Physics 9 Fall 2009
Midterm 2 Solutions

For the midterm, you may use one sheet of notes with whatever you want to put on it, front and back. Please sit every other seat, and please don’t cheat! If something isn’t clear, please ask. You may use calculators. All problems are weighted equally. PLEASE BOX YOUR FINAL ANSWERS! You have the full length of the class. If you attach any additional scratch work, then make sure that your name is on every sheet of your work. Good luck!

1. A square loop of wire (side \(a\)) lies on a table, a distance \(s\) from a very long straight wire, which carries current \(I\), as shown in the figure.

(a) Recalling that the magnetic field of an infinite wire is \(B = \frac{\mu_0 I}{2\pi r}\), find the flux of \(\vec{B}\) through the loop (notice - \(B\) changes with distance from the wire!).

\[
\Phi = \frac{\mu_0 I}{2\pi} \int_s^{s+a} \frac{1}{s} \, ds = \frac{\mu_0 I a}{2\pi} \log \left( \frac{s+a}{s} \right).
\]

(b) If someone now pulls the loop directly away from the wire, at speed \(v\), what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

(c) What if the loop is pulled to the right at speed \(v\), instead of away?

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Solution

(a) The magnetic field of the wire is \(B = \frac{\mu_0 I}{2\pi s}\). The flux through the loop is

\[
\Phi = \frac{\mu_0 I}{2\pi} \int_s^{s+a} \frac{1}{s} \, ds = \frac{\mu_0 I a}{2\pi} \log \left( \frac{s+a}{s} \right).
\]

(b) The induced emf, \(E\) is \(E = -\frac{d}{dt}\Phi\). So,

\[
E = -\frac{\mu_0 I a}{2\pi} \frac{d}{dt} \left[ \log \left( \frac{s+a}{s} \right) \right] = -\frac{\mu_0 I a}{2\pi} \left[ \frac{1}{s+a} \dot{s} - \frac{1}{s} \dot{s} \right] = \frac{-\mu_0 I a^2}{2\pi s(s+a)} v,
\]

where \(v = \dot{s}\) is the velocity of the loop. The induced emf is positive, and so the current flows counterclockwise.

(c) If the loop is pulled to the right, then the flux is constant and so \(E = 0\).
2. An inductor \((L = 500 \text{ mH})\), a capacitor \((C = 5 \mu \text{F})\), and a resistor \((R = 500 \Omega)\) are connected in series, as in the diagram. A 60.0 Hz AC generator produces a peak current of 1 A in the circuit.

(a) Determine the capacitive reactance for this circuit.
(b) Determine the inductive reactance for this circuit.
(c) What is the impedance of this circuit?
(d) Calculate the peak voltage, \(V_{\text{max}}\), that would give the indicated peak current.
(e) Determine the angle, \(\phi\), by which the current leads or lags the applied voltage. Does it lead, or lag?

**Solution**

(a) The capacitive reactance for this circuit is \(X_C = \frac{1}{\omega C}\), where \(\omega = 2\pi f\) is the angular frequency of the circuit, and \(C\) is the capacitance. Plugging in the numbers gives

\[
X_C = \frac{1}{\omega C} = \frac{1}{2\pi \times 60.0 \times 5 \times 10^{-6}} = 531 \Omega.
\]

(b) The inductive reactance for this circuit is \(X_L = \omega L\), where \(L\) is the inductance. Again, plugging in the numbers gives

\[
X_L = \omega L = 2\pi \times 60.0 \times 500 \times 10^{-3} = 189 \Omega.
\]

(c) The impedance is given by \(Z = \sqrt{R^2 + (X_L - X_C)^2}\), where \(R\) is the resistance, and so

\[
Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{500^2 + (189 - 531)^2} = 606 \Omega.
\]

(d) The peak voltage, \(V_{\text{max}}\) is given by Ohm's law, \(V_{\text{max}} = I_{\text{max}}Z\), where \(I_{\text{max}}\) is the peak current in the circuit. So,

\[
V_{\text{max}} = I_{\text{max}}Z = 1 \times 606 = 606 \text{ volts}.
\]

(e) Finally, the phase angle is given by \(\phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right)\). So, plugging in the values gives

\[
\phi = \tan^{-1} \left( \frac{189 - 531}{500} \right) = \tan^{-1} (-0.684) = -34.4^\circ.
\]

The angle is negative, so the current *leads* the voltage by 34.4°.
3. The toroid seen in the following diagram is a coil of wire wrapped around a doughnut-shaped ring (a torus) made of nonconducting material. Toroidal magnetic fields are used to confine fusion plasmas. The torus has an inner radius of $a$, an outer radius of $b$, and has $N$ total turns of wire carrying a current $I$.

(a) Using Ampere’s Law, find the magnetic field, $\vec{B}$, inside the torus at a distance $r$ from the axis of the toroid, as seen in the diagram.

(b) You know that the magnetic field inside a solenoid is uniform. Is the magnetic field you found in part (a) also uniform? Why or why not?

(c) Suppose we had an electron moving in the direction of the magnetic field with velocity $v$. What is the force on the electron due to the magnetic field?

Solution

(a) The toroid is basically a solenoid that has been bent around, connecting the ends, to form a doughnut shape. Ampere’s law says that the line integral of the magnetic field around a path depends on the current passing through an area bounded by the path, $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{through}}$. The path is shown in the diagram as the dotted line. The path encloses the center of the torus, and so includes all $N$ wire loops. So, the net current through the path is $I_{\text{through}} = NI$, where $I$ is the current in the wire. Because of the symmetry, the magnetic field is everywhere the same along the path, and so it can be taken outside the integral, $\oint \vec{B} \cdot d\vec{s} = B \int ds = Bs$, where $s = 2\pi r$ is the length of the path. So, altogether we find

$$\vec{B} = \frac{\mu_0 NI}{2\pi r} \hat{\phi}.$$  

(b) The magnetic field of a solenoid is uniform, but the result we found in part (a) depends on the distance away from the center of the toroid. This means that the magnetic field of the torus is not uniform.

(c) The magnetic force on a charge, $q$, moving with speed $v$ is $\vec{F} = q\vec{v} \times \vec{B}$. Since the electron is moving in the direction of the magnetic field, $\vec{v} \times \vec{B} = 0$, and so there is no force on the electron when it’s moving parallel to the magnetic field!
In 1965, Penzias and Wilson discovered the cosmic microwave background (CMB) radiation left over from the Big Bang expansion of the Universe, which subsequently earned them the Nobel Prize in physics in 1978. This radiation background behaves like a blackbody with a temperature of 2.725 K, as seen in the figure. The temperature is extremely uniform, with fluctuations less than one part in 100,000! Precise measurements of the CMB are critical to cosmology, since any proposed model of the Universe must explain this radiation. These fluctuations in the temperature are thought to have formed the seeds of galaxy formation in the early Universe, as the CMB dates to roughly 379,000 years after the Big Bang.

(a) For an electromagnetic wave, the magnitude of the magnetic field is related to that of the electric field by $E = cB$, where $c$ is the speed of light. Show that the energy densities of the electric and magnetic fields are equal in this case, and hence that an electromagnetic wave splits its energy evenly between the electric and magnetic fields.

(b) The energy density of a blackbody (such as the CMB) is given by the expression

$$u = \frac{\pi^2 k_B^4 T^4}{15 (hc)^3},$$

where $k_B$ is Boltzmann’s constant, and $T$ is the temperature. Plug in the values and show that the energy density is $4.235 \times 10^{-14} \text{ J/m}^3$.

(c) The energy density of the CMB comes from the electromagnetic waves in the microwave region of the electromagnetic spectrum. Determine the maximum values of the electric and magnetic fields of the CMB radiation.

**Solution**

(a) The energy density of the electric field is $u_E = \frac{\varepsilon_0}{2} E^2$, while the magnetic field energy density is $u_B = \frac{1}{2\mu_0} B^2$. In an electromagnetic wave, the electric and magnetic fields are related by $\frac{E}{B} = c$. So, we can write $u_B = \frac{1}{2\mu_0} \frac{E^2}{c^2} = \frac{1}{2\mu_0 c^2} E^2$. But, we know that, by definition, $c^2 = \frac{1}{\varepsilon_0\mu_0}$, and so we find that $u_B = \frac{\varepsilon_0}{2} E^2 \equiv u_E$. Since the total energy density is the sum of the electric and magnetic parts, $u_{\text{tot}} = u_E + u_B$, and since $u_E = u_B$, we have that $u_{\text{tot}} = u_E + u_E = 2u_E$, or $u_E = u_B = \frac{1}{2} u_{\text{tot}}$. 


(b) Starting with the given expression, we just plug in the correct numbers

\[ u = \frac{\pi^2 k_B^4 T^4}{15 (hc)^3} = \frac{\pi^2 (1.38 \times 10^{-23})^4 (2.725)^4}{15 (1.05 \times 10^{-34} \times 2.99 \times 10^8)^3} = 4.235 \times 10^{-14} \text{ J/m}^3. \]

(c) Because \( u = 2u_E = \epsilon_0 E^2 \), we just solve for the electric field. This gives, \( E = \sqrt{\frac{u}{\epsilon_0}} \).

Since \( E = cB \), then \( B = \frac{1}{c} \sqrt{\frac{u}{\epsilon_0}} = \sqrt{\mu_0 u} \). So, using our results from part (b), we find

\[ E = \sqrt{\frac{u}{\epsilon_0}} = \sqrt{\frac{4.235 \times 10^{-14}}{8.85 \times 10^{-12}}} = 0.0692 \text{ V/m}, \]

and

\[ B = \sqrt{\mu_0 u} = \sqrt{4\pi \times 10^{-7} \times 4.235 \times 10^{-14}} = 2.31 \times 10^{-10} \text{ T}. \]

These are the maximum values.
Extra Credit Question!!

The following is worth 10 extra credit points!

As a lecture demonstration a short cylindrical bar magnet is dropped down a vertical aluminum pipe of slightly larger diameter, about 2 meters long. It takes several seconds to emerge at the bottom, whereas an otherwise identical piece of unmagnetized iron makes the trip in a fraction of a second. Explain why the magnet falls more slowly.

Solution

As the magnet falls, the flux of the magnetic field through the walls of the pipe is changing. This induces an electric field in the pipe, creating a current. This current creates its own magnetic field in such a direction that it induces the change in magnetic flux in the pipe. In other words, the induced magnetic field opposes the old magnetic field. The falling magnet is repelled by this induced magnetic field, and causes an opposing force up! This slows the falling magnet. An unmagnetized piece of iron doesn’t change the magnetic flux in the pipe, and so there is no induced magnetic field and thus no opposing force.