A quantitative approach to soliton instability

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We present an approach for instabilities of solitons that is based on the spectrum of a fourth-order linearized operator. Unlike the standard approach which is based on the slope (Vakhito—Kolokolov) condition, this approach provides the quantitative value of the instability rate and the qualitative nature of the instability dynamics. © 2011 Optical Society of America

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Solitons are localized nonlinear waves that maintain their shape as they propagate. They arise in many physical fields, including nonlinear optics and Bose-Einstein condensates, where their dynamics is modeled by the nonlinear Schrödinger equation (NLS). The key question with regard to a soliton is whether it is stable. The standard approach for answering this question goes back to Vakhito and Kolokolov (VK), who showed that a necessary condition for stability is that the curve of the soliton power as a function of the propagation constant should have a nonnegative slope. Although the VK/slope condition has been used in hundreds if not thousands of studies, it has two major limitations. The first limitation is qualitative: The slope condition determines whether the soliton is stable, yet it can be experimentally inferred from the corresponding eigenfunctions: An even eigenfunction corresponds to a drift instability, whereas an odd eigenfunction corresponds to a drift instability. In both cases, the instability rate is given by the square root of the negative eigenvalue.

We briefly review the linear instability analysis that leads to the slope condition. See [4] for more details. We begin with the \((1+\text{d})\)-dimensional homogeneous NLS

\[ i\partial_t \psi(x, t) + \Delta \psi + F(|\psi|^2)\psi = 0, \quad \Delta \psi = \sum_{k=1}^d \partial^2_{x_k}, \tag{1} \]

where \(t > 0, x \in \mathbb{R}^d\), and \(F\) represents a power-law, a saturable, or a photorefractive nonlinearity. The NLS (1) admits the standing-wave solution \(\psi(x, t; \mu) = R(x; \mu) e^{i\mu t}\), where \(\mu\) is the frequency and \(R(x; \mu)\) satisfies

\[-\mu R + \Delta R + F(R^2) R = 0. \tag{2}\]

Let \(\psi(x, t) = [R(x; \mu) + e^{i\mu t}] e^{i\mu t}\) be a perturbed soliton solution of (1), where \(R\) is the positive (ground-state) solution of (3). The \(O(\epsilon)\) linearized equation for \(h\) is

\[ h_t = -i \{-\Delta + \mu - F(R^2)[h - R^2 F'(R^2) h^*]\}, \tag{3} \]

where \(h^*\) is the complex conjugate of \(h\) and \(F'\) denotes the derivative of \(F\) with respect to \(R^2\). Let

\[ h(x, t) = [u(x) + i v(x)] e^{i\Omega t}, \tag{4} \]

where \(u(x)\) and \(v(x)\) are real. If the ground state is linearly stable, there are no solutions of (3) with \(\Omega > 0\). In order to investigate whether there are such unstable modes, we substitute (3) into (1) and assume that \(\Omega \in \mathbb{R}\). This leads to

\[ L_- u = \Omega v, \quad L_- \varphi = -\Delta + \mu - F(R^2), \tag{5a} \]

\[ L_+ v = -\Omega u, \quad L_+ \varphi = L_- - 2R^2 F'(R^2). \tag{5b} \]

The eigenvalues of the system (3) were studied analytically in [4]. Alternatively, one can apply \(L_-\) to (3), which gives the fourth-order system
Let $R^\perp$ denote the subspace orthogonal to $R$. Then $L_-$ and $L_-^\perp$ are bounded and positive definite on $R^\perp$. Applying $L_-^\perp$ to (3) gives

$$L_+v = \lambda L_-^\perp v, \quad \forall v \in R^\perp. \quad (7)$$

Let $\lambda_{\text{min}}$ be the smallest eigenvalue of (4), with a corresponding eigenfunction $v_{\text{min}}$. Taking the inner product of (4) with $v$ leads to the variational characterization of $\lambda_{\text{min}}$

$$\lambda_{\text{min}} = \inf_{v \in R^\perp, ||v||_2=1} (v, L_+v) = \frac{(v_{\text{min}}, L_+v_{\text{min}})}{(v_{\text{min}}, L_-^\perp v_{\text{min}})}. \quad (8)$$

Hence, the necessary condition for stability becomes $\lambda_{\text{min}} \geq 0$. Since $L_-^\perp > 0$, $(v, L_-^\perp v) = ||L_-^\perp/2||_2^2 > 0$. Therefore,

$$\text{sgn}\lambda_{\text{min}} = \text{sgn}\alpha_-^\perp, \quad \alpha_-^\perp = \inf_{v \in R^\perp, ||v||_2=1} (v, L_+v). \quad (9)$$

Thus, the necessary condition for stability $\lambda_{\text{min}} \geq 0$ implies that $\alpha_-^\perp \geq 0$. By Lemma E.1, $(L_-^\perp R, R) \leq 0 \Rightarrow \alpha_-^\perp \geq 0$. Differentiating (4) with respect to $\mu$ gives $L_+ Q = -R$, where $Q = \partial_\mu R$. Therefore, $-(L_-^\perp R, R) = (Q, R) = \frac{1}{2} \partial_\mu R = \frac{1}{2} P'(\mu)$. Hence, the necessary condition for stability is satisfied when $P'(\mu) \geq 0$, which is known as the VK/slope condition. In (3), it was rigorously proved that the positive solitons of Eq. (4) are orbitally stable if $P'(\mu) > 0$ and unstable if $P'(\mu) < 0$. When $P'(\mu) < 0$, it follows from (3) that the instability rate is given by the maximal positive eigenvalue of (5), which can be expressed in terms of the most negative eigenvalue of $L_-^\perp$ as

$$\Omega_{\text{max}} \doteq \sqrt{-\lambda_{\text{min}}}. \quad (10)$$

To show analytically that the magnitude of $P'(\mu)$ is “unrelated” to the instability rate $\Omega_{\text{max}}$, we consider the case of a power-law nonlinearity $F(|\psi|^2) = |\psi|^{2\sigma}$. In this case, the NLS is invariant under the dilation symmetry $(x, t, \psi) \mapsto (\sqrt{\mu} x, \mu^{\sigma/2} \psi)$. Therefore, the ground state can be written as $R(x; \mu) = \mu^{\frac{\sigma}{2}} R(\sqrt{\mu} x; \mu = 1)$. Hence, the slope scales with $\mu$ as

$$P'(\mu) = \mu^{(\sigma-1)} c_{\sigma,d} ||R(\cdot, \mu = 1)||_2^2, \quad c_{\sigma,d} = \frac{2-\sigma d}{2\sigma}. \quad \text{imply that when } c_{\sigma,d} < 1 \text{ the slope's magnitude } |P'(\mu)| \text{ decreases with } \mu. \text{ On the other hand, Eq. (5) and the dilation invariance } \Omega t \mapsto \Omega t \mu \text{ imply that } \Omega_{\text{max}} \text{ increases linearly with } \mu \text{ for any } (\sigma, d). \text{ In particular, since the instability rate can be large when the slope is small and vice versa, the magnitude of the slope is unrelated to the instability rate.}$$

To better understand this surprising observation, we note that the derivation of the slope condition is based on the relations $\text{sgn} P'(\mu) \geq 0 \Rightarrow \text{sgn} \alpha_-^\perp \geq 0 \Rightarrow \text{sgn} \lambda_{\text{min}} \geq 0$. The magnitudes of $\lambda_{\text{min}}$ and $\alpha_-^\perp$ are “related,” as they are the minima of similar variational problems, Eqs. (3) and (4), respectively. In contrast, the magnitudes of $P'(\mu)$ and $\alpha_-^\perp$ are unrelated (see proof of Lemma E.1 in (4)).

To confirm numerically that the instability rate, which is associated with the violation of the slope condition is indeed given by Eq. (10), let us consider the NLS (1) with the initial condition

$$\psi_0(x) = R(x; \mu) + \varepsilon u(x), \quad |\varepsilon| \ll 1. \quad (11)$$

where $u$ is the eigenfunction of (1) that corresponds to $\Omega_{\text{max}}$. Because (1) is invariant under the transformation $(u, v, \Omega) \mapsto (u, -v, -\Omega)$, the linearized solution of (1) with the initial conditions (11) is given by

$$\psi(x, t) \doteq \left[R + \frac{\varepsilon}{2} (u + i v) e^{\Omega_{\text{max}} t} + \frac{\varepsilon^2}{2} (u - i v) e^{-\Omega_{\text{max}} t}\right] e^{i \mu t}. \quad \text{Hence, the on-axis amplitude of } \psi \text{ satisfies}

$$|\psi(0, t)| \approx |R(0) + \varepsilon |u(0)| \cosh(\Omega_{\text{max}} t)

+ \varepsilon |u(0)| \sinh(\Omega_{\text{max}} t)|. \quad (12)$$

We solve directly the supercritical NLS (4) with $d = 1, F(|\psi|^2) = |\psi|^6$, and the perturbed ground-state initial condition (11) with $\mu = 1$. Recall that $R(x; \mu)$ is given explicitly by $R = (4\mu)^{\frac{1}{2}} \text{sech}^2(3, \sqrt{\mu} x)$. Therefore, $P'(\mu) < 0$, showing that the soliton is unstable. As noted, however, the slope condition approach does not provide the instability rate. In contrast, computing the spectrum of $L_-^\perp$ yields the single negative eigenvalue $\lambda_{\text{min}} = -8.44$, which gives an instability rate of $\Omega_{\text{max}} = 2.9$; see (4). Figure 1 shows that the on-axis dynamics of the perturbed bound state agrees with the prediction (12) during the initial stage of the propagation, both for a focusing ($\varepsilon > 0$) and a defocusing ($\varepsilon < 0$) perturbation.

The above approach can be extended to the variable-coefficients NLS

$$i \psi_t(x, t) + \Delta \psi + F(|\psi|^2) \psi - V(x) \psi = 0, \quad x \in \mathbb{R}^d. \quad (13)$$

The linear stability analysis is the same as in the constant-coefficient case, except that now

$$L_+ \doteq -\Delta + \mu - V - F(R_V^2), \quad L_- \doteq L_+ - 2R_V^2 F'(R_V^2),$$

where $R_V$ is the bound state in the presence of the potential $V$. In the homogeneous case, the operators $L_+$ and $L_-$ have $d$ zero eigenvalues with corresponding eigenfunctions $\nabla R$. The inhomogeneous potential breaks up the translation invariance, leading to a bifurcation of

![Fig. 1. (Color online) On-axis amplitude of the perturbed unstable ground state of the supercritical NLS (blue, solid line) agrees with the prediction (12) with $\Omega_{\text{max}} = 2.9$ (red, dashed line) both for (a) a focusing perturbation $\varepsilon = 10^{-2}$ and (b) a defocusing perturbation $\varepsilon = -10^{-2}.$](Image 312x102 to 546x187)
litons of the quintic NLS with a periodic-lattice potential, any lattice configuration. To illustrate that, consider so-

compute the negative eigenvalues of stability:

turbed from 

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negative eigenvalues that induce an amplitude instability

ity. Fortunately, one can easily distinguish between

ations of both an amplitude instability and a drift instabil-

in any dimension, any nonlinearity

Under the standard approach, one plots the

that are centered at a lattice maximum, (b) the negative eigen-

values of \( L_{+} \), and (C) the instability rates computed from (b)

using \( \langle \rangle \) [same line types as in (b)]. The shapes in (c) are de-

linated for Fig. 4. Dotted curve is 100|\( P(\mu) \)|.

these \( d \) eigenvalues away from zero. The eigenvalues of

\( L_{+} \) that become negative are associated with a drift in-

stability, which typically occurs when the soliton is cen-

tered at a potential maximum or at a saddle point \( \langle \rangle \).

Therefore, negative eigenvalues of \( L_{+} \) can be indications of both an amplitude instability and a drift instability. Fortunately, one can easily distinguish between negative eigenvalues that induce an amplitude instability and those that induce a drift instability, as the former correspond to symmetric eigenmodes, and the latter corre-

spond to asymmetric eigenfunctions (as they are pertur-

bured from \( V R \)).

In summary, we propose the following approach for stability:

1. Compute the bound state (cf. \( \langle \rangle \)), and use it to compute the negative eigenvalues of \( L_{+} \).

2. A negative eigenvalue with a symmetric eigen-

mode indicates an amplitude instability.

3. A negative eigenvalue with an asymmetric eigen-

mode indicates a drift instability.

4. In both cases, the instability rate is given by

\[
\Omega = \sqrt{-\lambda}.
\]

This scheme thus completes the quantitative theory pre-

sented in \( \langle \rangle \). Our approach applies to positive solitons

in any dimension, any nonlinearity \( F(|\psi|^2) \), as well as for

any lattice configuration. To illustrate that, consider sol-

itons of the quintic NLS with a periodic-lattice potential,

\[
i\psi_t(t, x) + \psi_{xx} - 2 \cos(2\pi x)\psi + |\psi|^2\psi = 0,
\]

that are centered at \( x = 0 \) (i.e., at a potential maximum).

Under the standard approach, one plots the \( P(\mu) \) curve.

Since \( P(\mu) \) has a maximum at \( \mu_c \approx 3 \), see Fig. 2(a), the

solitons are unstable for \( \mu > \mu_c \). Alternatively, under

the new approach, one computes the negative eigenvalues

of \( L_{+} \), as a function of \( \mu \), see Fig. 2(b). Because

one of the negative eigenvalues exists for \( 0 < \mu \), these sol-

itons are unstable for all \( \mu > 0 \), and not just for \( \mu > \mu_c \).

This negative eigenvalue corresponds to an asymmetric eigen-

mode [see Figs. 2(a) and 2(b)] and is thus associated with a drift instability in the \( x \) direction, away from the potential maxima. The second negative eigenvalue

exists for \( \mu > \mu_c \). This eigenvalue corresponds to a sym-

metric eigenmode [see Fig. 2(b)] and is thus associated with an amplitude instability. Plotting the instability rates given by \( \langle \rangle \) in Fig. 2(c) shows that when \( \mu < 8.5 \) the drift instability dominates the initial dynamics, while for \( \mu > 8.5 \) the amplitude instability dominates. This plot also show that, as already noted, \( |\psi'(\mu)| \) is unrelated to the instability rate.

In conclusion, the standard VK/slope condition provides a yes/no answer to the existence of an amplitude instability. In contrast, the spectrum of \( L_{+} \) (1) detects both amplitude and drift instabilities, (2) provides the rates of these instabilities, and (3) determines which of them is dominant. Therefore, we propose that future studies will compute the negative spectrum of \( L_{+} \), rather than (or in addition to) the power curve \( P(\mu) \). Note that the eigenvalues of the system \( \langle \rangle \) were computed in several studies \( \langle \rangle \). However, to the best of our knowledge, these eigenvalues were never used to study the three aforementioned attributes but only to provide a yes/no answer to the question of stability.

References


