Carrier-envelope phase slip of ultrashort dispersion-managed solitons

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The carrier-envelope phase slip of an ultrashort pulse circulating in a mode-locked Ti:sapphire laser is analyzed. The laser cavity is modeled by a dispersion- and nonlinearity-managed nonlinear Schrödinger equation. The combined contributions to the phase slip induced by nonlinear phase and nonlinear dispersion are found to approach zero for strong dispersion maps. The dependence of the slip on third-order dispersion is found as well. The analytical results are verified using numerical simulations.

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Mode-locked Ti:sapphire lasers generate a regularly spaced train of pulses separated by one cavity round-trip time. The carrier-envelope phase slip (CEPS) is the change from pulse-to-pulse of the phase offset between the envelope and the carrier waves. Controlling the phase slip has been the subject of recent experimental efforts in optical frequency metrology, carrier-envelope phase coherence, and extreme nonlinear optics.[1–4] In a recent seminal contribution Haus and Ippen[5] studied the CEPS for classical and dispersion-managed (DM) solitons, the latter being waveguide solutions of a perturbed DM nonlinear Schrödinger equation. DM solitons are a key element to describing Ti:sapphire lasers.[6] In this Letter an asymptotic theory governing the propagation of DM and nonlinearity-managed solitons is applied to a model of the laser. Applying multiple-scales analysis yields a relation for the nonlinear change of the phase velocity over one cavity round trip. In our model we treat nonlinear dispersion (the shock term) and third-order dispersion (TOD) as small perturbations. They induce changes in the group velocity that are found using conservation-law methods. The nonlinear slip induced by the combined effects is found to approach zero as $O(1/s)$, where s is the cavity's map strength. In addition, it is found that the CEPS can be controlled by changing the average group-velocity dispersion (GVD) and TOD in the laser. Our analytical results agree with numerical simulations and display the explicit dependence of the phase slip on physical parameters.

Typically, the electromagnetic field of a pulse is decomposed into a rapidly oscillating carrier wave $\exp[i(kz - \omega t)] = \exp[i(1/v_p - t/z)\omega z]$ that is modulated by a slowly varying envelope. Here z is the propagation direction, t is time, $k(\omega) = \omega n(\omega)/c$ is the center wave number, where ω is the center frequency, $n(\omega)$ is the linear index of refraction, and c is light speed in vacuum. During propagation the carrier slips through the envelope, because the carrier propagates at phase velocity $v_p = \omega/k$ while the envelope propagates at group velocity $v_g = 1/k'(\omega)$. Thus the linear contribution to the slip of the carrier-envelope phase offset is given (mod 2π) by $\delta_{\text{linear}} = ({v_p}^{-1} - {v_g}^{-1})\omega L = -c^{-1}\omega^2 n'(\omega)L$, where L is the propagation distance. In addition, when an intense pulse propagates in a Kerr medium (such as sapphire) there is a nonlinear contribution to the phase slip. To study the nonlinear slip we recall that the propagation of the envelope is well described by the classical nonlinear Schrödinger (NLS) equation

$$iA_z - \frac{k''}{2}A_{\tau\tau} + \gamma|A|^2 A = -i\omega^{-1}\gamma(|A|^2 A)_\tau\,, \qquad (1)$$

where $A(z,\tau)$ is the slowly varying envelope, $\tau = t - z/v_g$ is the retarded-time frame, k'' is the GVD coefficient, $\gamma = n_2\omega/cA_{\text{eff}}$ is the nonlinear coefficient, where n_2 is the Kerr (nonlinear) refractive index, and A_{eff} is the effective cross-sectional area of the pulse, and the term on the right-hand side, often called the shock term, corresponds to nonlinear dispersion arising from the Kerr effect. We focus on the shock term first because it becomes larger with shorter pulses. Indeed, the shock term scales as $\epsilon|A|^3$, where $\epsilon \equiv 2\pi/\omega\tau_0$ and τ_0 is the pulse width. For example, at $\lambda = 800$ nm and $\tau_0 = 20$ fs one has $\omega/2\pi = 400$ THz and $\epsilon \approx 0.13$. In addition, the shock term induces a nonlinear change in the CEPS, a phenomenon that is consistent with the dependence of the CEPS on pulse energy (pump power).[1,4] Without the shock term and when k'' is a negative constant, Eq. (1) has the soliton solution

$$A(z,\tau) = A_0\,\text{sech}(\tau/\tau_0 - T)\exp[i\phi(z)],$$
$$\phi(z) \equiv \gamma|A_0|^2 z/2\,, \qquad (2)$$

where A_0 is amplitude and T corresponds to a time shift of the pulse center. Self-phase modulation is described by $\phi(z)$, which, in turn, induces a nonlinear change in the phase velocity as $\Delta(1/v_p) = \phi'(z)/\omega = \gamma|A_0|^2/2\omega$. The timing shift corresponds to a change in the group velocity as $\Delta(1/v_g) = T'(z)\tau_0$. The normalized nonlinear contribution to the CEPS was defined in Ref. 5 as

$$\tilde{\delta}_{\mathrm{NL}} \equiv \frac{\Delta(1/v_p) - \Delta(1/v_g)}{\Delta(1/v_p)} = 1 - \frac{T'(z)}{\epsilon\phi'(z)} \cdot \qquad (3)$$

For classical solitons the shock term gives[5] $T'(z) = \epsilon\gamma|A_0|^2$. Thus $\tilde{\delta}_{\mathrm{NL}} = -1$, which means that for classical solitons the nonlinear change in ${v_g}^{-1}$ is twice as large that of ${v_p}^{-1}$.

For a regularly spaced train of pulses, such as those emitted from a mode-locked Ti:sapphire laser, the CEPS refers to the change of the phase offset between carrier and envelope from pulse to pulse, which is the phase slip that each pulse accumulates over one cavity round trip, before being sampled at the output coupler. The linear contribution to the CEPS is thus $\delta_{\mathrm{linear}} = -c^{-1}\omega^2 \sum_m n_m'(\omega) L_m$, where the summation is carried over all the cavity elements (crystal, prisms, etc.). In addition, the nonlinear contribution to the CEPS is the difference between the nonlinear phase and the timing shifts accumulated over one cavity round trip. In normalized form it is given by the average of Eq. (3):

$$\tilde{\delta}_{\mathrm{NL}} = 1 - \frac{\langle T'(z)\rangle}{\epsilon\langle\phi'(z)\rangle}, \qquad (4)$$

where $\langle\rangle$ stands for the average over one cavity round trip. A typical Ti:sapphire laser consists of a Ti:sapphire crystal that has a Kerr response as well as large normal GVD and a set of prisms and (or) mirrors especially designed to have large anomalous GVD. Hence such lasers are well described as DM systems,[6] which have been heavily studied in telecommunications. Let us recall some of the results of these studies. For further analysis we normalize the variables as $\tilde{z} = z/\zeta^*$, $\tilde{t} = \tau/\tau_0$, $D(z) = -k''(\zeta)/k''^*$, and $u(z,t) = A(\zeta,\tau)/\sqrt{P^*}$, where P^* is the characteristic pulse peak power, $\zeta^* = 1/\gamma P^*$ is the (average) nonlinear length, and $k''^* = {\tau_0}^2/\zeta^*$. After dropping the tildes, we can describe the pulse propagation using the perturbed NLS[7]:

$$iu_z(z,t) + \frac{D(\zeta)}{2} u_{tt} + g(\zeta)|u|^2 u = -i\epsilon g(\zeta)(|u|^2 u)_t, \qquad (5)$$

where $\zeta = z/l_c$ is the fast variable, l_c is the normalized (with respect to ζ^*) optical length of the cavity, $D(\zeta) = \overline{D} + {l_c}^{-1}\Delta(\zeta)$ is the dispersion map, where $\overline{D}$ is the average dispersion and $\Delta(\zeta)/l_c$ is the large and rapidly varying dispersion with zero average path, and the right-hand side corresponds to the shock term. A lumped model of a cavity consists of a symmetric two-step dispersion map, i.e., $\Delta(\zeta) = \Delta_1 > 0$ for $\zeta \in \{[0, \theta/2), [1 - \theta/2, 1)\}$ and $\Delta(\zeta) = \Delta_2 < 0$ for $\zeta \in [\theta/2, 1 - \theta/2)$, subject to $\Delta_1\theta + \Delta_2(1-\theta) = 0$, with period extension for $\zeta > 1$. To model a Ti:sapphire laser we choose $\theta = 0.75$ and a managed nonlinearity: $g(z) = 1$ in the normal GVD section and $g(z) = 0$ in the anomalous GVD section [see Fig. 1(a)]. It is useful to define $C(\zeta) = \int_0^\zeta \Delta(\zeta')d\zeta'$ as well as to map strength $s = \Delta_1\theta/2$. Classical NLS equation (1) corresponds to $s = 0$ and $C(\zeta) \equiv 0$. Ti:sapphire systems, however, operate in the strong DM regime, which corresponds to large-variance GVD (s) and small average GVD ($\overline{D}$).

The two small parameters in Eq. (5) are ϵ (weak nonlinear dispersion) and l_c (short nonlinear length). Let us treat the shock term perturbatively by first considering the unperturbed model, i.e., when $\epsilon = 0$. Since $l_c << 1$ one can apply the method of multiple scales to Eq. (5). With this method it was previously shown[7] that, to leading order, $\hat{u}(z,\omega) \sim \hat{U}(\omega,z)\exp[-i/2\,\omega^2 C(\zeta)]$, where $\hat{h}(\omega) \equiv F\{h(t)\} = \int h(t)\exp(i\omega t)dt$. A solvability condition for $O(l_c)$ leads to the DMNLS (averaged) equation:

$$i\frac{\partial\hat{U}}{\partial z} - \frac{\overline{D}}{2}\omega^2\hat{U} + \langle J[\hat{u}]\rangle = 0, \qquad (6)$$

where $J[\hat{u}] = g(\zeta)\exp[i/2\,\omega^2 C(\zeta)]F\{|u|^2 u\}$. Equation (6) is a nonlocal (integral) equation that governs the averaged dynamics of the solutions of Eq. (5). Looking for a DM soliton of the form $\hat{U}(z,\omega) = \hat{f}(\omega)\exp(i\phi z)$, where $\phi(z) = \lambda^2 z/2$, leads to the following equation:

$$-\frac{\lambda^2}{2}\hat{f}(\omega) - \frac{\overline{D}}{2}\omega^2\hat{f} + \langle J[\hat{f}]\rangle\exp(-i\lambda^2 z/2) = 0. \qquad (7)$$

Taking the inverse Fourier transform, multiplying by $(f + tf_t)$, and integrating leads to

$$\frac{\lambda^2}{2} = \frac{2}{W}\left\langle \int F^{-1}\{J[\hat{f}]\}(f + tf_t)dt \right\rangle - \frac{3\overline{D}}{2W}\int (f_t)^2 dt, \qquad (8)$$

where $W = \int f^2 dt$ is energy. With strong dispersion management $f(t)$ can be approximated with a Gaussian,[7] which is helpful for gaining insight into the physics, by assigning specific pulse parameters. Thus, substituting $f(t) \approx a(2\pi b)^{-1/2}\exp(-t^2/2b)$ into Eq. (8) gives the result that

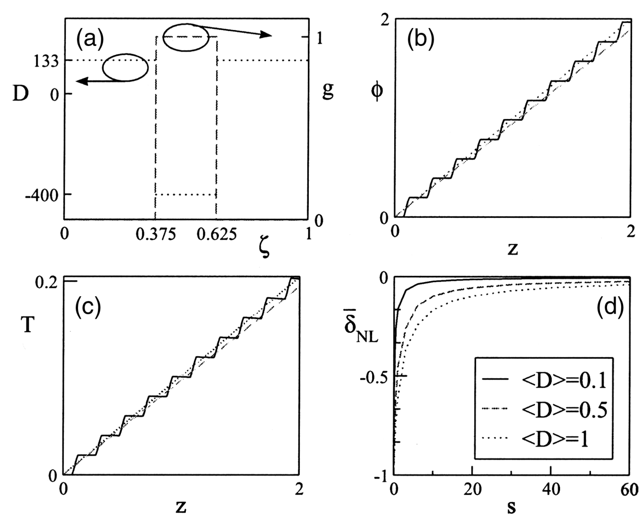

Fig. 1. (a) Dispersion [$D(\zeta)$, dotted lines] and nonlinearity maps [$g(\zeta)$, dashed lines] used in (b) and (c). (b) Numerical phase [Eq. (6), solid curve], $\lambda^2 z/2$ (dotted curve) and ${\lambda_G}^2 z/2$ (dashed curve). (c) Numerical timing shift [Eq. (6), solid curve], average slope with Eq. (10) (dotted curve), and with Eq. (11), below (dashed curve). (d) Normalized CEPS (12) with $\lambda = \sqrt{2}$, $\theta = 0.75$, for three values of $\overline{D}$.

$$\langle \phi'(z) \rangle = \frac{\lambda^2}{2} \approx \frac{\lambda_G{}^2}{2} \equiv \frac{a^2(1-\theta)q(x)}{\sqrt{8}\,\pi b} - \frac{3\overline{D}}{4b}, \qquad (9)$$

where

$$q(x) \equiv \frac{\sinh^{-1}(x/2)}{x} + \frac{2}{(4+x^2)^{1/2}}, \qquad x = \frac{2s}{b}.$$

Equation (7) can be solved numerically using a fixed-point method,[7] which for strong dispersion management gives the result that x grows monotonically to 7.9. Therefore, $q(x)$ decreases monotonically to 0.51, $b \approx s/3.95$, and $a \approx 1.46\lambda[s/(1-\theta)]^{1/2}$. Equation (9) thus shows that with strong dispersion management the average nonlinear phase shift decreases to a (energy-dependent) constant.

To find the timing shift induced by the shock term we use the conservation law for timing corresponding to Eq. (5), which, after averaging, gives us, to leading order,

$$\langle T'(z) \rangle = -\frac{2\epsilon}{W l_c}\left\langle \Delta(\zeta)\mathrm{Im}\iint |u|^2 u u_{tt}{}^{*}\mathrm{d}t\mathrm{d}z \right\rangle + \frac{3\epsilon}{2W}\left\langle \int |u|^4 \mathrm{d}t \right\rangle, \qquad (10)$$

where $u(z,t) = F^{-1}\{\hat{f}(\omega)\exp[-i\omega^2 C(\zeta)/2]\}$. Note that the shock term conserves energy, i.e., W = constant. Using the Gaussian ansatz leads to

$$\langle T'(z) \rangle \approx \frac{\epsilon a^2(1-\theta)q(x)}{\sqrt{8}\,\pi b}, \qquad (11)$$

which shows that the average timing shift decreases to a constant with strong dispersion management. We note that the shock term induces an $O(l_c)$ phase change as well, which, however, is negligible compared with Eq. (9).

To verify the analytic results we solve Eq. (5) numerically with $\epsilon = 0.1$, $l_c = 0.2$, $\overline{D} = 0.1$, $s = 10$, $\theta = 0.75$, and initial conditions $\hat{u}(0,\omega) = \hat{f}(\omega)$, where $\hat{f}(\omega)$ is the solution of Eq. (7) with $\overline{D} = 0.1$, $s = 10$, and $\lambda = \sqrt{2}$. Figure 1(b) shows that the slope of the numerically recovered phase (after unwrapping), i.e., $\phi(z) = \arg[\hat{u}(z,0)]$, is almost indistinguishable from $\lambda^2/2$ during propagation and agrees well with $\lambda_G{}^2/2$ calculated using Eq. (9). Figure 1(c) shows that the averaged slope of the numerical timing shift $[T(z) \equiv W^{-1}\int t|u|^2]$ is precisely that obtained with Eq. (10) and is in good agreement with Eq. (11). Note that the CEPS (4) depends on the phase and timing shifts accumulated over one round trip and hence on the average slopes of $\phi(z)$ and $T(z)$.

Remarkably, the first term on the right-hand side of Eq. (9) times ϵ is the same as Eq. (11). Thus, combining these results with Eq. (4) we arrive at

$$\tilde{\delta}_{\mathrm{NL}} \approx \frac{1}{1-\sqrt{2}\,a^2(1-\theta)q(x)/3\pi\overline{D}} \approx \frac{1}{1-\lambda^2 s/6\overline{D}}. \qquad (12)$$

Hence $\tilde{\delta}_{\mathrm{NL}}$ approaches zero monotonically as $O(1/s)$ [see Fig. 1(d)]. This means that, unlike classical solitons, the combined contributions of the nonlinear phase and the shock term to the phase slip nearly cancel each other with strong dispersion management. We note that Ref. 5 obtained $\tilde{\delta}_{\mathrm{NL}}$ approaches -0.1, presumably because a rather large $\overline{D}$ was used. Indeed, Fig. 1(d) shows that with larger values of $\overline{D}$ the slip saturates at larger s, whereas with smaller $\overline{D}$ the slip becomes roughly independent of map strength, a conclusion that may be useful for controlling the slip. The unnormalized form of the slip equation (12) is $\delta_{\mathrm{NL}} \equiv (\langle\phi'\rangle - \epsilon^{-1}\langle T'\rangle)\gamma P^{*} L \approx -3\overline{k}''L/\tau_0{}^2 s$, where $\overline{k}''$ is the average-cavity GVD and L is the cavity length. This result is consistent with the insensitivity of the slip to pulse energy with strong dispersion management.[4] In that case, however, one should consider additional effects on the slip. One such effect is TOD, which can be modeled by adding $[i\overline{k}'''/(6\gamma P^{*}\tau_0{}^3)]u_{ttt}$ to the right-hand side of Eq. (5), where $\overline{k}'''$ is the average TOD coefficient. Using a similar analysis leads to $\delta_{\mathrm{TOD}} \approx -\omega\overline{k}'''L/\tau_0{}^2 s$.

We remark that the asymptotic methods used here on a nonlinear DM model are different from the analysis in Ref. 5, which was based on a linear DM model with effective parameters. Our methods are accurate and display an explicit dependence of the CEPS on physical parameters.

To conclude, our results indicate that for stronger dispersion management the nonlinear phase slip becomes insensitive to map strength and tends to zero.

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