# Noise-induced linewidth in frequency combs

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Frequency combs generated by trains of pulses emitted from mode-locked lasers are analyzed when the center time and phase of the pulses undergo noise-induced random walk, which broadens the comb lines. Asymptotic analysis and computation reveal that, when the standard deviation of the center-time jitter of the $n$th pulse scales as $n^{p/2}$, where $p$ is a jitter exponent, the linewidth of the $k$th comb line scales as $k^{2/p}$. The linear-dispersionless ($p=1$) and pure-soliton ($p=3$) dynamics in lasers are derived as special cases of this time-frequency duality relation. In addition, the linewidth induced by phase jitter decreases with power $P_{\rm out}$, as $(P_{\rm out})^{-1/p}$. © 2006 Optical Society of America

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Frequency combs are a ubiquitous tool in time and frequency metrology. They are generated by trains of evenly spaced pulses, whose spectrum consists of a comb of oscillators with evenly spaced frequencies, or comb lines. Advanced ultrafast mode-locked lasers can generate broad optical frequency combs with unprecedented stability. This breakthrough has revolutionized the field of frequency metrology, since it allows for bidirectional conversion between high (optical) and low (radio, microwave) frequencies.[1,2] However, noise sources in the lasing medium broaden the comb lines.[3,4] Schawlow and Townes (S–T) derived a formula for the linewidth of cw lasers.[5] With the advent of pulsed mode-locked lasers, quantum-noise theory developed by Haus and collaborators has been useful for studying the stability of the ensuing modes.[6] Many theoretical studies have since been devoted to the understanding of the pulse dynamics (see Ref. 4 and references therein). It is known that many physical mechanisms can broaden the comb lines; see, for example, Takushima *et al.*[3] and Kärtner *et al.*,[4] who recently studied the linewidth experimentally and theoretically in passively and actively mode-locked lasers. In this Letter, the pulse's jitter is treated as a generalized random walk, from which the induced linewidth is derived. Asymptotic analysis reveals that, when the standard deviation of the $n$th pulse's time-jitter scales $n^{p/2}$, where $p$ is a jitter exponent, the linewidth of the $k$th comb line scales like $k^{2/p}$. The S–T (linear-dispersionless) and pure-soliton (nonlinear-dispersive) dynamics are derived as two special cases of this scaling law. These results have potential implications for current research in ultrafast spectroscopy.[1,2,7]

A mode-locked laser emits an ultrashort pulse each time the intracavity pulse arrives at the output coupler. The spectrum of $N$ successively emitted pulses is

$$\mathcal{F}\left\{\sum_{n=1}^{N} E(t-T_n)e^{i\phi_n}e^{-i\omega_c t}+\text{c.c.}\right\}=\hat{E}(\tilde{\omega})\hat{S}(\tilde{\omega})+\text{c.c.}, \quad (1)$$

where $E(t-T_n)e^{i\phi_n}$ is the complex, slowly varying electric-field envelope of the $n$th pulse, where $T_n$ and $\phi_n$ are the arrival time and overall phase of the $n$th pulse, respectively, and $\omega_c$ is the carrier frequency; we define $\tilde{\omega}\equiv\omega-\omega_c$, c.c. stands for the complex conjugate, $\hat{u}=\mathcal{F}[u]=\int u(t)e^{i\omega t}\mathrm{d}t$ denotes the Fourier transform, $\hat{E}(\tilde{\omega})$ is the single-pulse spectrum, and the comb function is

$$\hat{S}(\tilde{\omega})=\sum_{n=1}^{N} e^{i\tilde{\omega}T_n+i\phi_n}. \quad (2)$$

In the absence of noise, the time between successive pulses is $T_{\rm rep}=T_{n+1}-T_n$; the repetition frequency is $\omega_{\rm rep}=2\pi/T_{\rm rep}$; and the pulse-to-pulse (overall) phase change is $\Delta\phi=\phi_{n+1}-\phi_n$, which is related to the carrier-envelope phase change, $\Delta\phi_{\rm CE}$, as $\Delta\phi=\Delta\phi_{\rm CE}+\omega_c T_{\rm rep}$. Therefore apart from absolute time and phase offsets, $T_n=nT_{\rm rep}$, $\phi_n=n\Delta\phi$, and the measurement time is $NT_{\rm rep}$. In the limit of an infinite number of pulses, the comb function approaches the ideal frequency comb (see Fig. 1), i.e., $|\hat{S}(\tilde{\omega})|\to\sum_{k=-\infty}^{\infty}\delta(\tilde{\omega}-\tilde{\omega}_k)$ as $N\to\infty$, where $\delta(\omega)$ is the Dirac delta function, and the $k$th comb line's frequency is given by

$$\tilde{\omega}_k=k\omega_{\rm rep}+\tilde{\omega}_{\rm o}, \quad \tilde{\omega}_{\rm o}=-\frac{\Delta\phi}{2\pi}\omega_{\rm rep}, \quad (3)$$

where $\tilde{\omega}_{\rm o}$ is the offset frequency and, for convenience, the comb lines are enumerated around $\omega_c$.

When noise is considered, let $T_n=nT_{\rm rep}+x$ and $\phi_n=n\Delta\phi+y$ be the center time and phase of the $n$th

 

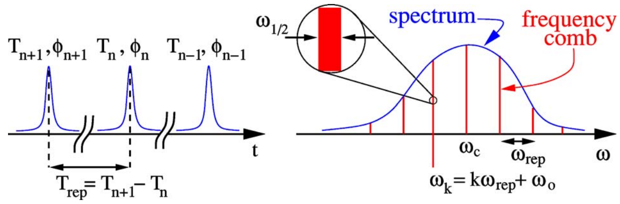


Fig. 1. (Color online) Schematic of a pulse train (left) and its spectrum (right). In the absence of noise, the pulse's spectrum determines the bandwidth, while the repetition time, $T_{\rm rep}=T_{n+1}-T_n$, and overall phase change, $\Delta\phi=\phi_{n+1}-\phi_n$, determine the comb function [Eqs. (1)–(3)]. The frequency of the $k$th comb line (enumerated around $\omega_c$) is $\tilde{\omega}_k=k\omega_{\rm rep}+\tilde{\omega}_{\rm o}$, where $\omega_{\rm rep}$ and $\tilde{\omega}_{\rm o}$ are the repetition and offset frequencies. Noise induces a random jitter in the center time and phase, $T_n$ and $\phi_n$, which broadens the comb lines. The linewidth ($\omega_{1/2}$ in the inset) is the FWHM of the comb function [Eq. (2)] around a comb frequency [Eq. (7)].

pulse, respectively, where $T_{\rm rep}$ and $\Delta\phi$ are the average repetition time and phase change, and $x=\Delta T_n$ and $y=\Delta\phi_n$ correspond to zero-mean random jitter. The probability density of the $n$th center-time and phase jitter is assumed to be joint Gaussian[8]:

$$g_n(x,y)=Ae^{-[1/2(1-r^2)][(x/\sigma_x)^2-(2r^2xy/C_{x,y})+(y/\sigma_y)^2]}, \quad (4)$$

where $\sigma_x$ and $\sigma_y$ are the standard deviations of the center-time and phase jitter, respectively; the covariance is $C_{x,y}=\langle\Delta T_n\Delta\phi_n\rangle=r\sigma_x\sigma_y$, where $r$ is the correlation coefficient; and $A=(2\pi\sigma_x\sigma_y\sqrt{1-r^2})^{-1}$. We consider pulses that undergo generalized random walks; i.e., their center-time and phase jitter satisfy

$$\sigma_x^2=c_x(\sigma\tau)^2n^p, \quad C_{x,y}=c_{xy}\sigma\tau n^p, \quad \sigma_y^2=c_y\sigma^2n^p, \quad (5)$$

where $c_x, c_y, c_{xy}$ are $O(1)$ constants, $\sigma$ is nondimensional noise strength, $\tau$ is the pulse temporal FWHM, and $p$ is the jitter exponent. Our analysis holds for all $p\geqslant 1$; however, we focus on two physically special cases:

Linear-dispersionless dynamics ($p=1$). This corresponds to cw laser beams, whose dynamics are governed by a linear-dispersionless wave equation. Spontaneous emission leads to a random jitter of the phase of the carrier wave, which, in turn, broadens the spectrum.[5] It can be shown that the phase jitter is a first integral of the noise along the intracavity propagation distance, $z$, which leads to a (standard) random walk of the phase. Therefore considering a wave train that is emitted from the laser cavity at every $z_n=nL$, for cavity length $L$, the standard deviation of the phase of the $n$th wave packet scales as $\sigma_y\sim\sqrt{z_n}\sim\sqrt{n}$; i.e., Eq. (5) with $p=1$. We note that this scaling law has also been found to describe center-time jitter of certain mode-locked lasers.[3,4]

Pure-soliton dynamics ($p=3$). Pulse propagation in mode-locked lasers is described by the master equation; i.e., a nonlinear Schrödinger equation for the electric field in the laser cavity that takes gain and loss mechanisms into account.[6] In many types of mode-locked lasers, the dispersion and nonlinear coefficients vary along $z$, generating dispersion and nonlinear managed solitons.[9,10] SE occurs in the gain medium, e.g., a Ti:sapphire crystal. These models resemble soliton propagation in amplified telecommunication fibers, which have been studied extensively. Based on such models, Gordon and Haus[11] and Gordon and Mollenauer[12] showed that the center-time and phase jitter of solitons scale as $z^{3/2}$; i.e., Eq. (5) with $p=3$. This is because the frequency jitter is a first integral (along $z$) of the noise, and, therefore, a standard random walk, whereas nonlinearity and dispersion cause the center-time and phase jitter to be integrals of the frequency jitter.

Taking the average of the comb function Eq. (2) as $\bar{S}(\tilde{\omega})=\int\int\hat{S}(\tilde{\omega})g_n(x,y)\mathrm{d}x\mathrm{d}y$, and using Eqs. (4) and (5), yields the averaged comb function as

$$\bar{S}(\tilde{\omega})=\sum_{n=1}^{N}\underbrace{e^{i(\tilde{\omega}T_{\rm rep}+\Delta\phi)n}}_{\text{deterministic}}\underbrace{e^{-(1/2)\epsilon(\tilde{\omega})n^p}}_{\text{averaged noise}},$$

$$\epsilon(\tilde{\omega})=(c_x\tau^2\tilde{\omega}^2+2c_{xy}\tau\tilde{\omega}+c_y)\sigma^2, \quad (6)$$

where $\epsilon(\tilde{\omega})$ is dimensionless.

Our goal is to find the linewidth of the $k$th comb line, defined as the FWHM of $|\bar{S}(\tilde{\omega})|^2$ around $\tilde{\omega}_k$. It is convenient to normalize this linewidth with respect to $\omega_{\rm rep}$, in which case the normalized $\delta\omega_{1/2}(k)$ satisfies

$$\left|\bar{S}\left(\tilde{\omega}_k+\frac{1}{2}\omega_{\rm rep}\delta\omega_{1/2}\right)\right|^2=\frac{1}{2}|\bar{S}(\tilde{\omega}_k)|^2. \quad (7)$$

We first analyze the broadening induced solely by center-time jitter; i.e., when $c_{xy}=c_y=0$, $c_x=1$, and $\epsilon(\tilde{\omega})=(\sigma\tau\tilde{\omega})^2$. Recalling that $\tilde{\omega}=\omega-\omega_c$, the averaged comb function [Eq. (6)] admits the following frequency regimes. Near the carrier frequency, i.e., for $|\omega-\omega_c|\ll(\sigma\tau N^{p/2})^{-1}$ [or $N^p\epsilon(\tilde{\omega})\ll 1$], the broadening is due only to the finite number of pulses. This gives the time-measurement-limited (TML) linewidth that scales like the inverse of the measurement time; i.e., $\delta\omega_{1/2}\sim(NT_{\rm rep})^{-1}$ (see Fig. 2A). On the other hand, for $|\omega-\omega_c|\gg(\sigma\tau)^{-1}$, the noise-induced exponent in Eq. (6) is so large that the noise smears out the comb lines. In between there is an asymptotic frequency regime, which is characterized by weak noise, i.e., $\epsilon(\tilde{\omega})=(\sigma\tau\tilde{\omega})^2\ll 1$, and many pulses (or long measurement time); i.e., $N^p\epsilon(\tilde{\omega})\gg 1$. These asymptotic conditions bound the frequency as

$$\frac{1}{\sigma\tau N^{p/2}}\ll|\omega-\omega_c|\ll\frac{1}{\sigma\tau}. \quad (8)$$

Within this range, careful asymptotic analysis of Eq. (6) reveals that the linewidth scales as $\delta\omega_{1/2}\sim\epsilon^{1/p}(\tilde{\omega})$. Since $\tilde{\omega}_k\sim k\omega_{\rm rep}$, it follows that the fundamental relation between center-time jitter and linewidth is given by

$$\sigma_x\sim\sigma\tau n^{p/2}\Rightarrow\delta\omega_{1/2}\sim(\sigma\tau\omega_{\rm rep})^{2/p}k^{2/p}, \quad (9)$$

where $n$ is pulse number and $k$ is comb-line number. Equation (9) shows that the standard deviation of center-time jitter and the broadening of the comb lines have reciprocal exponents. Since $k$ is enumerated with respect to $\omega_c$, the broadening Eq. (9) is the largest at the outer edges of the spectrum. Of particular interest are the linear-dispersionless ($p=1$) and pure-soliton ($p=3$) dynamics, which yield $\delta\omega_{1/2}^{\text{linear}} \sim k^2$ and $\delta\omega_{1/2}^{\text{soliton}} \sim k^{2/3}$, respectively (see Fig. 2A). To generate Fig. 2, the averaged comb function [Eq. (6)] is summed in the vicinity of a given comb line [Eq. (3)], and the FWHM [Eq. (7)] of the resulting function is calculated to obtain the linewidth of that comb line. This is repeated for many comb lines to generate the graphs. It follows that the linear-dispersionless linewidth increases parabolically around $\omega_c$, whereas the soliton linewidth increases as $(\omega-\omega_c)^{2/3}$ (see Fig. 2B). We note that Eq. (9) gives the limiting value of the linewidth for long measurement time; i.e., after $N(\omega) \gg |\sigma\tau(\omega-\omega_c)|^{-2/p}$ pulses. All subsequent pulses have a negligible effect on the linewidth. Indeed, we verified that the asymptotic linewidth in Fig. 2 is almost the same with $10^3$ and $10^4$ pulses.

Returning to the averaged comb function [Eq. (6)], for pure phase jitter, $c_x = c_{xy} = 0$, and $\epsilon = c_y\sigma^2$ is independent of $\omega$. Thus phase jitter causes all the comb lines to drift together, maintaining their relative spacing. Using similar analysis as above yields $\delta\omega_{1/2} \sim \sigma^{2/p}$, when $N^{-p/2} \ll \sigma \ll 1$. Since $\sigma^2$ scales as the ratio of lasing-threshold power, $P_{\text{th}}$, to mode power, $P_{\text{out}}$, the fundamental relation between phase jitter and linewidth is given by

$$\sigma_y^2 \sim \frac{P_{\text{th}}}{P_{\text{out}}} n^p \Rightarrow \delta\omega_{1/2} \sim \left(\frac{P_{\text{th}}}{P_{\text{out}}}\right)^{1/p}. \tag{10}$$

For $p=1$ the well-known S–T scaling law is recovered, i.e., $\delta\omega_{1/2}^{\text{S–T}} \sim (P_{\text{out}})^{-1}$, whereas for solitons we obtain that $\delta\omega_{1/2}^{\text{soliton}} \sim (P_{\text{out}})^{-1/3}$. Since $P_{\text{out}} \gg P_{\text{th}}$, this suggests that phase-jitter broadening is more sensitive to power changes with soliton dynamics than with linear-dispersionless dynamics.

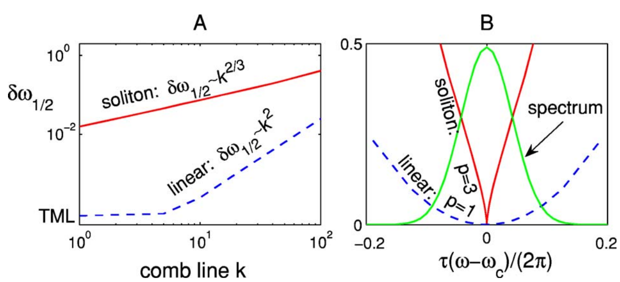


Fig. 2. (Color online) Relative linewidth [i.e., Eq. (7)] induced solely by center-time jitter [i.e., Eq. (6) with $c_{xy} = c_y = 0$ and $c_x = 1$] with $\omega_{\text{rep}} = 2\pi[\text{GHz}]$, $\tilde{\omega}_o = 0$, pulse width $\tau = 100$ fs, $N = 10^4$ pulses, and noise strength $\sigma = 1$. A, Log-log plot, where $k=0$ corresponds to $\omega_c$. Within the asymptotic regime [Eq. (8)], the linewidths computed for the linear dispersionless ($p=1$, dashes) and pure-soliton ($p=3$, solid) dynamics obey the scaling law [Eq. (9)]; i.e., they fit the power laws $\delta\omega_{1/2}^{\text{linear}} \sim k^{2.0}$ and $\delta\omega_{1/2}^{\text{soliton}} \sim k^{0.67}$, respectively. Near the center frequency for $p=1$, the TML linewidth scales as $1/N$. B, Same as A using normalized absolute frequency. The linear-dispersionless and soliton linewidths increase as $(\omega-\omega_c)^2$ and $(\omega-\omega_c)^{2/3}$, respectively. For reference, a single-pulse spectrum with a dimensional FWHM of $1/(10\tau)$ is depicted.

When both center-time and phase jitter are considered, it follows from the comb function [Eq. (6)] that the contribution from center-time jitter dominates at the outer edges of the spectrum, i.e., when $|\omega-\omega_c| \gg 1/\tau$; the phase-jitter contribution dominates near $\omega_c$, i.e., when $|\omega-\omega_c| \ll 1/\tau$; while center-time, phase, and cross-correlation jitter can have comparable effects when $|\omega-\omega_c| \approx 1/\tau$.

In summary, the broadening induced by center-time jitter of linear-dispersionless waves increases parabolically around the center frequency, whereas for solitons, it grows as $(\omega-\omega_c)^{2/3}$. In general, the center-time jitter and linewidth have reciprocal exponents. This result can be understood as a noise-induced manifestation of time-frequency duality. In addition, phase jitter induces a linewidth that scales like $P_{\text{out}}^{-1/p}$, which yields a $P_{\text{out}}^{-1/3}$ scaling law for solitons. We remark that due to the wide range of operating regimes of mode-locked lasers,[3,4,6] one should not rule out any value of the jitter exponent from consideration. In addition, technical noise could change the carrier-envelope phase (instead of the overall phase), which would call for a different choice of random variables.

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## References

1. Th. Udem, R. Holzwarth, and T. W. Häncsh, Nature **416**, 233 (2002).
2. S. T. Cundiff and J. Ye, eds. *Femtosecond Optical Frequency Comb Technology* (Springer, 2005).
3. Y. Takushima, H. Sotobayashi, M. E. Grein, E. P. Ippen, and H. A. Haus, in Proc. SPIE **5595**, 213 (2004).
4. F. X. Kärtner, U. Morgner, T. Schibli, R. Ell, H. A. Haus, J. G. Fujimoto, and E. P. Ippen, Vol. 95 of *Topics in Applied Physics XIV* (Springer-Verlag, 2004), p. 73.
5. A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).
6. H. A. Haus and A. Mecozzi, IEEE J. Quantum Electron. **29**, 983 (1993).
7. A. Marian, M. C. Stowe, D. Felinto, and J. Ye, Phys. Rev. Lett. **95**, 023001 (2005).
8. A. Papoulis and S. Unnikrishna Pillai, *Probability, Random Variables and Stochastic Processes* (McGraw-Hill, 2001).
9. Y. Chen, F. X. Kärtner, U. Morgner, S. H. Cho, H. A. Haus, E. P. Ippen, and J. G. Fujimoto, J. Opt. Soc. Am. B **16**, 1999 (1999).
10. Q. Quraishi, S. T. Cundiff, B. Ilan, and M. J. Ablowitz, Phys. Rev. Lett. **94**, 243904 (2005).
11. J. P. Gordon and H. A. Haus, Opt. Lett. **11**, 665 (1986).
12. J. P. Gordon and L. F. Mollenauer, Opt. Lett. **15**, 1351 (1990).