Make sure your name is on your homework, and please **box** your final answer. Because we will be giving partial credit, be sure to attempt all the problems, even if you don’t finish them. The homework is due at the beginning of class on **Monday, March 30th**. Because the solutions will be posted immediately after class, *no late homeworks can be accepted!* You are welcome to ask questions during the discussion session or during office hours.

1. *Why* is the gravitational potential energy of two masses negative? Note that saying “because that’s what the equation gives” is *not* an explanation!

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**Solution**

The gravitational potential energy represents the energy of the system, which you can think of as the amount of work that it took to put the system together. Because the masses attract each other gravitationally, then they naturally fall together. This means that the work was done for us, and so the system has negative energy. This is also the amount of energy that we’d need to *add* to the system to break it apart. Since we’d need to do work on the system to break it apart (ending up with zero total energy), we had to start with a negative energy. The same thing occurs with oppositely-charged particles like protons and electrons, which also have negative electrostatic potential energy.
2. Some people think that the shuttle astronauts are “weightless” because they are “beyond the pull of Earth’s gravity.” In fact, this is completely untrue.

(a) What is the magnitude of the gravitational field in the vicinity of a shuttle orbit? A shuttle orbit is about 400 km above the ground.

(b) Given the answer in Part (a), explain why shuttle astronauts suffer from adverse biological effects such as muscle atrophy even though they are not actually “weightless.”

Solution

(a) The shuttle orbits at a distance of \( r = R_E + h \) from the center of the Earth, where \( R_E \) is the radius of the Earth, and \( h \) is the height above the surface. If the height is \( h = 400 \) km, then \( r = 6400 + 400 = 6800 \) km, or \( 6.8 \times 10^6 \) meters. At this point, the gravitational field has a magnitude

\[
g = \frac{GM_E}{r^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.8 \times 10^6)^2} = 8.6 \text{ m/s}^2,
\]

which is still almost 90\% of the acceleration at the surface of the Earth.

(b) Remember that the weight that we feel is due to the normal force. The astronauts in orbit are in constant free-fall, and don’t feel their weight. So, without the compensating normal force to fight against the muscles begin to weaken, not needing to do as much anymore.
3. Calculate the mass of Earth from the period of the moon, \( T = 27.3 \text{ d} \); its mean orbital radius \( r_m = 3.84 \times 10^8 \text{ m} \); and the known value of \( G \).

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Solution

The moon is held to the Earth by the gravitational force, \( F_G = G \frac{m_{\text{moon}} M_{\text{Earth}}}{r^2} \), where \( r \) is the orbital radius. Now, because the orbit is (pretty much) circular, then the net force on the moon is the centripetal force, \( F_{\text{cent}} = \frac{m_{\text{moon}} v^2}{r} \). But, for an object moving in a circle, then \( v = \frac{2\pi r}{T} \), and so

\[
F_{\text{cent}} = \frac{m_{\text{moon}}}{r} \left( \frac{2\pi r}{T^2} \right)^2 = \frac{4\pi^2}{T^2} m_{\text{moon}} r.
\]

Because the orbit is stable this force is equal to the gravitational force. Setting the two forces equal gives

\[
\frac{4\pi^2}{T^2} m_{\text{moon}} r = G \frac{m_{\text{moon}} M_{\text{Earth}}}{r^2}.
\]

Solving for the mass of the Earth gives

\[
M_{\text{Earth}} = \frac{4\pi^2}{T^2 G} r^3.
\]

The period is \( T = 27.3 \times 3600 \times 24 = 2.36 \times 10^6 \text{ seconds} \), and so

\[
M_{\text{Earth}} = \frac{4\pi^2}{T^2 G} r^3 = \frac{4\pi^2}{(2.36 \times 10^6)^2 (6.672 \times 10^{-11}) (3.84 \times 10^8)^3} = 6 \times 10^{24} \text{ kg}.
\]
4. (a) If we take the potential energy of a 100 kg object and Earth to be zero when the two are separated by an infinite distance, what is the potential energy when the object is at the surface of Earth?

(b) Find the potential energy of the same object at a height above Earth’s surface equal to Earth’s radius.

(c) Find the escape speed for a body projected from this height.

Solution

(a) The gravitational potential energy of two objects of masses $m$ and $M$, separated by a distance $r$ is

$$PE_{grav} = -G \frac{mM}{r},$$

which sets the potential energy to zero when the objects are infinitely far apart ($r \rightarrow \infty$). So, if the object is on the surface of the Earth, then $r = R_E$, and $M = M_E$, and

$$PE_g = -G \frac{mM_E}{R_E}.$$  

Now, we can rewrite this recalling that for Earth, $g = \frac{GM_E}{R_E^2}$. Thus,

$$PE_g = -G \frac{mM_E}{R_E} = -m \frac{GM_E}{R_E^2} R_E = -mgR_E.$$  

Now, if the radius of the Earth is 6400 km, or $6.4 \times 10^6$ meters, then

$$PE_g = -mgR_E = -(100)(9.8)(6.4 \times 10^6) = -6.3 \times 10^9 \text{ J}.$$  

(b) The potential energy at a distance of $r = R_E + R_E = 2R_E$ will just be

$$PE_g = -G \frac{mM_E}{2R_E} = -\frac{1}{2} mgR_E = -3.15 \times 10^9 \text{ J}.$$  

(c) To determine the escape velocity from this distance we just recall the usual method setting $KE + PE = 0$, so $\frac{1}{2}mv^2 = \frac{1}{2} mgR_E$, giving

$$v_{esc} = \sqrt{gR_E} = \sqrt{(9.8)(6.4 \times 10^3)} = 8 \text{ km/s},$$

which is smaller than the escape velocity from the surface of the earth ($v_{esc} \approx 11 \text{ km/s}$), as we should expect.
5. **Black holes** are objects whose gravitational field is so strong that not even light can escape. One way of thinking about this is to consider a spherical object whose density is so large that the escape speed at its surface is greater than the speed of light, \( c \). If a star’s radius is smaller than a value called the *Schwarzschild radius* \( R_s \), then the star will be a black hole, that is, light originating from its surface cannot escape.

(a) For a nonrotating black hole, the Schwarzschild radius depends only upon the mass of the black hole. Show that it is related to that mass \( M \) by \( R_s = \frac{(2GM)}{c^2} \).

(b) Calculate the value of the Schwarzschild radius for a black hole whose mass is ten solar masses.

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**Solution**

(a) The escape velocity of a spherical mass \( M \) with radius \( R \) is given by

\[
v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.
\]

If we want light to be unable to escape from the surface, then the escape velocity has to be greater than, or equal to, the speed of light. Thus, setting the escape velocity \( v_{\text{esc}} = c \) and solving for the radius gives

\[
R_s = \frac{2GM}{c^2}.
\]

(b) The mass of the sun, \( M_{\text{sun}} = 2 \times 10^{30} \) kg. So, we can determine the Schwarzschild radius for this black hole,

\[
R_s = \frac{2GM}{c^2} = \frac{2 \times 6.67 \times 10^{-11} \times 10 \times 2 \times 10^{30}}{(3 \times 10^8)^2} = 29.6 \text{ km}.
\]

So, the mass of ten suns is squashed into a ball about the size of a large city.