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Critical exponents and collapse of nonlinear Schrödinger equations with anisotropic fourth-order dispersion

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Abstract
We calculate the critical exponent of nonlinear Schrödinger (NLS) equations with anisotropic negative fourth-order dispersion using an anisotropic Gagliardo–Nirenberg inequality. We also prove global existence, and in some cases uniqueness, for subcritical solutions and for critical solutions with small L^2 norm, without making use of Strichartz-type estimates for the linear operator. At exponents equal to or above critical, the blowup profile is anisotropic. Our results imply, in particular, that negative fourth-order temporal dispersion arrests spatio-temporal collapse in Kerr media with anomalous time-dispersion in one transverse dimension but not in two transverse dimensions. We also show that a small negative anisotropic fourth-order dispersion stabilizes the (otherwise unstable) waveguide solutions of the two-dimensional critical NLS.

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1. Introduction

The nonlinear Schrödinger equation (NLS)

$$\mathrm{i}u_t(\boldsymbol{x}, t) + \Delta u + \kappa|u|^{2\sigma}u = 0, \qquad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $\Delta = \sum_{i=1}^{d} \partial_{x_i x_i}$, arises in a variety of physical applications. Most notably, it models the propagation of laser beams in nonlinear Kerr media. The NLS is called focusing or defocusing when $\kappa > 0$ or $\kappa < 0$, respectively. It is well known that all solutions of the defocusing NLS exist globally. Similarly, when $\kappa > 0$ and $\sigma d < 2$, the subcritical focusing NLS, all solutions exist globally. The focusing NLS, however, admits solutions that become singular in finite time when $\sigma d = 2$ or $\sigma d > 2$, the critical and supercritical cases, respectively. Therefore, the critical exponent of the NLS is $\sigma^*_{\mathrm{NLS}} = 2/d$. In the critical case, a sufficient condition for global existence is that $\|u_0\|_2^2 < N_{\mathrm{c}}$, where $\|u_0\|_2^2 = \int |u_0(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$, and N_{c}, the threshold power for singularity formation, is a number that depends only on the dimension. For more information on singularity formation in the NLS (see [7, 14, 15]).

If one replaces the Laplacian with the biharmonic operator $\Delta^2 = \Delta\Delta$, the resulting equation is the biharmonic NLS,

$$\mathrm{i}u_t(\boldsymbol{x}, t) + \varepsilon\Delta^2 u + |u|^{2\sigma} u = 0. \tag{2}$$

In [5] it was shown that some NLS properties, but not all, remain true for the biharmonic NLS. Thus, all solutions of the biharmonic NLS exist globally in the defocusing case $\epsilon > 0$. In the focusing case $\epsilon < 0$, all solutions exist globally provided that either $\sigma d < 4$, or that $\sigma d = 4$ and $\|u_0\|_2^2 < N_{\mathrm{c}}^{\mathrm{B}}$, where $N_{\mathrm{c}}^{\mathrm{B}}$, the threshold power for singularity formation in the biharmonic NLS, depends only on the dimension. In addition, numerical simulations suggest that when $\sigma d \geqslant 4$ the biharmonic NLS admits solutions that become singular in finite time[3]. Therefore, the critical exponent of the biharmonic NLS is $\sigma^*_{\mathrm{biharmonic}} = 4/d$.

In [5] it was also shown that the conditions for global existence in the isotropic mixed-dispersion NLS

$$\mathrm{i}u_t(\boldsymbol{x}, t) + \Delta u + \varepsilon\Delta^2 u + |u|^{2\sigma} u = 0 \tag{3}$$

are the same as for the biharmonic NLS, i.e. all solutions exists globally in the defocusing case $\epsilon > 0$, in the focusing case $\epsilon < 0$ the critical exponent is $\sigma^*_{\mathrm{mixed}} = 4/d$, and in the critical case $\sigma d = 4$ all solutions exist globally when $\|u_0\|_2^2 < N_{\mathrm{c}}^{\mathrm{B}}$, where $N_{\mathrm{c}}^{\mathrm{B}}$, the threshold power for singularity formation in the biharmonic NLS, depends only on the dimension. This result is not surprising, as global existence is determined by the highest-order derivatives. The above result also holds for the weakly anisotropic, mixed-dispersion NLS

$$\mathrm{i}u_t(\boldsymbol{x}, t) + \Delta u + |u|^{2\sigma} u + \varepsilon \sum_{i=1}^{d} u_{x_i x_i x_i x_i} = 0, \tag{4}$$

which is the modified equation of the semi-discrete second-order NLS [4].

The picture becomes considerably more complex, however, when fourth-order dispersion is truly anisotropic. Such equations arise in various applications. For example, the propagation of ultrashort laser pulses in a medium with anomalous time-dispersion in the presence of fourth-order time-dispersion is given, in dimensionless form[4], by [21]

$$\mathrm{i}u_t(x, y, z, t) + \Delta u + |u|^2 u + \epsilon u_{xxxx} = 0, \qquad \Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}, \tag{5}$$

in a bulk medium, and by

$$\mathrm{i}u_t(x, y, t) + \Delta u + |u|^2 u + \varepsilon u_{xxxx} = 0, \qquad \Delta = \partial_{xx} + \partial_{yy}, \tag{6}$$

in a planar waveguide geometry. Equation (6) arises also in models of propagation in fibre arrays [1, 5].

Equations (5) and (6) are special cases of the focusing NLS with anisotropic fourth-order dispersion

$$\mathrm{i}u_t(\boldsymbol{x}, t) + \Delta u + |u|^{2\sigma} u + \varepsilon \sum_{i=1}^{k} u_{x_i x_i x_i x_i} = 0, \tag{7}$$

[3] Blowup of NLS solutions can be rigorously proved when $\sigma d \geqslant 2$ using the variance identity [18]. No equivalent result is known, at present, for the biharmonic NLS.

[4] Here, t is the distance in direction of propagation, x is time, and y and z are spatial coordinates in the transverse plane.

Table 1. Critical exponent $\sigma^*_{\text{anisotropic}}$ as a function of d and k.

	$k=0$	$k=1$	$k=2$	$k=3$
$d=2$	1	$\frac{4}{3}$	2	—
$d=3$	$\frac{2}{3}$	$\frac{4}{5}$	1	$\frac{4}{3}$

where $0 \leqslant k \leqslant d$. In this paper we consider equation (7) in the case $\epsilon < 0$. Since when $\epsilon < 0$ the second-order derivatives 'work together' with the fourth-order ones[5], one can expect the critical exponent to be higher than σ^*_{NLS}. Indeed, the main result of this paper is that the critical exponent for singularity formation in (7) is[6]

$$\sigma^*_{\text{anisotropic}} = \frac{2}{d - k/2}.$$

Therefore,

$$\begin{cases} \sigma^*_{\text{anisotropic}} = \sigma^*_{\text{NLS}}, & k = 0, \\ \sigma^*_{\text{NLS}} < \sigma^*_{\text{anisotropic}} < \sigma^*_{\text{biharmonic}}, & 0 < k < d, \\ \sigma^*_{\text{anisotropic}} = \sigma^*_{\text{biharmonic}}, & k = d. \end{cases}$$

Some specific values of the critical exponent are listed in table 1. These values show, in particular, that a cubic nonlinearity is supercritical for collapse in equation (5), but is subcritical for collapse in equation (6). In other words, a negative fourth-order temporal dispersion arrests the spatio-temporal collapse of ultrashort pulses in one transverse spatial dimension, but not in two transverse spatial dimensions.

The paper is organized as follows. In section 2 we use an anisotropic Gagliardo–Nirenberg inequality to show that if u is a solution of (7), if $\epsilon < 0$, and if either $\sigma < \sigma^*_{\text{anisotropic}}$ or $\sigma = \sigma^*_{\text{anisotropic}}$ and $\|u_0\|_2^2$ is sufficiently small, then for all time

$$\|u\|^2_{H^{(2,1)}_{(k,d-k)}} \leqslant c, \tag{8}$$

where

$$\|u\|^2_{H^{(2,1)}_{(k,d-k)}} = \|u\|_2^2 + \|\nabla u\|_2^2 - \varepsilon \sum_{i=1}^{k} \|u_{x_i x_i}\|_2^2. \tag{9}$$

In theorem 2 we use this estimate to prove the existence of weak solutions to (7) in the anisotropic Sobolev space $H^{(2,1)}_{(k,d-k)}$ equipped with the norm (9). Uniqueness of the weak solution of theorem 2 is proved for some special cases in theorem 3. In section 3 we use asymptotic analysis to show that a small negative fourth-order dispersion stabilizes the (otherwise unstable) waveguide solutions of the critical NLS. Simulation results are presented in section 4.

We recall that the local existence and uniqueness theory for the isotropic NLS in $\mathbb{R}^d$ [10,11] and for the isotropic biharmonic NLS and the mixed-dispersion NLS (3) in $\mathbb{R}^d$ [2] is based on propagation estimates for the linear portion of the equation. In this paper we use a different approach which follows the proofs of [12, 16] for nonlinear wave equations and for smooth solutions of a variety of equations, respectively. This approach had the advantage that it does not require estimates of the linear portion of the equation. However, this approach gives uniqueness only in some cases. Our proof of global existence and uniqueness is also valid

[5] That is, the corresponding terms in the Hamiltonian (11) have the same sign.

[6] To be precise, we prove global existence for $\sigma < \sigma^*_{\text{anisotropic}}$. As in the isotropic case (the biharmonic NLS), at present there is numerical evidence, but no rigorous proof, that there exist blowup solutions for $\sigma \geqslant \sigma^*_{\text{anisotropic}}$.

for periodic domains, as well as for the isotropic biharmonic NLS (2) for $d \leqslant 3$, for both the focusing and the defocusing case.

2. Critical exponent

Solutions of (7) conserve both the power and Hamiltonian, i.e.

$$\|u\|_2^2 \equiv \|u_0\|_2^2, \qquad H(t) = H(0), \tag{10}$$

where

$$H = \|\nabla u\|_2^2 - \varepsilon \sum_{i=1}^{k} \|u_{x_i x_i}\|_2^2 - \frac{1}{2\sigma + 2}\|u\|_{2\sigma+2}^{2\sigma+2}. \tag{11}$$

Since $\varepsilon < 0$, both derivative terms in (11) have the same sign. Provided that σ is not too large, the conserved Hamiltonian (11) can be used to derive an *a priori* bound for $\|u\|^2_{H^{(2,1)}_{(k,d-k)}}$. To do that, we need the following anisotropic Sobolev inequality.

Lemma 1. *Suppose that*

$$\begin{cases} 0 \leqslant \sigma < \infty, & d - \dfrac{k}{2} \leqslant 2, \\ 0 \leqslant \sigma \leqslant \dfrac{2}{d - k/2 - 2}, & d - \dfrac{k}{2} > 2. \end{cases} \tag{12}$$

Then there is a constant $C_{d,k,\sigma}$ such that

$$\|u\|_{2\sigma+2}^{2\sigma+2} \leqslant C_{d,k,\sigma} \|u\|_2^{2-\sigma(d-k/2-2)} \prod_{i=1}^{k} \|u_{x_i x_i}\|_2^{\sigma/2} \prod_{i=k+1}^{d} \|u_{x_i}\|_2^{\sigma} \tag{13}$$

and the embedding of the anisotropic Sobolev space equipped with the norm $\|u\|^2_{H^{(2,1)}_{(k,d-k)}}$ into $L^{2\sigma+2}$ is compact.

Lemma 1 is a special case of estimate (12) on p 325 in [3]. Still, a fairly simple and direct proof can be given.

Proof. In proving (13) we may assume that u belongs to C_0^∞, because that space is dense in both $H^{(2,1)}_{(k,d-k)}$ and $L^{2\sigma+2}$. Then

$$\begin{aligned} |u(x)|^p &= p\int_{-\infty}^{x_i} |u|^{p-2} u u_{x_i}\, \mathrm{d}x_i \leqslant p \int_{-\infty}^{\infty} |u|^{p-1} |u_{x_i}|\, \mathrm{d}x_i \\ &\leqslant p \left[\int_{-\infty}^{\infty} |u|^{2(p-1)}\, \mathrm{d}x_i\right]^{1/2} \left[\int_{-\infty}^{\infty} |u_{x_i}|^2\, \mathrm{d}x_i\right]^{1/2}. \end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned} |u|^q &\leqslant q\int_{-\infty}^{\infty} |u|^{q-1}|u_{x_i}|\, \mathrm{d}x_i = q\int_{-\infty}^{\infty} |u|^{q-r-1}[|u|^r |u_{x_i}|]\, \mathrm{d}x_i \\ &\leqslant c\left[\int_{-\infty}^{\infty} |u|^{2(q-r-1)}\, \mathrm{d}x_i\right]^{1/2} \left[\int_{-\infty}^{\infty} (|u|^{2r}u)_{x_i} u_{x_i}\, \mathrm{d}x_i\right]^{1/2} \\ &= c\left[\int_{-\infty}^{\infty} |u|^{2(q-r-1)}\, \mathrm{d}x_i\right]^{1/2} \left[-\int_{-\infty}^{\infty} |u|^{2r} u u_{x_i x_i}\, \mathrm{d}x_i\right]^{1/2} \\ &\leqslant c\left[\int_{-\infty}^{\infty} |u|^{2(q-r-1)}\, \mathrm{d}x_i\right]^{1/2} \left[\int_{-\infty}^{\infty} |u|^{4r+2}\, \mathrm{d}x_i\right]^{1/4} \left[\int_{-\infty}^{\infty} u_{x_i x_i}^2\, \mathrm{d}x_i\right]^{1/4}, \end{aligned} \tag{15}$$

where r is an arbitrary number between zero, inclusive, and $q-1$, noninclusive, that will be chosen shortly; it is easy to check that the chosen value indeed lies in the required interval.

Take the product of (15) over $1 \leqslant i \leqslant k$ times the product of (14) over $k+1 \leqslant i \leqslant d$ to obtain

$$|u|^{kq+(d-k)p} \leqslant c \prod_{i=1}^{k} \left[\int_{-\infty}^{\infty} |u|^{2(q-r-1)}\, \mathrm{d}x_i \right]^{1/2} \left[\int_{-\infty}^{\infty} |u|^{4r+2}\, \mathrm{d}x_i \right]^{1/4} \left[\int_{-\infty}^{\infty} u_{x_i x_i}^2\, \mathrm{d}x_i \right]^{1/4}$$
$$\times \prod_{i=k+1}^{d} \left[\int_{-\infty}^{\infty} |u|^{2(p-1)}\, \mathrm{d}x_i \right]^{1/2} \left[\int_{-\infty}^{\infty} |u_{x_i}|^2\, \mathrm{d}x_i \right]^{1/2}. \tag{16}$$

We would now like to integrate (16) over all the x_i and use the many-factor Hölder inequality $\int \prod f_i^{\alpha_i} \leqslant \prod [\int f_i]^{\alpha_i}$ to replace integrals over products on the right-hand side of the result by products of integrals, so that each factor on the right becomes some power of an integral over all the x_i. Now the sum of powers in each of (14) and (15) equals one, and for any variable x_i there are $d-1$ factors that must be included in the product, because one factor was already integrated over that variable, so the sum of powers inside the integral with respect to any x_i equals $d-1$. Since the many-factor Hölder inequality is valid only when the α_i sum to 1, we must therefore raise (16) to the power $1/(d-1)$ before integrating over all the x_i to obtain

$$\|u\|_{(kq+(d-k)p)/(d-1)}^{kq+(d-k)p} \leqslant \|u\|_{2(q-r-1)}^{k(q-r-1)} \|u\|_{2(2r+1)}^{k(2r+1)/2} \|u\|_{2(p-1)}^{(d-k)(p-1)} \prod_{i=1}^{k} \|u_{x_i x_i}\|_2^{1/2} \prod_{i=k+1}^{d} \|u_{x_i}\|_2, \tag{17}$$

where for notational convenience we have raised the result back to the $d-1$ power.

The $d+1$ powers of the norms appearing on the right-hand side of (13) can be uniquely determined by the condition that the estimate be invariant when any of the $d+1$ independent and dependent variables are rescaled by a multiplicative constant, while estimate (17), since it is valid, is necessarily also invariant under such rescaling.

Since the right-hand side of (17) involves L^{s_i} norms whereas the right-hand side of (13) involves an L^2 norm, we use interpolation inequalities for L^p norms to estimate the L^{s_i} norms on the right-hand side of (17) in terms of the L^2 and the $L^{2\sigma+2}$ norms. Therefore, we should have that $2\sigma + 2 = (kq + (d-k)p)/(d-1)$ and that for all three L^{s_i} norms we have $2 \leqslant s_i \leqslant 2\sigma+2$, so that the interpolation inequality is valid. For a given σ, the maximum of the three s_i is minimized when they are all equal, and making that choice determines p, q, r, and s as functions of σ, d, and k. Straightforward calculations yield the formulae

$$p = 2(r+1), \qquad q = 3r+2, \qquad r = \frac{\sigma(d-1)-1}{d+k/2}, \qquad \frac{s}{2} - 1 = 2r.$$

The above condition on s is equivalent to $0 \leqslant s/2 - 1 \leqslant \sigma$, which reduces to $\sigma \geqslant 1/(d-1)$ and $(d - k/2 - 2)\sigma \leqslant 2$. The latter is equivalent to (12). The former can be weakened to $\sigma \geqslant 0$ by noting that (13) holds trivially for $\sigma = 0$, hence we can use interpolation inequalities for $L^{2\sigma+2}$ for $0 < \sigma < 1/(d-1)$ to interpolate between the L^2 norm (i.e. $\sigma = 0$) and the $L^{2d/(d-1)}$ norm (i.e. $\sigma = 1/(d-1)$).

Finally, the compactness of the embedding follows from standard compactness and interpolation arguments: if u_n is a sequence uniformly bounded in $H^{(2,1)}_{(k,d-k)}$, then by the compactness of H^1 in L^2 some subsequence converges in L^2. Applying (13) to differences $u_m - u_n$ restricted to that subsequence shows that the subsequence is Cauchy in $L^{2\sigma+2}$. □

Theorem 1. *Let u be a solution of (7) such that $\|u_0\|^2_{H^{(2,1)}_{(k,d-k)}} < \infty$, let $\epsilon < 0$, and let either $0 < \sigma < \sigma^*_{\mathrm{anisotropic}}$, or $\sigma = \sigma^*_{\mathrm{anisotropic}}$ and $\|u_0\|_2$ be sufficiently small. Then, (8) holds for all time.*

Proof. Note first that the assumption $\sigma \leqslant \sigma^*_{\text{anisotropic}}$ implies that (12) holds. By (10), (11), and (13),

$$\|u\|^2_{H^1} - \varepsilon \sum_{i=1}^{k} \|u_{x_i x_i}\|_2^2 = \|u_0\|_2^2 + H(0) + \frac{1}{\sigma+1}\|u\|_{2\sigma+2}^{2\sigma+2}$$
$$\leqslant \|u_0\|_2^2 + H(0) + \kappa \prod_{i=1}^{k} \|u_{x_i x_i}\|_2^{\sigma/2} \prod_{i=k+1}^{d} \|u_{x_i}\|_2^{\sigma},$$

where

$$\kappa = \frac{C_{d,k,\sigma}}{\sigma+1}\|u_0\|_2^{2-\sigma(d-k/2-2)}.$$

By Young's inequality,

$$\prod_{i=1}^{k} \|u_{x_i x_i}\|_2^{\sigma/2} \prod_{i=k+1}^{d} \|u_{x_i}\|_2^{\sigma} \leqslant \frac{1}{2d-k}\sum_{i=1}^{k} \|u_{x_i x_i}\|_2^{\sigma(d-k/2)} + \frac{2}{2d-k}\sum_{i=k+1}^{d} \|u_{x_i}\|_2^{\sigma(d-k/2)}.$$

Therefore,

$$\|u\|^2_{H^1} - \varepsilon \sum_{i=1}^{k} \|u_{x_i x_i}\|_2^2 \leqslant \|u_0\|_2^2 + H(0) + \frac{2\kappa}{2d-k}\left(\sum_{i=1}^{k} \|u_{x_i x_i}\|_2^{\sigma(d-k/2)} + \sum_{i=k+1}^{d} \|u_{x_i}\|_2^{\sigma(d-k/2)}\right).$$

We thus see that the right-hand side is bounded if either $\sigma(d-k/2) < 2$ or if $\sigma(d-k/2) = 2$ and $2\kappa/(2d-k) < 1$. □

2.1. Existence and uniqueness results

Theorem 2. *Under the conditions of theorem 1, there exists a global-in-time weak solution to (7).*

Proof. Let J_ε be a symmetric mollifier, i.e. a convolution operator whose kernel is $(1/\varepsilon^d)\phi(x/\varepsilon)$, where ϕ is a smooth, compactly supported, nonnegative function whose integral equals one, and which is symmetric in the sense that $\phi(-x) = \phi(x)$. Then (e.g. [16]) J_ε is a symmetric operator that is continuous for fixed ε from any L^p space to any Sobolev space, is uniformly bounded in ε on L^p and Sobolev spaces, and tends to the identity as $\varepsilon \to 0$.

We will obtain solutions to (7) as limits of solutions to the mollified equation

$$\mathrm{i}u_t + J_\varepsilon\left[|J_\varepsilon u|^{2\sigma} J_\varepsilon u + \Delta(J_\varepsilon u) + \varepsilon \sum_{i=1}^{k} (J_\varepsilon u)_{x_i x_i x_i x_i}\right] = 0. \tag{18}$$

Here u should really be denoted u_ε but the subscript will henceforth be omitted for notational simplicity. Thanks to the smoothing property of the mollifier, (18) may be viewed as an ODE on L^2. The placement of the mollifiers in (18) has been chosen so that the solutions of that equation will inherit the conservation of power and Hamiltonian: multiplying (18) by the factor $-\mathrm{i}\bar{u}$ and adding $\mathrm{i}u$ times the conjugate of (18), then integrating over space and using the symmetry of J_ε to transfer the outer occurrence of that operator from the term appearing in (18) to the above-mentioned factors, and integrating by parts as for the NLS equation yields the conservation of $\|u\|_2^2$. This ensures that the solutions to (18) exist for all time, since the only way for an ODE solution not to exist globally is for its norm to become infinite in finite time. Multiplying by $\bar{u}_t$ instead and following the rest of this procedure yields the conservation of the Hamiltonian

$$H = \|\nabla(J_\varepsilon u)\|_2^2 - \varepsilon \sum_{i=1}^{k} \|(J_\varepsilon u)_{x_i x_i}\|_2^2 - \frac{1}{2\sigma+2}\|J_\varepsilon u\|_{2\sigma+2}^{2\sigma+2}.$$

The subcriticality or criticality with small power means that these bounds imply the uniform boundedness of $J_\varepsilon u_\varepsilon$ in the $H^{(2,1)}_{(k,d-k)}$ and $L^{2\sigma+2}$ norms. Given these uniform bounds, the convergence of some sequence of the $J_\varepsilon u_\varepsilon$ to a solution u of (7) can be obtained by a standard compactness argument. (In fact, the situation here is even simpler than that treated in [14, p 22], which includes the defocusing case with large σ.)

First, for some sequence of values of ε tending to zero, u_ε tends weak-$*$ in $L^\infty(L^2)$ and $J_\varepsilon u_\varepsilon$ converges weak-$*$ in $L^\infty(H^{(2,1)}_{(k,d-k)})$ to the same limit u, which suffices to take the weak limit of the linear terms. Furthermore, since $\sigma \leqslant \sigma^*_{\text{anisotropic}}$, condition (12) holds, so the embedding of the anisotropic Sobolev space $H^{(2,1)}_{(k,d-k)}$ into $L^{2\sigma+2}$ is compact. Since equation (18) plus its conserved quantities show that $\partial_t(J_\varepsilon u_\varepsilon)$ is uniformly bounded in $L^\infty(H^{-s})$ for s sufficiently large, the Lions–Aubin compactness theorem (e.g. [17, theorem III.2.1]) shows that some subsequence of the $J_\varepsilon u_\varepsilon$ converges in $L^p_{\text{loc}}(L^{2\sigma+2})$ for $1 < p < \infty$, which yields the convergence of the nonlinear term in (18) to the corresponding term in (7). □

In general, the weak solution obtained in theorem 2 is not known to be unique. That uniqueness can, however, be proven in some special cases.

Theorem 3. *Let*

$$d = 1, \quad k = 0, 1 \qquad \text{or} \qquad d = 2, \quad k = 1, 2 \qquad \text{or} \qquad d = 3, \quad k = 3. \tag{19}$$

Then, the weak solution constructed in theorem 2 is unique.

Proof. By Parseval's theorem, the boundedness of the power and Hamiltonian imply in the subcritical or critical cases that $\|w(\kappa)\hat{u}(\kappa)\|_{L^2} < \infty$, where $w(\kappa) = (1 + |\kappa|^2 + \sum_{i=1}^{k} \kappa_i^4)^{1/2}$. In addition, $w(\kappa)^{-1}$ belongs to L^2 if and only if (19) holds. The standard argument $\|\hat{u}\|_{L^1} = \|w^{-1} \cdot w\hat{u}\|_{L^1} \leqslant \|w^{-1}\|^{1/2}_{L^2} \|w\hat{u}\|^{1/2}_{L^2}$ therefore shows that the Fourier transform of u lies in L^1, and hence that u itself belongs to L^∞. Since the nonlinear terms in (7) are not differentiated, this amount of smoothness is formally sufficient to obtain uniqueness: if u_1 and u_2 are two solutions then their difference $u = u_2 - u_1$ satisfies

$$\mathrm{i}u_t(\boldsymbol{x}, t) + |u_2|^{2\sigma} u + [|u_1 + u|^{2\sigma} - |u_1|^{2\sigma}]u_1 - \Delta u + \varepsilon \sum_{i=1}^{k} u_{x_i x_i x_i x_i} = 0. \tag{20}$$

Applying formally to this equation the procedure used to obtain the conservation of power for (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_2^2 \leqslant c \int [|u_1| + |u_2|]^{2\sigma} |u|^2 \leqslant c\|u\|_2^2. \tag{21}$$

This differential inequality implies that $\|u(t)\|_2^2 \leqslant \|u(0)\|_2^2 \mathrm{e}^{ct}$ so that if $u = 0$ initially then it remains zero for all time.

Since the u_j are only weak solutions, this formal calculation requires justification, which can be achieved by applying the mollifier J_ε to (20) and applying the procedure used to obtain the conservation of power for (18). After using the boundedness of J_ε and u, we obtain the modified estimate $(\mathrm{d}/\mathrm{d}t)\|J_\varepsilon u\|_2^2 \leqslant c\|u\|_2^2$. Integrating this equation over time and only then taking ε to zero yields $\|u(t)\|_2^2 \leqslant \|u(0)\|_2^2 + c \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s$, which is equivalent to the differential inequality (21) obtained formally. □

Remark. When $d - k/2 \leqslant 2$, i.e. the first alternative in (12), lemma 1 ensures that u is in L^p for all $0 \leqslant p < \infty$. However, it is well known that an L^∞ bound does not always follow, which is why the cases for which we prove uniqueness (i.e. that u is in L^∞) only form a proper subset of $d - k/2 \leqslant 2$.

3. Critical NLS

We now analyse the effect of small negative one-dimensional fourth-order dispersion on two-dimensional critical self-focusing, i.e.

$$\mathrm{i}u_t(x, y, t) + \Delta u + |u|^2 u + \varepsilon u_{xxxx} = 0. \qquad (22)$$

When $\epsilon = 0$, equation (22) is the critical focusing NLS, which can admit blowup solutions. We have proved that the addition of negative fourth-order dispersion arrests the collapse. Our proof, however, does not provide any insight as to the dynamics of the globally existing solutions. To do that, we can use modulation theory [7, 8], which is an asymptotic theory for analysing the effects of small perturbations on self-focusing in the critical NLS

$$\mathrm{i}u_t(x, y, t) + \Delta u + |u|^2 u = 0. \qquad (23)$$

Modulation theory is based on the observation that as the solution undergoes self-focusing, the solution core rearranges itself as modulated ground state, i.e.

$$|u| \sim \frac{1}{L(t)} R\left(\frac{r}{L(t)}\right), \qquad (24)$$

where $r = \sqrt{x^2 + y^2}$ and $R(r)$ is the ground-state solution of

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \qquad R'(0) = 0, \qquad R(\infty) = 0. \qquad (25)$$

Therefore, self-focusing dynamics is described by the modulation variable $L(t)$. In particular, $L \to 0$ and $L \to \infty$ correspond to singularity formation and to complete defocusing, respectively. Application of modulation theory to (22) leads to the following ODE for $L(t)$ [7]:

$$-L^3 L_{tt} = \beta_0 + \frac{9\epsilon N_c}{4M} \frac{1}{L^2},$$

where $N_c = \|R\|_2^2 \approx 2\pi \times 1.86$ is the threshold power for singularity formation in the NLS (23), and $M = (1/4)\|rR\|_2^2 \approx 2\pi \times 0.55$. Let us consider an initial condition for which the solution of the NLS (23) becomes singular. This solution has input power above critical, i.e. $\|u_0\|_2^2 > N_c$, which in modulation theory variables amounts to $\beta_0 > 0$. Hence, by proposition 4.3 in [7], when $\epsilon < 0$ the beam width $L(t)$ does not shrink to zero (i.e. the NLS solution does not become singular), but rather undergoes nearly periodic focusing defocusing oscillations.

In most NLS equations the ground-state waveguide solutions are modulationally stable if and only if the NLS is focusing and subcritical[7]. Therefore, we can expect the waveguide solutions of (6) to be stable. To see that, let us consider waveguide solutions of (6) of the form

$$u(x, y, t) = \mathrm{e}^{\mathrm{i}\lambda t} Q_\lambda(x, y).$$

In the following lemma we prove the generic condition for stability of waveguides [13, 20].

Lemma 2. *Let* $\epsilon < 0$ *and let* $\lambda \ll -1/\epsilon$. *Then,* $(\mathrm{d}/\mathrm{d}\lambda)\|Q_\lambda\|_2^2 > 0$.

Proof. Q_λ satisfies

$$-\lambda Q_\lambda + \Delta Q_\lambda + Q_\lambda^3 + \epsilon Q_{\lambda,xxxx} = 0.$$

Let

$$Q_\lambda = \lambda^{1/2} Q(\lambda^{1/2}x, \lambda^{1/2}y), \qquad \delta = \lambda\epsilon.$$

[7] Notable exceptions are the critical NLS on bounded domains [6] and the critical NLS with inhomogeneous nonlinearity [9].

Then

$$-Q + \Delta Q + Q^3 + \delta Q_{xxxx} = 0, \qquad 0 < -\delta \ll 1.$$

Expand $Q(x, y) = R(r) + \delta h(x, y) + \mathrm{O}(\delta^2)$, where R is the solution of (25). Then h satisfies

$$(3R^2 - 1)h(x, y) + \Delta h = -R_{xxxx}.$$

The equation for h can be rewritten using polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ as

$$(3R^2 - 1)h(r, \theta) + h_{rr} + \frac{1}{r}h_r + \frac{1}{r^2}h_{\theta\theta} = -R'''' \cos^4\theta - \frac{6}{r}R''' \sin^2\theta\cos^2\theta$$
$$-3\left(\frac{R''}{r^2} - \frac{R'}{r^3}\right)(\sin^4\theta - 4\sin^2\theta\cos^2\theta). \tag{26}$$

Note that the terms on the right-hand side are not singular at the origin. Indeed, by L'Hôpital's rule and using the fact that R is even in r,

$$\lim_{r\to 0}\frac{R'''}{r} = R''''(0), \qquad \lim_{r\to 0}\left(\frac{R''}{r^2} - \frac{R'}{r^3}\right) = \frac{1}{3}R''''(0).$$

Let $h(r, \theta) = f_0(r) + f_1(r)\cos(2\theta) + f_2(r)\cos(4\theta)$. Using the trigonometric identities

$$\cos^4\theta \equiv \tfrac{3}{8} + \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{8}\cos(4\theta), \qquad \sin^2\theta\cos^2\theta \equiv \tfrac{1}{8} - \tfrac{1}{8}\cos(4\theta),$$
$$\sin^4\theta - 4\sin^2\theta\cos^2\theta \equiv -\tfrac{1}{8} - \tfrac{1}{2}\cos(2\theta) + \tfrac{5}{8}\cos(4\theta),$$

we get that f_0, f_1, f_2 satisfy the three decoupled ODEs

$$(3R^2 - 1)f_0 + f_{0,rr} + \frac{1}{r}f_{0,r} = -\frac{3}{8}R'''' - \frac{3}{4}\frac{R'''}{r} + \frac{3}{8}\left(\frac{R''}{r^2} - \frac{R'}{r^3}\right),$$
$$\left(3R^2 - 1 - \frac{4}{r^2}\right)f_1 + f_{1,rr} + \frac{1}{r}f_{1,r} = -\frac{1}{2}R'''' + \frac{3}{2}\left(\frac{R''}{r^2} - \frac{R'}{r^3}\right),$$
$$\left(3R^2 - 1 - \frac{16}{r^2}\right)f_2 + f_{2,rr} + \frac{1}{r}f_{2,r} = -\frac{1}{8}R'''' + \frac{3}{4}\frac{R'''}{r} - \frac{15}{8}\left(\frac{R''}{r^2} - \frac{R'}{r^3}\right),$$

subject to the boundary conditions

$$\frac{\mathrm{d}f_j}{\mathrm{d}r}(0) = 0, \quad \lim_{r\to\infty} f_j(r) = 0, \qquad j = 0, 1, 2.$$

To finish the proof, note that

$$\|Q_\lambda\|_2^2 = \|Q\|_2^2 = \|R\|_2^2 + 2\delta\int Rh\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}(\delta^2) = N_{\mathrm{c}} + 2\delta\int Rf_0\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}(\delta^2).$$

The equation for f_0 can be solved using a shooting method, where one searches for the value of $f_0(0)$. Numerical calculation gives that $f_0(0) \approx 2.8084$ and that $\int Rf_0 r\,\mathrm{d}r \approx -2.14$. Therefore, when $\epsilon < 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\|Q_\lambda\|_2^2 \approx 2\epsilon\cdot(-2\pi\times 2.14) > 0. \qquad \square$$

For completeness, the profiles of f_0, f_1, and f_2 and a contour plot of h are given in figure 1. Note that no shooting is needed for f_1 and f_2, since $f_1(0) = f_2(0) = 0$.

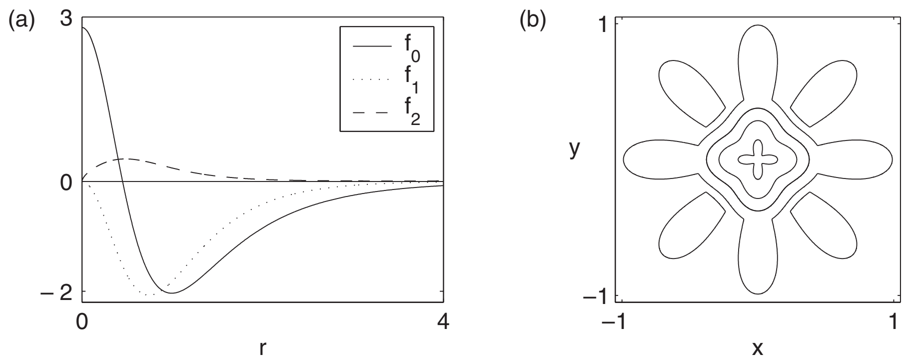

Figure 1. (*a*) f_0, f_1, and f_2 as a function of r. (*b*) Contour plot of $h(x, y)$.

4. Simulations

We solve the two-dimensional NLS with anisotropic negative fourth-order dispersion,

$$\mathrm{i}u_t(x, y, t) + |u|^{2\sigma}u + \Delta u + \varepsilon u_{xxxx} = 0, \qquad \epsilon < 0, \tag{27}$$

using finite differences in space and fourth-order Runge–Kutta in time. As pointed out in [4], the approximation of the Laplacian with a finite difference scheme adds numerical high-order dispersion to the computed solution. Therefore, one has to make sure that the numerical high-order dispersion does not corrupt the computed solution by conducting grid-convergence tests. We use Dirichlet boundary conditions at the boundary of the computational domain. Because of the strong radiation due to the fourth-order dispersion, one has to take a sufficiently large domain in the x direction (see section 8 in [5]) and to verify that reflections from the boundaries do not corrupt the solution. We use radially symmetric initial condition $u_0 = c\exp(-x^2 - y^2)$. The value of c is chosen so that the initial condition has 1.5 times the critical power for collapse, i.e. $\|u_0\|_2^2 = 1.5N_{\mathrm{c}}$. Similar results were observed with other initial conditions.

The solution of the unperturbed critical NLS (i.e. when $\sigma = 1$ and $\varepsilon = 0$) with this initial condition becomes singular in a finite time (see dashed line in figure 2(*a*)). The addition of one-dimensional fourth-order dispersion changes the critical exponent from $\sigma^*_{\mathrm{NLS}} = 1$ to $\sigma^*_{\mathrm{anisotropic}} = \frac{4}{3}$ (see theorem 1). Indeed, collapse is arrested when $\sigma = 1$ and $\epsilon < 0$ (figure 2(*a*)). The subsequent focusing–defocusing oscillations agree with the predictions of the asymptotic analysis in section 3.

When σ is equal to or above the critical exponent $\sigma^*_{\mathrm{anisotropic}} = \frac{4}{3}$, the solution becomes singular (figures 2(*b*) and (*c*)). In these cases the blowup profile near the singularity is anisotropic, and its level sets are roughly ellipses which are more elongated along the x-axis (figures 3 and 4). The last observation can be motivated using the following hand-waving argument. Assume that near the singularity

$$|u| \sim \frac{1}{L_x^{1/2}(t)L_y^{1/2}(t)}Q\left(\frac{x}{L_x(t)}, \frac{y}{L_y(t)}\right), \qquad \text{where } L_x, L_y \to 0.$$

Since the balance in equation (27) is between the highest-order derivatives in x and in y, it follows that $L_x^4 \sim L_y^2$, hence that $L_x \gg L_y$.

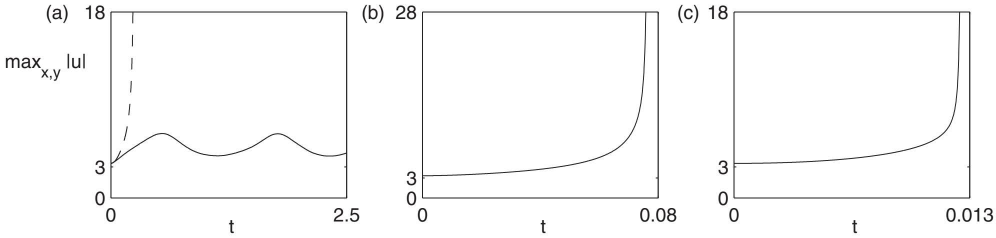

Figure 2. Solution of equation (27) with $\varepsilon = -0.1$. (*a*) $\sigma = 1$ (dashed line is $\varepsilon = 0$), (*b*) $\sigma = \frac{4}{3}$, and (*c*) $\sigma = 2$.

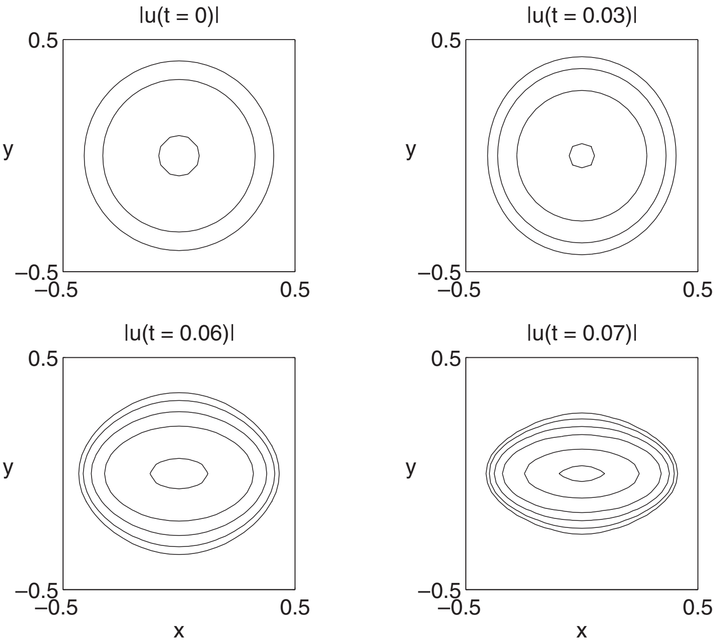

Figure 3. Intensity contours corresponding to the solution shown in figure 2(*b*). In all four plots the outermost contour is $|u| = 8$. Subsequent contour levels are 9, 11, 15, 32, and 71 (when applicable).

5. Final remarks

There are many issues related to the NLS with anisotropic fourth-order dispersion which have not been considered in this paper. For example, we did not calculate the threshold input power in the critical case $\sigma = \sigma^*_{\text{anisotropic}}$, or equivalently, the optimal constant $C_{d,k,\sigma}$. Recall that in the case of the isotropic NLS (1), Weinstein [19] calculated the corresponding optimal constant using a variational formulation. This calculation showed that the threshold power N_c is equal to the power of the ground-state waveguide solution of the NLS. Proving a similar result for the (isotropic) biharmonic NLS is, however, considerably harder, as its waveguide solutions are not positive [5]. Clearly, extending this result to equation (7) is even harder, as its waveguide solutions are also not radially symmetric.

The dynamics of equation (7) is quite different when $\epsilon > 0$. In the isotropic case, equation (3), $\epsilon > 0$ is the defocusing case, hence its solutions exist globally. Indeed, in this

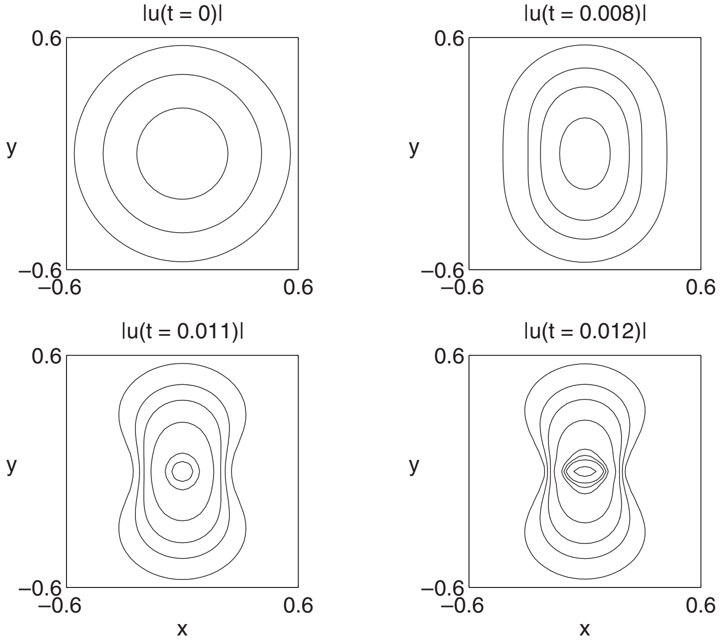

Figure 4. Intensity contours corresponding to the solution shown in figure 2(*c*). In all four plots outermost contour is $|u| = 6$. Subsequent contour levels are 8, 10, 14, 24, 28, 32, and 45 (when applicable).

case self-focusing is initially accelerated compared with the case $\epsilon = 0$, but subsequently collapse is arrested and the solution defocuses 'to infinity' [5]. A similar dynamics was observed numerically in the truly anisotropic case, equation (6) with $\epsilon > 0$. In that case, however, proof of global existence is still an open question.

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