec{x}|$ for $d\vec{x}$ representing a small displacement along the boundary.

In the normal direction, taking advantage of $\vec{n} \cdot \vec{t} = 0$ we have

$$\vec{n}ds = dx_2\vec{e}_1 - dx_1\vec{e}_2.$$

The divergence theorem then becomes

$$\int_A \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dA = \int_C \vec{n} \cdot \vec{f}\, ds = \int_C f_1 dx_2 - f_2 dx_1.$$

where dA is an area element that may also be written as $dx_1\, dx_2$.

If we now denote $f_1 = N(x_1, x_2)$ and $f_2 = -M(x_1, x_2)$, we then have

$$\int_A \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right) dA = \int_C \vec{n} \cdot \vec{f}\, ds = \int_C M dx_1 + N dx_2.$$

If you consider the vector field $\vec{F} = (M, N)$, you can now see that we have

$$\int_C \vec{F} \cdot \vec{t}\, ds = \int_A (\vec{\nabla} \wedge \vec{F}) \cdot \vec{e}_3\, dA$$

which is the planar version of the Stokes Theorem (which we will return to shortly).

3 Theorems Following from the Divergence Theorem

We begin with

$$\int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S n_i f_i\, dS$$

and consider some special cases of $\vec{f}$. *If you read up to here email me with the second letter of your family name as the subject.*

1. Let $\vec{f} = \phi\vec{b}$ for a scalar function ϕ and an arbitrary constant vector $\vec{b}$. We then have
$$\int_V \left(\frac{\partial \phi}{\partial x_i}\, dV\right) b_i = \int_S (n_i \phi\, dS) b_i$$
Because $\vec{b}$ is arbitrary, we must have that
$$\int_V \left(\frac{\partial \phi}{\partial x_i}\, dV\right) = \int_S (n_i \phi\, dS)$$
or in vector notation
$$\int_V \vec{\nabla}\phi\, dV = \int_S \vec{n}\phi\, dS$$
which is Gauss' theorem for a scalar function. Note that this is a vector equality, so it holds for *each* component.

2. We now let $\vec{f} = \vec{\nabla}\phi$, meaning that $\vec{f}$ is a conservative field and ϕ is its scalar potential. We then have that
$$\vec{\nabla}\cdot\vec{f} = \vec{\nabla}\cdot\nabla\phi = \nabla_i\nabla_i\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \nabla^2\phi$$
The quantity $\nabla^2\phi$ is called the *Laplacian* of ϕ.
The divergence theorem then becomes
$$\int_V \nabla^2\phi\, dV = \int_S \vec{n}\vec{\nabla}\phi\, dS = \int_S \frac{\partial\phi}{\partial n}\, dS$$
where $\frac{\partial\phi}{\partial n}$ is the *normal derivative* of ϕ, or its directional derivative in the direction of the normal.

4 Green's theorem

We may also recover Green's First identity by letting $\vec{f} = \Psi\vec{\nabla}\phi$ for Ψ and ϕ some scalar functions. We have:
$$\begin{aligned}\int_S (\vec{n}\cdot\vec{\nabla}\phi)\Psi\, dS &= \int_V \vec{\nabla}(\vec{\nabla}\phi\Psi)\, dV\\ &= \int_V \nabla_i(\nabla_i\phi\Psi)\, dV\\ &= \int_V \Psi\nabla_i\nabla_i\phi + \nabla_i\phi\nabla_i\Psi\, dV\end{aligned}$$

So we get Green's first identity:

$$\int_S \Psi \frac{\partial \phi}{\partial n}\, dS = \int_V [\vec{\nabla}\Psi \cdot \vec{\nabla}\phi + \Psi \nabla^2 \phi]\, dV$$

If we now interchange Ψ and ϕ in the expression above and subtract the result from what we just obtained, we find Green's second identity:

$$\int_S \left(\Psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \Psi}{\partial n} \right)\, dS = \int_V (\Psi \nabla^2 \phi - \phi \nabla^2 \Psi)\, dV$$

As an interesting aside, Green's identities are often useful in proving very general results. For example, if we begin with Green's first identity and let $\Psi = \phi$, we get

$$\int_S \phi \frac{\partial \phi}{\partial n}\, dS = \int_V [\vec{\nabla}\phi \cdot \vec{\nabla}\phi + \phi \nabla^2 \phi]\, dV$$

This can be useful if we are trying to solve Laplace's equation: $\nabla^2 \phi = 0$, subject to homogeneous boundary conditions: $\phi = 0$ on S.

In that case, the LHS is zero because of the boundary conditions. Because of the PDE, we then have

$$\int_V \vec{\nabla}\phi \cdot \vec{\nabla}\phi\, dV = 0$$

Because the integrand is always positive, we must have $\vec{\nabla}\phi = 0$. Therefore $\phi = C$, a constant. But since on the boundary we have that $\phi = 0$, that constant must be 0 and we have that $\phi = 0$ everywhere as the only solution.

5 A further generalization of the Divergence Theorem

We began by recalling a vector equality we obtained earlier

$$\int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_V \nabla_i \phi\, dV = \int_S (n_i \phi\, dS) \tag{1}$$

We then look for a similar result involving a cross product: $\int_V \vec{\nabla} \wedge \vec{f}\, dV$.

Rewriting this in index notation gives

$$\int_V \vec{\nabla} \wedge \vec{f}\, dV = \int_V \nabla_i f_j \epsilon_{ijk}\, dV = \epsilon_{ijk} \vec{e}_k \int_V \nabla_i f_j\, dV$$

Now for each component f_j, we can use equation (1)

$$\int_V \frac{\partial f_j}{\partial x_i}\, dV = \int_V \nabla_i f_j\, dV = \int_S (n_i f_j\, dS)$$

We combine this result with the previous equation to find

$$\int_V \vec{\nabla} \wedge \vec{f}\, dV = \int_V \nabla_i f_j \epsilon_{ijk}\, dV = \epsilon_{ijk} \vec{e}_k \int_S n_i f_j\, dS = \vec{e}_k \int_S \epsilon_{ijk} n_i f_j\, dS$$

so finally we get

$$\int_V \vec{\nabla} \wedge \vec{f}\, dV = \int_S \vec{n} \wedge \vec{f}\, dS$$

So we can get the VERY general result that

$$\int_V \vec{\nabla} * \Phi\, dV = \int_S \vec{n} * \Phi\, dS \tag{2}$$

for any differentiable quantity Φ, scalar, vector, or even tensor, and any operation $*$ that makes mathematical sense (product, scalar product, cross product, gradient operation)

5.1 Examples

Consider a constant vector $\vec{a}$. Then

$$\int_S \vec{n} \cdot \vec{a}\, dS = \int_V \vec{\nabla} \cdot \vec{a}\, dV = 0.$$

Evaluate $\int_S \vec{n} \cdot (\vec{\nabla} \wedge \vec{f})\, dS$. We use the general divergence theorem

$$\int_S \vec{n} \cdot (\vec{\nabla} \wedge \vec{f})\, dS = \int_V \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{f})\, dV$$

But this last integrand is 0 for any twice differentiable vector field.

Consider the distance function r with $r^2 = \vec{x} \cdot \vec{x}$. Compute $\int_S \vec{n} \cdot \vec{\nabla} r^2\, dS$

$$\begin{aligned}\int_S \vec{n} \cdot \vec{\nabla} r^2\, dS &= \int_V \vec{\nabla} \cdot \vec{\nabla} r^2\, dV \\ &= \int_V \nabla^2 (x_1^2 + x_2^2 + x_3^2)\, dV \\ &= \int_V 6\, dV = 6\mathcal{V}\end{aligned}$$

where $\mathcal{V}$ is the volume of V.

You could also show (as an exercise?) that

$$\int_S \vec{n} \wedge \vec{\nabla} r^2\, dS = 0$$

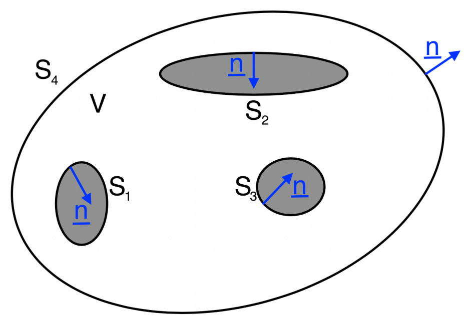
and

$$\int_S \vec{n} \cdot \vec{\nabla}(\vec{x} \wedge \vec{a})\, dS = 0$$

where $\vec{x}$ is the position vector and $\vec{a}$ is constant.

6 Multiple bounding surfaces

This is a bit beyond what will be needed in this class, but it is a natural extension of the Divergence Theorem we have seen so far. Consider a volume V that is bounded by several, disconnected, surfaces, S_1, S_2, etc. We denote by S the union of all the bounding surfaces. In other words, the Divergence theorem is then

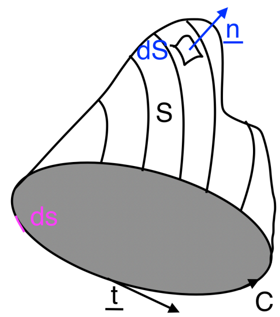


$$\int_V \vec{\nabla} \cdot \vec{f}\, dV = \int_S \vec{n} \cdot \vec{f}\, dS = \int_{S_1+S_2+S_3+S_4} \vec{n} \cdot \vec{f}\, dS$$

In this case, the unit normal associated to each bounding surface is always pointing away from the volume V.

7 Stokes' theorem

This theorem allows us to express the integral along a curve C as an integral over the area of any surface S that has that curve as its (only) boundary. Let C be a closed curve with a given orientation and S be a surface whose only boundary is C. Note that here S is NOT a closed surface, unlike in most prior examples. You can think of S as a "hat" and of C as its "rim". Consider $\vec{n}$ a unit normal to S in the direction obtained by the right-hand-rule applied to C and a vector tangent to S that starts on the curve C. Denote by $\vec{t}$ a unit tangent vector to C.



Stokes' theorem then states that

$$\oint_C \vec{f} \cdot \vec{t}\, ds = \int_S \vec{n} \cdot (\vec{\nabla} \wedge \vec{f})\, dS$$

Note that here ds is a linear element of length along C and dS is a surface element along S.

In index notation, this becomes

$$\oint_C f_i t_i \, ds = \int_S n_k (\nabla_i f_j \epsilon_{ijk})\, dS$$

Importantly, S can be ANY surface whose boundary is the closed curve C. This can sometimes be chosen to make your life easier.

We do not present a proof of this theorem here, but vector calculus textbooks nearly all contain one.

1.1 Lagrangian vs. Eulerian points of view

In fluid mechanics we describe the motion of liquids and gases (such as water and air) using the approach of continuum mechanics, wherein the fluid is characterized by properties that are aggregates over a large number of individual molecules. When we talk about a 'fluid particle', we mean an infinitesimally small region of fluid when discussing mathematical formulations (when taking limits for derivatives, for example) but we understand that the region is nevertheless large in comparison with the mean spacing between molecules. Each fluid particle has associated with it various physical properties, such as temperature and density, and is assumed to have a well defined position and velocity.

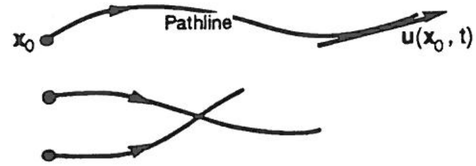
There are two different mathematical representations of fluid flow: the **Lagrangian** picture in which we keep track of the locations of individual fluid particles; and the **Eulerian** picture in which coordinates are fixed in space (the laboratory frame).

The Lagrangian picture is not often used for theoretical developments but can provide a useful picture of fluid flow in experiments. For example, in oceanography, buoys and patches of dye are deposited on the sea surface and their positions are noted as they vary in time. The density ρ and velocity $\mathbf{u}$ are described mathematically by

Velocity $$\mathbf{u} = \mathbf{u}(\mathbf{x}_0, t),$$

Density $$\rho = \rho(\mathbf{x}_0, t),$$

i.e., the field values are those of a fluid particle at some time t after the particle was 'released' at the initial position $\mathbf{x}_0$.



The loci of fluid particles are called 'pathlines' and it is clear that these lines may cross, since two different fluid particles may occupy the same position in space at different times.

Since these coordinates describe the motion of individual particles, the acceleration of a particle is given simply by

Acceleration
$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t}.$$

If the fluid is incompressible then the density of each fluid particle remains constant in time, which is expressed mathematically as

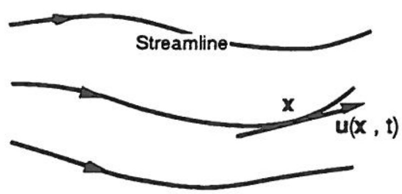
Incompressibility
$$\frac{\partial \rho}{\partial t} = 0.$$

In the Eulerian picture, the velocity and density are given by

Velocity
$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t),$$

Density
$$\rho = \rho(\mathbf{x}, t),$$

where $\mathbf{x}$ is a fixed location in the laboratory frame, and thus $\mathbf{u}$ and ρ are the velocity and density of the fluid particle that is instantaneously at position $\mathbf{x}$ at time t.



The velocity vectors form a vector field that is assumed to be differentiable and hence there are 'streamlines' that are everywhere parallel to the local velocity vector. Streamlines can never cross except at point sources or sinks of fluid.

In order to compute the acceleration of a fluid particle with these coordinates, we must realise that after a small time δt the particle is at the new position $\mathbf{x} + \delta\mathbf{x}$ with velocity

$$\mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = \mathbf{u}(\mathbf{x}, t) + (\delta\mathbf{x} \cdot \nabla)\mathbf{u} + \delta t \frac{\partial \mathbf{u}}{\partial t} + O(\delta\mathbf{x}^2, \delta t^2).$$

Thus the acceleration of the fluid particle is

Acceleration $$\lim_{\delta t \to 0} \frac{\mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - \mathbf{u}(\mathbf{x}, t)}{\delta t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \equiv \frac{D\mathbf{u}}{Dt}.$$

The operator $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is called the 'material derivative' or 'substantial derivati-vative'. It is the rate of change with time following a fluid particle.

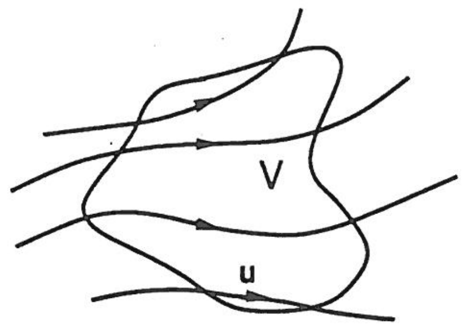
In the Eulerian picture, incompressibility is expressed by

Incompressibility $$\frac{D\rho}{Dt} = 0,$$

since it is the density of a fluid particle that remains constant, not the density of the fluid at a fixed position in space.

1.2 Conservation of mass

Consider an arbitrary fixed control volume V in the laboratory frame



The rate of change of the mass of fluid contained within V is equal to the mass inflow through the boundary ∂V of V. Thus

$$\frac{d}{dt}\int_V \rho \, dV = -\int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the outward normal to ∂V. Applying the divergence theorem to this equation, we obtain

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \mathbf{u}) \, dV.$$

Since these integrals are equal for arbitrary control volumes, it can be deduced that the integrands must also be equal. Thus the differential equation expressing conservation of mass is

Mass conservaton $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

This equation is readily rearranged into the form

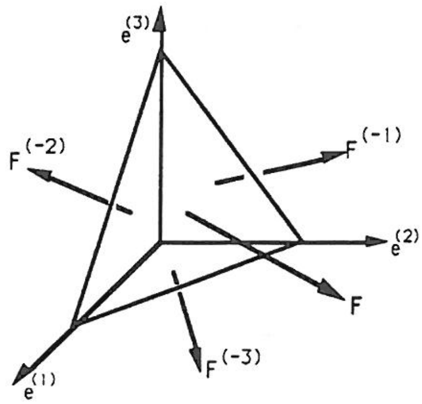
$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0,$$

from which we see that if the fluid is incompressible

Incompressibility $$\nabla \cdot \mathbf{u} = 0.$$

1.3 The Stress Tensor

Consider a small tetrahedron of fluid aligned with local, rectangular coordinate axes $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$. The forces exerted by the fluid exterior to the tetrahedron on the surfaces of the tetrahedron are $\mathbf{F}^{(-i)}$ on the three surfaces having outward normals in the three negative coordinate directions $-\mathbf{e}_i$ and $\mathbf{F}$ on the sloping face of the tetrahedron, which has outward normal $\mathbf{n}$.



The magnitude of the surface forces, which are due to molecular jostling and to short-range van der Waals forces, are proportional to the surface area of the tetrahedron, which is of order $V^{2/3}$, where V is the volume of the tetrahedron, whereas the inertial forces (mass $\times$ acceleration) and long-range body forces, such as gravity, are proportional to V. Thus the surface forces must balance by themselves in the limit as $V \to 0$ and we obtain

$$\begin{aligned} \mathbf{F} &= -\sum_k \mathbf{F}^{(-k)} \\ &= \sum_k \mathbf{F}^{(k)} \qquad \text{(by Newton's 3rd law)} \\ \Rightarrow A\tau &= \sum_k A^{(k)}\tau^{(k)} \end{aligned}$$

where τ is the **stress**, which is the force per unit area acting on a surface, and $A^{(k)}$ is the area of the k^{th} surface of the tetrahedron. From projective geometry, we have that

$A^{(k)} = A\mathbf{n} \cdot \mathbf{e}^{(k)}$. Thus the stress can be written as

$$\begin{aligned}\tau &= \left(\sum_k \tau^{(k)} \mathbf{e}^{(k)} \right) \cdot \mathbf{n} \\ &= \sigma \cdot \mathbf{n}\end{aligned}$$

where $\sigma = \sum_k \tau^{(k)} \mathbf{e}^{(k)}$ is the **stress tensor**, which is independent of the direction $\mathbf{n}$. The components of the stress tensor are given by

$$\sigma_{ij} = \sum_k \tau_i^{(k)} \mathbf{e}_j^{(k)}.$$

But $\mathbf{e}_j^{(k)} = \delta_{jk}$, so

$$\sigma_{ij} = \tau_i^{(j)},$$

which is the i th component of the force per unit area exerted by the fluid on a surface with normal in the j th coordinate direction.

The most important statement relating to the stress tensor is that the force per unit area (stress) exerted by the fluid on a surface with unit normal $\mathbf{n}$ pointing into the fluid is given by

Stress $$\tau = \sigma \cdot \mathbf{n}$$

1.4 The momentum equation

Consider the arbitrary fixed control volume of section 1.2. The rate of change of the total momentum within the control volume is effected by the inflow of momentum through the boundary, and the forces acting on the fluid, which comprise both body forces (total per unit volume) and surface forces. Thus

$$\begin{aligned} \frac{d}{dt}\int_V \rho\mathbf{u}\,dV = &-\int_{\partial V}(\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS && \text{momentum flux} \\ &+\int_V \mathbf{f}\,dV && \text{body forces} \\ &+\int_{\partial V} \sigma\cdot\mathbf{n}\,dS && \text{surface forces} \end{aligned}$$

Use of the divergence theorem gives

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\,dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\,dV + \int_V f_i\,dV + \int_V \frac{\partial}{\partial x_j}(\sigma_{ij})\,dV$$

Again, since this expression holds for arbitrary control volumes, the integrands must equate to give

$$\rho\frac{D\mathbf{u}}{Dt} + \mathbf{u}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right) = \mathbf{f} + \nabla\cdot\sigma.$$

The second term is zero by conservation of mass, so

The momentum equation $$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla\cdot\sigma.$$

1.5 Stress tensor for a Newtonian fluid

In this course, we shall be concerned solely with **Newtonian** fluids, which are those that are assumed to have two fundamental properties: the fluid should be isotropic; and there should be a linear relationship between stress and the rate of strain of the fluid. In addition, we require that the long-range forces exert no couple on individual molecules (a counter example to this last requirement is provided by magnetic fluids – see homework). With this latter condition, we can show that the stress tensor is symmetric as follows.

The rate of change of the angular momentum of a fluid particle is equal to the moment of the forces acting on the particle. Thus

$$\frac{d}{dt}\int_V \mathbf{x}\wedge(\rho\mathbf{u})\,dV = \int_V \mathbf{x}\wedge\mathbf{f}\,dV + \int_{\partial V} \mathbf{x}\wedge(\sigma\cdot\mathbf{n})\,dS$$

The term on the left-hand side and the first term on the right-hand side are each of order $V^{4/3}$ as $V\to 0$, while the last term, representing the couple exerted by the surface forces is of order V. Thus the surface moments dominate the equation and must tend to zero as $V\to 0$. We can apply the divergence theorem to this equation to give

$$\begin{aligned}
0 &= \int_{\partial V} \mathbf{x}\wedge(\sigma\cdot\mathbf{n})\,dS \\
&= \int_V \frac{\partial}{\partial x_m}(\epsilon_{ijk}x_j\sigma_{km})\,dV \\
&= \int_V \epsilon_{ijk}\delta_{jm}\sigma_{km}\,dV + \int_V \epsilon_{ijk}x_j\frac{\partial}{\partial x_m}\sigma_{km}\,dV \\
&= \int_V \epsilon_{ijk}\sigma_{kj}\,dV + \int_V \epsilon_{ijk}x_j\frac{\partial}{\partial x_m}\sigma_{km}\,dV
\end{aligned}$$

Now, provided that the stress tensor is differentiable so that $\nabla\cdot\sigma$ is finite, the second term in this last equation is of order $V^{4/3}$ while the first term is of order V as $V\to 0$. So the first term dominates the equation and shows that

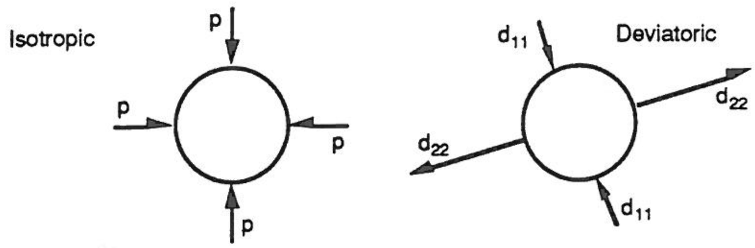
$$\epsilon_{ijk}\sigma_{kj} = 0$$

i.e., that the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$).

Next, we note that we can always write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \qquad \text{with } d_{ii} = 0$$

thus splitting σ into an isotropic part and a non-isotropic part called the deviatoric stress tensor.



The isotropic part of the stress tensor gives a force that pushes equally in all directions and so we interpret the constant p as a pressure. The deviatoric stress arises from deviations of the flow local to a fluid particle and we assume therefore that $\mathbf{d}$ is a function of the velocity gradient $\nabla \mathbf{u}$ with $\mathbf{d} = 0$ when $\nabla \mathbf{u} = 0$.

Here is where we assume that Newtonian fluids are **linear**, by which we mean that $\mathbf{d}$ is a linear function of $\nabla \mathbf{u}$, so that

$$d_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}.$$

Finally, we assume that the fluid is isotropic so that $\mathbf{A}$ is isotropic and hence is given by

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{kj},$$

where λ, μ, ν are constants, this being the most general isotropic fourth-rank tensor.

From symmetry, we deduce that

$$d_{ij} = d_{ji} \Rightarrow A_{ijkl} = A_{jikl} \Rightarrow \mu = \nu,$$

while the fact that $\mathbf{d}$ is traceless gives

$$d_{ii} = 0 \Rightarrow A_{iikl} = 0 \Rightarrow 3\lambda + \mu + \nu = 0,$$

whence $\lambda = -\frac{2}{3}\mu$. Hence

$$d_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \tfrac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

or

$$\mathbf{d} = 2\mu\mathbf{e} - \tfrac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I},$$

where $\mathbf{e}$ is the symmetric part of the velocity gradient $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. If the fluid is incompressible, so that $\nabla\cdot\mathbf{u} = 0$ then

$$\sigma = -p\mathbf{I} + 2\mu\mathbf{e}.$$

The constant μ is called the **dynamic viscosity** of the fluid.

Putting this stress tensor in the general momentum equation yield the Navier-Stokes equations

Navier Stokes

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla\mathbf{p} + \mu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0.$$

We interpret dx/dt as the time rate of change of the x-coordinate position of our observer, i.e., dx/dt is the x-component of the velocity, $\mathbf{w}$, of our observer. Thus

$$w_x = \frac{dx}{dt},$$

$$w_y = \frac{dy}{dt},$$

and

$$w_z = \frac{dz}{dt},$$

and Eq. 4.1-4 becomes

$$\frac{dS}{dt} = \left(\frac{\partial S}{\partial t}\right) + w_x\left(\frac{\partial S}{\partial x}\right) + w_y\left(\frac{\partial S}{\partial y}\right) + w_z\left(\frac{\partial S}{\partial z}\right). \tag{4.1-5}$$

In vector notation this becomes,

$$\frac{dS}{dt} = \left(\frac{\partial S}{\partial t}\right) + \mathbf{w}\cdot\nabla S, \tag{4.1-6}$$

and in index notation we express this result as

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + w_i\left(\frac{\partial S}{\partial x_i}\right). \tag{4.1-7}$$

Here the repeated indices are summed from 1 to 3 in accordance with the summation convention[2]. If our observer moves *with the fluid*, i.e., $\mathbf{w} = \mathbf{v}$ the time derivative is called the *material derivative* and is denoted by

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \mathbf{v}\cdot\nabla S. \tag{4.1-8}$$

If our observer fixes himself in space, $\mathbf{w} = 0$, and the *total time derivative* is simply equal to the partial time derivative

$$\frac{dS}{dt} = \frac{\partial S}{\partial t}, \quad \text{for } \mathbf{w} = 0 \tag{4.1-9}$$

Now we wish to consider the total time derivative of the volume integral of S over the region $\mathscr{V}_a(t)$. Here $\mathscr{V}_a(t)$ represents an *arbitrary* (hence the subscript a) volume moving through space in some specified manner. The time derivative we seek is given by

$$\frac{d}{dt}\int_{\mathscr{V}_a(t)} S\,dV = \lim_{\Delta t\to 0}\left\{\frac{\int_{\mathscr{V}_a(t+\Delta t)} S(t+\Delta t)\,dV - \int_{\mathscr{V}_a(t)} S(t)\,dV}{\Delta t}\right\}. \tag{4.1-10}$$

DERIVATION OF REYNOLDS TRANSPORT THEOREM

To visualize the process under consideration, we must think of a volume, such as a sphere, moving through space so that the velocity of each point on the surface of the volume is given by $\mathbf{w}$. The velocity $\mathbf{w}$ may be a function of the spatial coordinates (if the volume is deforming) and time (if the volume is accelerating or decelerating). At every instant of time some quantity, denoted by S, is measured throughout the region occupied by the volume $\mathscr{V}_a(t)$. The volume integral can then be evaluated at each point in time and the time derivative obtained by Eq. 4.1-10.

In Fig. 4.1.1 we have shown a volume at the times t and $t + \Delta t$ as it moves and deforms in space. During the time interval Δt the volume sweeps out a "new" region designated by $V_{II}(\Delta t)$ and leaves behind an "old" region designated by $V_I(\Delta t)$. Clearly we can express the volume $\mathscr{V}_a(t+\Delta t)$ as

$$\mathscr{V}_a(t+\Delta t) = \mathscr{V}_a(t) + V_{II}(\Delta t) - V_I(\Delta t), \tag{4.1-11}$$

reference : S. Whitaker, Elementary Heat Transfer Analysis, Pergamon Press 1976.

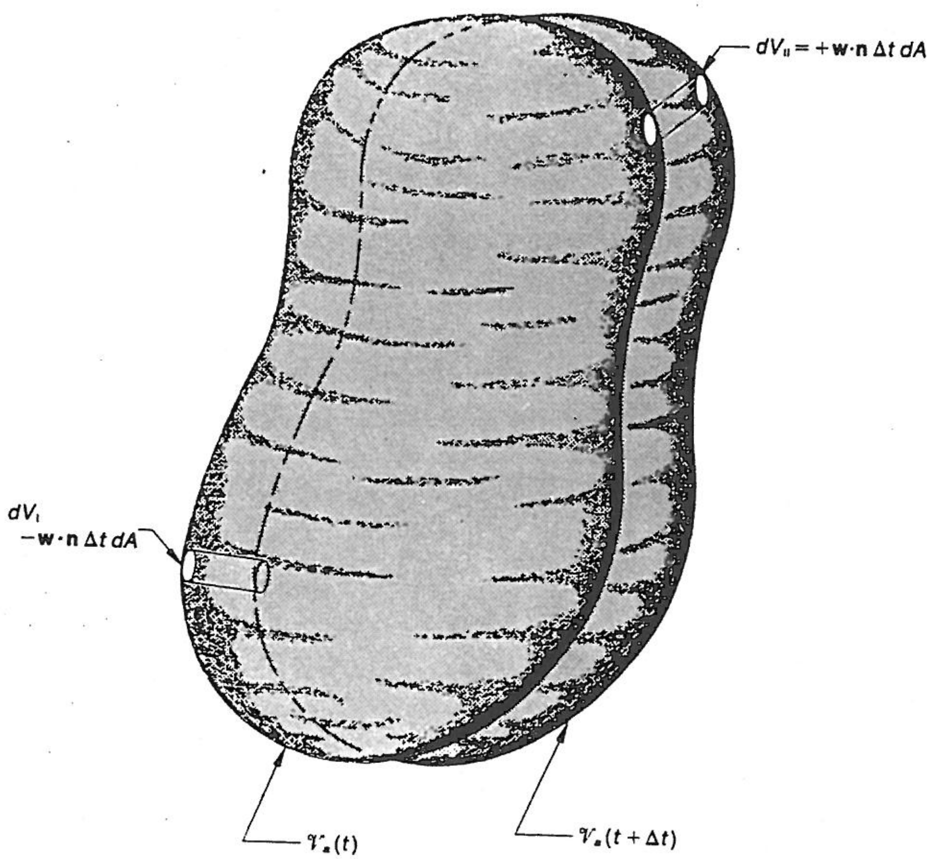


Fig. 4.1.1 A moving volume $\mathscr{V}_a(t)$.

so that the integral of $S(t+\Delta t)$ in Eq. 4.1-10 can be put in the form

$$\int_{\mathscr{V}_a(t+\Delta t)} S(t+\Delta t)\,dV = \int_{\mathscr{V}_a(t)} S(t+\Delta t)\,dV + \int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I. \tag{4.1-12}$$

Substitution of Eq. 4.1-12 into Eq. 4.1-10 leads to

$$\frac{d}{dt}\int_{\mathscr{V}_a(t)} S\,dV = \lim_{\Delta t\to 0}\left\{\frac{\int_{\mathscr{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathscr{V}_a(t)} S(t)\,dV}{\Delta t}\right\} + \lim_{\Delta t\to 0}\left\{\frac{\int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I}{\Delta t}\right\} \tag{4.1-13}$$

In treating the first term on the right-hand-side of Eq. 4.1-13 we note that limits of integration are the same so that the two terms can be combined to give

$$\lim_{\Delta t\to 0}\left\{\frac{\int_{\mathscr{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathscr{V}_a(t)} S(t)\,dV}{\Delta t}\right\} = \lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\int_{\mathscr{V}_a(t)} [S(t+\Delta t) - S(t)]\,dV\right\}. \tag{4.1-14}$$

Since the limits of integration are independent of Δt the limit can be taken inside the integral sign so that Eq. 4.1-14 takes the form

$$\lim_{\Delta t\to 0}\left\{\frac{\int_{\mathscr{V}_a(t)} S(t+\Delta t)\,dV - \int_{V_a(t)} S(t)\,dV}{\Delta t}\right\} = \int_{\mathscr{V}_a(t)} \lim_{\Delta t\to 0}\left[\frac{S(t+\Delta t) - S(t)}{\Delta t}\right]. \tag{4.1-15}$$

Here we must recognize that $S(t+\Delta t)$ and $S(t)$ are evaluated at the *same point in space* so that the integrand on the right-hand-side of Eq. 4.1-15 is the partial derivative and Eq. 4.1-15 takes the form

$$\lim_{\Delta\to 0}\left\{\frac{\int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathcal{V}_a(t)} S(t)\,dV}{\Delta t}\right\} = \int_{\mathcal{V}_a(t)} \frac{\partial S}{\partial t}\,dV. \tag{4.1-16}$$

We can now return to Eq. 4.1-13 and express the time rate of change of the volume integral as

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \lim_{\Delta t\to 0}\left\{\frac{\int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I}{\Delta t}\right\}. \tag{4.1-17}$$

From Fig. 4.1.1 we note that the differential volume elements of the "new" and "old" regions can be expressed as†

$$dV_{II} = +\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_{II}, \tag{4.1-18}$$

and

$$dV_I = -\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_I. \tag{4.1-19}$$

Use of Eqs. 4.1-18 and 4.1-19 allows us to express the volume integrals as area integrals, thus leading to

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \lim_{\Delta t\to 0}\left\{\frac{\int_{A_{II}} S(t+\Delta t)\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_{II} + \int_{A_I} S(t+\Delta t)\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_I}{\Delta t}\right\}. \tag{4.1-20}$$

On the right-hand-side of Eq. 4.1-20 we can cancel Δt in the numerator and denominator and note that

$$A_{II} + A_I \to \mathcal{A}_a(t) \quad \text{as} \quad \Delta t \to 0,$$

so that Eq. 4.1-20 takes the form

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \int_{\mathcal{A}_a(t)} S\,\mathbf{w}\cdot\mathbf{n}\,dA. \tag{4.1-21}$$

This is known as the *general transport theorem*. A more rigorous derivation is given by Slattery [3]. If we let our arbitrary volume $\mathcal{V}_a(t)$ move *with the fluid*, the velocity $\mathbf{w}$ is equal to the fluid velocity $\mathbf{v}$, the volume $\mathcal{V}_a(t)$ becomes a *material volume* designated by $\mathcal{V}_m(t)$, and the total derivative becomes the material derivative. Under these circumstances Eq. 4.1-21 takes the form

$$\frac{D}{Dt}\int_{\mathcal{V}_m(t)} S\,dV = \int_{\mathcal{V}_m(t)} \left(\frac{\partial S}{\partial t}\right) dV + \int_{\mathcal{A}_m(t)} S\,\mathbf{v}\cdot\mathbf{n}\,dA, \tag{4.1-22}$$

and is called the *Reynolds transport theorem*.

Conservation of mass

The principle of conservation of mass can be stated as,

$$\{\text{the mass of a body}\} = \text{constant}, \tag{4.1-23}$$

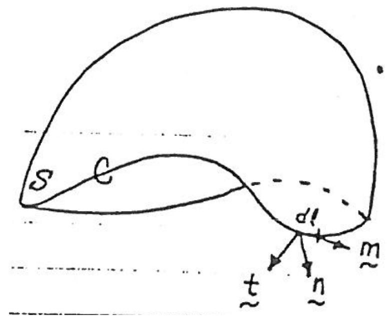
or in the rate form

$$\{\text{time rate of change of the mass of a body}\} = 0. \tag{4.1-24}$$

Using the language of calculus we express Eq. 4.1-24 as

$$\frac{D}{Dt}\int_{\mathcal{V}_m(t)} \rho\,dV = 0. \tag{4.1-25}$$

†See Reference 2, Sec. 3.4 for a detailed discussion of this point.



In the Figure above, we define the following quantities:

1. S is a two-dimensional surface.
2. C is the curve bounding the surface S.
3. dl is a arclength element along C.
4. $\hat{n}$ is a unit vector normal to S.
5. $\hat{t}$ is a unit vector tangent to C (and so to C also).
6. $\hat{\lambda}$ is a unit vector tangent to S and normal to C. It is known as the binormal vector and defined as $\hat{\lambda} = \hat{t} \times \hat{n}$.

We begin by recalling Stokes' Theorem:

$$\oint_C \vec{F} \cdot \hat{t}\ dl = \int_S \hat{n} \cdot (\nabla \times \vec{F}) dS$$

In order to develop a generalization of this theorem, we let $\vec{F} = \vec{f} \times \vec{b}$, for $\vec{b}$ an arbitrary constant vector. We then have

$$\oint_C (\vec{f} \times \vec{b}) \cdot \hat{t}\ dl = \int_S \hat{n} \cdot (\nabla \times (\vec{f} \times \vec{b})) dS.$$

We now use the vector identity

$$\nabla \times (\vec{f} \times \vec{b}) = \vec{f}(\nabla \cdot \vec{b}) - \vec{b}(\nabla \cdot \vec{f}) + \vec{b} \cdot \nabla \vec{f} - \vec{f} \cdot \nabla \vec{b} = -\vec{b}(\nabla \cdot \vec{f}) + \vec{b} \cdot \nabla \vec{f}$$

where the last equality follows from $\vec{b}$ being constant.

Moreover, $(\vec{f} \times \vec{b}) \cdot \hat{t} = -\vec{b} \cdot (\vec{f} \times \hat{t})$, so that we may write

$$\vec{b} \cdot \oint_C (\vec{f} \times \hat{t})\ dl = \vec{b} \cdot \int_S \hat{n}(\nabla \cdot \vec{f}) - \nabla \vec{f} \cdot \hat{n}\ dS.$$

Since the vector $\vec{b}$ is arbitrary, we have

$$\oint_C (\vec{f} \times \hat{t})\ dl = \int_S \hat{n}(\nabla \cdot \vec{f}) - \nabla \vec{f} \cdot \hat{n}\ dS.$$

In particular, if we consider $\vec{f} = \sigma\hat{n}$ and recall that $\hat{n} \times \hat{t} = -\hat{\lambda}$, we find

$$\begin{aligned}
-\oint_C \sigma\hat{\lambda}\, dl &= \int_S \hat{n}(\nabla \cdot (\sigma\hat{n})) - \nabla(\sigma\hat{n}) \cdot \hat{n}\, dS. \\
&= \int_S \hat{n}(\nabla\sigma \cdot \hat{n}) + \sigma\hat{n}(\nabla \cdot \hat{n})) - (\nabla\sigma)\hat{n} \cdot \hat{n} - \sigma(\nabla\hat{n}) \cdot \hat{n}\, dS.
\end{aligned}$$

We note that $\vec{0} = \nabla(\hat{n} \cdot \hat{n}) = 2\nabla\hat{n} \cdot \hat{n}$, so that $(\nabla\hat{n}) \cdot \hat{n} = 0$. Finally, because σ is only defined on the interface S, we have that $(\nabla\sigma \cdot \hat{n}) = 0$. This leaves only

$$\oint_C \sigma\hat{\lambda}\, dl = \int_S -\sigma\hat{n}(\nabla \cdot \hat{n}) + (\nabla\sigma)\, dS.$$