Physics 161 Fall 2011
Extra Credit 2
Investigating Black Holes
The Following is Worth 50 Points!!!

This extra credit assignment will investigate various properties of black holes that we didn’t have time to investigate in lecture. We will answer the following questions:

1. What is the closest orbit that a massive body can have around a black hole, and is it stable?

2. Can light orbit around a black hole, and, if so, what is the distance and is it stable?

Let’s begin with the Schwarzschild metric, expressed in terms of the proper time,

$$d\tau^2 = \left(1 - \frac{R_S}{r}\right) dt^2 - \frac{1}{c^2} \left[\frac{dr^2}{1 - R_S/r} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right], \quad (1)$$

where $R_S = 2G_NM/c^2$ is the Schwarzschild radius. Now, since we are interested in orbits, which occur in a plane, we can choose coordinates such that $θ = \pi/2$, which gives

$$d\tau^2 = \left(1 - \frac{R_S}{r}\right) dt^2 - \frac{1}{c^2} \left[\frac{dr^2}{1 - R_S/r} + r^2 d\phi^2\right], \quad (2)$$

Dividing through by $d\tau^2$ gives

$$1 = \left(1 - \frac{R_S}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2} \left[\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\phi}{d\tau}\right)^2\right]. \quad (3)$$

Now, the complete theory of General Relativity tells us that there are some constants of motion,

$$\frac{E}{mc^2} = (1 - \frac{R_S}{r}) \frac{dt}{d\tau} \quad \text{and} \quad \frac{L}{m^2 c^2} = r^2 \frac{d\phi}{d\tau}, \quad (4)$$

where $E$ is the energy of the body of mass $m$, and $L$ is the angular momentum. Here’s where your work begins:

1. Plug in above constants to Eq. (3) to find

$$c^2 \left(1 - \frac{R_S}{r}\right) = \frac{E^2}{m^2c^2} - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{m^2r^2} \left(1 - \frac{R_S}{r}\right). \quad (5)$$

Next, use the chain rule to write

$$\frac{dr}{d\tau} = \frac{dr}{d\phi} \frac{d\phi}{d\tau},$$

and plug in again for $d\phi/d\tau$ to find

$$c^2 \left(1 - \frac{R_S}{r}\right) = \frac{E^2}{m^2c^2} - \frac{L^2}{m^2r^4} \left(\frac{dr}{d\phi}\right)^2 - \frac{L^2}{m^2r^2} \left(1 - \frac{R_S}{r}\right). \quad (5)$$
2. The expression in Eq. (5) is a mess, full of nonlinearities. The standard way of handling this sort of orbital expression is to make the substitution

\[ u = \frac{1}{r}. \]

Using this, show that Eq. (5) becomes

\[ \frac{m^2 c^2}{L^2} (1 - R_S u) = \left( \frac{E}{L} \right)^2 - \left( \frac{du}{d\phi} \right)^2 - u^2 (1 - R_S u). \] (6)

3. Now, we could solve for the derivative and integrate, but a simpler way of proceeding is to differentiate everything with respect to \( \phi \), noting that \( u \) is the only variable. Do this, and cancel off a common \( du/d\phi \) to find a much simpler equation,

\[ \frac{d^2 u}{d\phi^2} + u = \frac{3}{2} R_S u^2 + \frac{R_S m^2 c^2}{2L^2}. \] (7)

4. Determine the radius of the constant orbit, \( r_0 = u_0^{-1} \). Note: you’ll need to solve a quadratic equation. Explain which sign for the square root you need to take.

5. We now want to investigate the stability of this orbit, looking at small perturbations about a constant orbit,

\[ u (\phi) = u_0 + \delta u (\phi), \] (8)

where \( \delta u \ll u_0 \). Plug Eq. (8) into Eq. (7), expand out and neglect terms of order \( \delta u^2 \). Show that you get an equation of the form

\[ \frac{d^2}{d\phi^2} \delta u + \omega_0^2 \delta u = 0, \] (9)

and determine the harmonic oscillator frequency, \( \omega_0 \). Determine the smallest radius that keeps the orbit stable to these small perturbations. What does the angular momentum need to be for this orbit?

6. The above analysis has been performed for a massive body. Now we want to look at the orbit of light. Fortunately, we can simply use Eq. (7) setting \( m \equiv 0 \). What is the radius of the constant orbit, \( r_0 = u_0^{-1} \)?

7. Finally, perform the same stability analysis as for the massive body, and show that there is no stable orbit for light!