Quantum-Noise Limit on the Linewidth of Frequency Combs

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Abstract: The linewidth of a mode-locked laser's frequency comb induced by spontaneous emission, or quantum noise, is shown to obey different scaling laws in the linear-dispersionless, i.e., Schawlow-Townes, nonlinear-dispersive, or pure soliton, and intermediate operating regimes.

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Advanced ultrafast mode-locked lasers can generate broad optical frequency combs with unprecedented stability. This breakthrough has revolutionized the field of frequency metrology, as it allows for bi-directional conversion between high (optical) and low (radio, microwave) frequencies (cf. [1, 2]). However, noise sources in the lasing medium induce a jitter in the pulses, which, in turn, broadens the comb lines.

We derive the fundamental scaling laws of the linewidth induced by Spontaneous Emission (**SE**) or quantum noise for a general jitter scaling law. These scaling laws show that the quantum-noise floor behaves differently in the different operating regimes of mode-locked lasers: the linear-dispersionless (which corresponds to the Schawlow-Townes formula for cw lasers), nonlinear dispersive (or pure-soliton), and novel intermediate regimes.

The spectrum of N pulses emitted from a mode-locked laser is given by

$$\mathcal{F}\left\{\sum_{n=1}^{N} E(t-T_n)e^{-i\omega_c t+i\phi_n}+c.c.\right\}=\hat{E}(\tilde{\omega})\hat{S}(\tilde{\omega},N)+c.c.\,,\quad \hat{S}(\tilde{\omega},N):=\sum_{n=1}^{N} e^{i\tilde{\omega}T_n+i\phi_n}\,, \tag{1}$$

where $E(t-T_n)e^{i\phi_n}$ is the complex slowly-varying electric-field envelope of the n'th pulse that arrives at the detector at time T_n with overall phase ϕ_n, $\hat{u}\equiv\mathcal{F}[u]$ denotes the Fourier transform, $\hat{E}(\tilde{\omega})$ is the single-pulse spectrum; where $\tilde{\omega}\equiv\omega-\omega_c$, ω_c is carrier frequency, $c.c.$ stands for complex conjugate, and $\hat{S}(\tilde{\omega},N)$ is the "comb function".

In the absence of noise (or for sufficiently short measurement time) the comb function can be summed explicitly:

$$|\hat{S}(\tilde{\omega},N)|=\frac{\sin\left[\frac{1}{2}N(\tilde{\omega}T_{\rm rep}+\Delta\phi)\right]}{\sin\left[\frac{1}{2}(\tilde{\omega}T_{\rm rep}+\Delta\phi)\right]}\quad\text{(noiseless/short-time)} \tag{2}$$

for repetition time $T_{\rm rep}=T_{n+1}-T_n$ and pulse-to-pulse (overall) phase change $\Delta\phi=\phi_{n+1}-\phi_n$. The comb function is periodic in the frequency, its k'th comb line (enumerated around ω_c) given by $\tilde{\omega}_k=k\omega_{\rm rep}+\tilde{\omega}_o$, where $\omega_{\rm rep}=\frac{2\pi}{T_{\rm rep}}$ and $\tilde{\omega}_o=-\frac{\Delta\phi}{2\pi}\,\omega_{\rm rep}$ are the repetition and offset frequencies, respectively. It follows from (2) that the linewidth (FWHM) of each comb line is independent of frequency and scales with the measurement time, $T_N\equiv NT_{\rm rep}$, as

$$\omega_{\frac{1}{2}}(T_N)\sim\frac{2\pi}{NT_{\rm rep}}=\frac{2\pi}{T_N}\quad\text{(noiseless/short-time)}\,. \tag{3}$$

In the limit of an infinite number of pulses or measurement time, $T_N\to\infty$, the comb lines have zero width and the comb function approaches a sum of evenly-spaced Dirac delta functions, $|\hat{S}|\to\sum_{k=-\infty}^{\infty}\delta(\tilde{\omega}-\tilde{\omega}_k)$.

SE noise broadens the comb lines randomly, which raises the question what is the SE-limited linewidth and after what measurement time is this limit reached? In [3] we used asymptotic analysis to show that the effect of additive white noise on the jitter of the pulse's center time and overall phase is to set a linewidth floor for sufficiently long measurement time. More precisely, we define the nondimensional quantity $\varepsilon(\tilde{\omega})\equiv(c_x\tau^2\tilde{\omega}^2+2c_{xy}\tau\tilde{\omega}+c_y)\sigma^2$ with $\tilde{\omega}\equiv\omega-\omega_c$ for pulse duration τ, normalized noise variance σ^2 (see below), and normalized variance and cross correlation (c_{xy}) coefficients of the time-center (c_x) and overall phase (c_y) jitter. The SE induced linewidth floor is found to scale like

$$\omega_{\frac{1}{2}}^{\rm SE}(\tilde{\omega})\sim\varepsilon^{2/p}(\tilde{\omega})\omega_{\rm rep}\quad\text{for measurement time}\quad T_N>T_{\rm SE}(\tilde{\omega})=\frac{2\pi}{\varepsilon^{2/p}(\tilde{\omega})\omega_{\rm rep}}\quad\text{(SE-limited)}\,, \tag{4}$$

where $p = 1$ corresponds to the linear-dispersionless regime for cw lasers (as first derived by Schawlow and Townes [4]) as well as certain mode-locked lasers [5, 6], whereas $p = 3$ corresponds to the pure soliton regime. Due to the wide range of operating regimes of mode-locked lasers other values of the jitter exponent (e.g., $p = 2$) should be considered as well. Furthermore, this broadening scales with lasing-threshold and output powers as $\omega_{\frac{1}{2}} \sim \sigma^2 \sim \left(\frac{P_{\rm th}}{P_{\rm out}}\right)^{1/p}$, which shows that the broadening is more sensitive to power changes in the soliton regime $(p = 3)$ than in the linear-dispersionless $(p = 1)$ one.

It is instructive estimate the SE-noise linewidth floors. For a Ti:sapphire laser we estimate a gain coefficient $f = 1.25$, loss coefficient $\alpha = 5 \cdot 10^{-8}{\rm mm}^{-1}$, and nonlinear coefficient $\gamma = 0.01({\rm MWmm})^{-1}$, which yields (cf. [7]) $\sigma^2 = \hbar\omega_c\alpha f\gamma\tau^3/|\beta_2|^2 \approx 10^{-20}$ for pulse duration τ =10fs at center wavelength λ =800nm (ω_c =2360THz) and average GVD $|\beta_2| = 100{\rm fs}^2{\rm mm}^{-1}$. Using a 100MHz repetition rate ($\omega_{\rm rep} = 2\pi \cdot 100$MHz), Eq. (4 [see Fig. 1(a)] shows that the SE-noise limited linewidth floor (as a function of spectral frequency) significantly increases with the jitter exponent p. In fact, in the soliton regime ($p = 3$) the linewidth is on the order of 100Hz; in the case of $p = 2$ a linewidth of 0.1Hz is calculated; whereas in the linear-dispersionless regime ($p = 1$) the linewidth is so small [10^{-11}Hz, see Fig. 1(b)] that the data results are not shown.

Focusing on a particular comb line as a function of measurement time, Fig. 1(b) shows that the linewidth is initially limited by the measurement time as in Eq. (3) but eventually, i.e., for measurement time greater than $T_{\rm SE}$ in Eq. (4), the noise sets a plato floor. This plato is reached much earlier for solitons (approximately after 0.01s), whereas, in the linear-dispersionless regime, it would take an impractically long time (thousands of years).

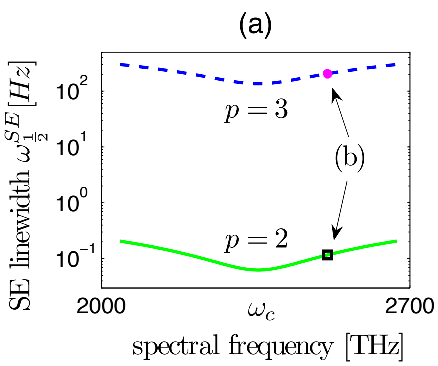

Fig. 1. (a) The linewidth floor induced by spontaneous emission (SE) noise increases near the wings of the spectrum as $(\omega - \omega_c)^{1/p}$ [i.e., Eq. (4); see also [3]], where in the soliton regime [$(p = 3)$, dashes] the linewidth is much greater than in the $p = 2$ (dots) case, which, in turn, is much greater than the linear-dispersionless case [$(p = 1)$, not shown]. (b) The linewidth is initially limited by the measurement time [i.e., the decreasing part of the lines correspond to Eq. (3)] until the SE-noise floor is reached [i.e., the platos, see Eq. (4) at ω =2500THz], which is reached after approximately 0.01s in the soliton $(p = 3)$ regime, but is practically unreachable in the linear-dispersionless $(p = 1)$ case.

In summary, quantum noise sets the fundamental limit on the linewidth of the frequency combs of mode-locked lasers and the precision of optical clocks. The scaling laws of this linewidth show that the noise floor is much larger, and sets in much earlier, when a mode-locked laser operates in the soliton regime than when it operates in the linear-dispersionless regime.

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