1. (a) Assuming that the surface temperature of the Sun is $T_\odot = 5800$ K, use Stefan’s law to determine the rest mass lost per second to radiation by the Sun. Take the Sun’s radius to be $R_\odot = 7.0 \times 10^8$ meters.

   (b) What fraction of the Sun’s rest mass is lost each year from electromagnetic radiation? Take the Sun’s rest mass to be $M_\odot = 2.0 \times 10^{30}$ kg.

---

### Solution

(a) The intensity of radiation (in watts per square meter) from the Sun is $I = P/A = \sigma T^4$, where $\sigma$ is the Stefan-Boltzmann constant and $T$ is the temperature. Thus the power is $P = AI = 4\pi R_\odot^2 \sigma T^4$, where $R_\odot$ is the radius of the Sun. Now, the power is $P = \dot{E}$, and since the energy is coming from the rest mass, $E = mc^2$, then $\dot{E} = \dot{m}c^2$. Thus, the rate of mass loss is

$$\frac{dm}{dt} = \frac{4\pi R_\odot^2 \sigma}{c^2} T^4.$$

So, plugging in the constants gives

$$\frac{dm}{dt} = \frac{4\pi R_\odot^2 \sigma}{c^2} T^4 = \frac{4\pi (7 \times 10^8)^2 (5.67 \times 10^{-8})}{(3 \times 10^8)^2} (5800)^4 = 4.39 \times 10^9 \text{ kg/sec}.$$

(b) So, in one year the Sun burns through

$$\Delta m = \frac{dm}{dt} \cdot \Delta t = (4.39 \times 10^9) \times (3.15 \times 10^7) = 1.38 \times 10^{17} \text{ kg}.$$

This is a fraction

$$\frac{\Delta m}{M_\odot} = \frac{1.38 \times 10^{17}}{2 \times 10^{30}} = 6.9 \times 10^{-14},$$

meaning that the Sun loses roughly $10^{-13}\%$ of its mass per year.
2. Emission lines of hydrogen, H\(\beta\) \((n = 4 \rightarrow 2\) and \(\lambda_{\text{rest}} = 4861\,\text{Å}\) are observed in the spectrum of a spiral galaxy at redshift \(z = 0.9\). The galaxy disk is inclined by 45\(^\circ\) to the line of sight (if the inclination was 0\(^\circ\) then we’d see the galaxy from a top view, while if it was 90\(^\circ\), we’d see it edge-on). The H\(\beta\) wavelength of lines from one side of the galaxy are shifted to the blue by 5\,\text{Å} relative to the emission line from the center of the galaxy, and to the red by 5\,\text{Å} on the other side.

(a) What is the galaxy’s rotation speed?

(b) Find the age of the Universe when the light was emitted from this galaxy.

---

**Solution**

(a) There are two Doppler shifts in this problem, one from the redshift \(z\), and the other from the rotational motion of the galaxy. The redshift from the galaxy shifts the wavelength

\[
\frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = 1 + z = 1.9.
\]

Thus, the observed wavelength to the center of the galaxy is \(\lambda_{\text{obs}} = (1 + z)\lambda_{\text{emit}} = 1.9\lambda_{\text{emit}} = 9236\,\text{Å}\).

Next, the rotation of the galaxy adds an additional redshift, depending on the rotational velocity. This velocity also depends on the angle of orientation. If the angle was zero, giving a top view of the galaxy, then we wouldn’t measure any rotational velocity, since it would be rotating perpendicular to the line of sight. If the angle was 90\(^\circ\), then we’d see the galaxy edge-on, and would measure the full rotational velocity. Thus, we see that we care about the sine component of the velocity, \(v\sin\theta\).

Now, the Doppler effect in the light emitted from the rotational velocity is (taking the specific case of the edge that is moving away from us),

\[
\frac{\lambda_{\text{obs}}'}{\lambda_{\text{obs}}} = \sqrt{\frac{c + v\sin\theta}{c - v\sin\theta}},
\]

where \(\lambda_{\text{obs}}'\) is the wavelength seen at the edge of the galaxy, and \(\lambda_{\text{obs}}\) is the wavelength at the center of the galaxy \(((1 + z)\lambda_{\text{emit}})\). Since the rotation velocity is much smaller that light we can expand

\[
\frac{\lambda_{\text{obs}}'}{\lambda_{\text{obs}}} \approx \sqrt{\left(1 + \frac{v}{c}\sin\theta\right)^2} = 1 + \frac{v}{c}\sin\theta.
\]

So,

\[
\frac{\lambda_{\text{obs}}' - \lambda_{\text{obs}}}{\lambda_{\text{obs}}} = \frac{\Delta\lambda}{\lambda_{\text{obs}}} = \frac{v}{c}\sin\theta.
\]
where $\Delta \lambda$ is the difference in wavelength between the edge and center of the galaxy (5 Å). Now, $\lambda_{\text{obs}} = (1 + z) \lambda_{\text{emit}}$, and so, all together the velocity is

$$v = \frac{c}{\sin \theta} \frac{\Delta \lambda}{(1 + z) \lambda_{\text{emit}}}.$$ 

Plugging in the numbers gives $v = 230 \text{ km/s}$ (which shows that we were justified in assuming $v$ small in our derivation).

(b) The redshift is defined in terms of the scale factor as

$$\frac{a(t)}{a_0} = \frac{1}{1 + z}.$$ 

The galaxy emitted light in the matter-dominated era, and so $a \sim t^{2/3}$. Thus,

$$\frac{a(t)}{a_0} = \left( \frac{t}{t_0} \right)^{2/3} = \frac{1}{1 + z},$$

where $t_0$ is the current age of the Universe. Solving for the emission time, $t$ we find

$$t = t_0 (1 + z)^{-3/2}.$$ 

Taking $t_0 = 13.7$ billion years, then

$$t = \frac{t_0}{(1 + z)^{3/2}} = \frac{13.7 \times 10^9}{(1 + 0.9)^{3/2}} = 5.23 \text{ billion years}.$$
3. We have seen that, for a star composed of a classical, nonrelativistic, ideal gas, \( E_{\text{total}} = E_{\text{th}} + E_{\text{grav}} = -E_{\text{th}} \), and therefore the star is bound. Repeat the derivation, but for a classical \textit{relativistic} gas of particles. Recall that the equation of state of a relativistic gas is \( P = \frac{1}{3} E_{\text{th}} \). Show that, in this case, \( E_{\text{grav}} = -E_{\text{total}} \), and therefore \( E_{\text{total}} = E_{\text{th}} + E_{\text{grav}} = 0 \); i.e., the star is marginally bound. As a result, stars dominated by radiation pressure are unstable.

\[ \textbf{Solution} \]

The pressure of a relativistic gas is \( P = E_{\text{th}}/3V \), which means that

\[ \langle P \rangle = \frac{1}{3V} E_{\text{th}}. \]

Furthermore, we’ve seen that

\[ \langle P \rangle = -\frac{1}{3} \frac{E_{\text{grav}}}{V}, \]

which tells us that \( E_{\text{grav}} = -E_{\text{th}} \). Since \( E_{\text{tot}} = E_{\text{th}} + E_{\text{grav}} = -E_{\text{grav}} + E_{\text{grav}} = 0 \).
4. Because of the destabilizing influence of radiation pressure, the most massive stars that can form are those in which the radiation pressure and the nonrelativistic kinetic pressure are approximately equal. Estimate the mass of the most massive stars, as follows:

(a) Assume that the gravitational binding energy of a star of mass $M$ and radius $R$ is $E_{\text{grav}} \sim G_N M^2 / R$. Use the virial theorem

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\text{grav}}}{V},$$

and show that

$$P \sim \left( \frac{4\pi}{3} \right)^{1/3} G_N M^{2/3} \rho^{4/3},$$

where $\rho$ is the typical density.

(b) Show that if the radiation pressure, $P_{\text{rad}} = \frac{1}{3} aT^4$, equals the kinetic pressure, then the total pressure is

$$P = 2 \left( \frac{3}{a} \right)^{1/3} \left( \frac{k_B \rho}{\bar{m}} \right)^{4/3},$$

where $\bar{m} = m_H/2$ is the mean mass.

(c) Equate the expressions for the pressure in parts (a) and (b), to obtain an expression for the maximal mass of a star. Find its value, in solar masses, assuming a fully ionized hydrogen composition.

---

Solution

(a) The pressure

$$P \sim \frac{G_N M^2}{3RV}.$$

Now, $1/V = \rho/M$, while $R = (3M/4\pi\rho)^{1/3}$, and so

$$P \sim \frac{G_N M \rho}{3} \left( \frac{4\pi\rho}{3M} \right)^{1/3} = \left( \frac{4\pi}{3} \right)^{1/3} G_N M^{2/3} \rho^{4/3}. $$

(b) The total pressure is $P = P_{\text{grav}} + P_{\text{rad}} = \frac{\rho k_B T}{\bar{m}} + \frac{1}{3} a T^4$. If the kinetic and radiation pressures are equal then

$$\frac{\rho k_B T}{\bar{m}} = \frac{1}{3} a T^4 \Rightarrow T = \left( \frac{3 \rho k_B}{\bar{m} a} \right)^{1/3}.$$

Then, the total pressure is

$$P = P_g + \frac{1}{3} a T^4 = \frac{2}{3} a T^4 = \frac{2}{3} a \left( \frac{3 \rho k_B}{\bar{m} a} \right)^{4/3} = 2 \left( \frac{3}{a} \right)^{1/3} \left( \frac{k_B \rho}{\bar{m}} \right)^{4/3}. $$
(c) Equating the two expressions gives

\[
\left( \frac{4\pi}{3^4} \right)^{1/3} G_N M^{2/3} \rho^{4/3} \sim 2 \left( \frac{3}{a} \right)^{1/3} \left( \frac{k_B \rho}{\bar{m}} \right)^{4/3}.
\]

This gives for the mass, after plugging in for \( a = 8\pi^5k_B^4/15\hbar^3c^3 \),

\[
M \sim \sqrt{\frac{14580}{\pi^6 m_H^4} \left( \frac{\hbar c}{G_N} \right)^3} = 2.27 \times 10^{32} \text{ kg},
\]

which is \( M \sim 113M_\odot \). So, the largest stars have a mass of roughly 100 solar masses.
5. Suppose a star of total mass $M$ and radius $R$ has a density profile

$$\rho = \rho_C \left(1 - \frac{r}{R}\right),$$

where $\rho_C$ is the central density (assume constant).

(a) Find $M(r)$.

(b) Express the total mass $M$ in terms of $R$ and $\rho_C$.

(c) Solve for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.

---

**Solution**

(a) We can determine $M(r)$ via the mass continuity equation,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) = 4\pi \rho_C \left(r^2 - \frac{r^3}{R}\right).$$

Separating and integrating noting that $M(r = 0) = 0$, we find

$$\int_0^{M(r)} dM = 4\pi \rho_C \int_0^r dr \left(r^2 - \frac{r^3}{R}\right).$$

Integrating both sides gives

$$M(r) = \frac{4\pi}{3} \rho_C r^3 \left(1 - \frac{3r}{4R}\right).$$

(b) Finding the total mass is very simple. We can either repeat the above derivation in part (a), integrating to $r = R$, or just plug in $r = R$ in to our result for $M(r)$. Then,

$$M = M(R) = \frac{4\pi}{3} \rho_C R^3 \left(1 - \frac{3}{4}\right) = \frac{\pi}{3} \rho_C R^3.$$

(c) Now, to determine the pressure, $P(r)$, we use the hydrostatic equilibrium equation,

$$\frac{dP(r)}{dr} = -\frac{G_N M(r) \rho(r)}{r^2}.$$

The density $\rho(r)$ was given, and we have just found $M(r)$, so we find

$$\frac{dP(r)}{dr} = -\frac{4\pi G_N}{3} \rho_C^2 r^3 \left(1 - \frac{3r}{4R}\right) \times \left(1 - \frac{r}{R}\right) = -\frac{4\pi G_N}{3} \rho_C^2 r \left(1 - \frac{r}{R} - \frac{3r}{4R} + \frac{3r^2}{R^2}\right) = -\frac{4\pi G_N}{3} \rho_C^2 \left(r - \frac{7r^2}{4R} + \frac{3r^3}{4R^2}\right).$$
Now, since we know that $P(R) = 0$, we integrate from $r = r$ to $r = R$,

$$
\int_{P(r)}^{0} dP(r) = -\frac{4\pi G_N}{3} \rho_C^2 \int_{r}^{R} dr \left( r - \frac{7r^2}{4R} + \frac{3r^3}{4R^2} \right)
$$

Integrating gives

$$
-P(r) = -\frac{4\pi G_N}{3} \rho_C^2 \left( \frac{5}{48} R^2 - \frac{r^2}{2} + \frac{7}{12} R \right) \left( \frac{7}{2} R - \frac{3}{16} R^2 \right),
$$

or,

$$
P(r) = \pi G_N \rho_C^2 R^2 \left[ \frac{5}{36} - \frac{2}{3} \left( \frac{r}{R} \right)^2 + \frac{7}{9} \left( \frac{r}{R} \right)^3 - \frac{1}{4} \left( \frac{r}{R} \right)^4 \right].
$$

Notice that the pressure at the center of the star (in this model) is

$$
P(0) = \frac{5\pi G_N}{36} \rho_C^2 R^2.
$$