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Green's function formalism for nonimaging optics

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ABSTRACT

Nonimaging optics is described in the context of radiative transfer theory. The radiative transport equation reduces to a local equation, whose solution can be expressed using the Green's function formalism. This yields a surface integral equation for the radiance, which can be useful for analytical and numerical calculations.

Keywords: Nonimaging concentrators, Radiative transport equation

1. INTRODUCTION

The field of nonimaging optics is young and rapidly growing, with applications to solar energy harvesting and illumination engineering.[1–3] There are few analytical design principles and methods, mostly for planar designs. We present a formalism based on the theory of radiative transfer.

2. NONIMAGING OPTICS CONCENTRATORS

An example of nonimaging reflective concentrator, known as a Winston cone, is shown in Fig. 1 along with a typical design setup. Such concentrators consist of an entry aperture (a planar surface), a curved reflector, and a receiver, typically with free space in between.

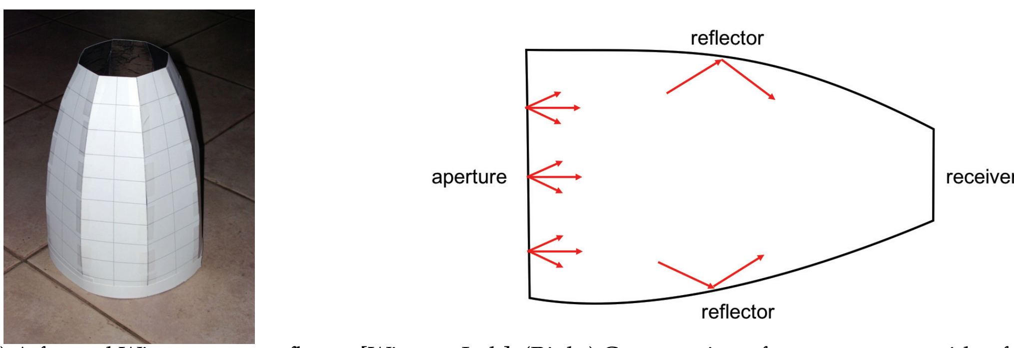

Figure 1. (Left) A faceted Winston cone reflector [Winston Lab]. (Right) Cross section of a concentrator with a free-form reflector.

An ideal concentrator would transport all the light that enters the aperture to the receiver (see also [4] for the conditions of ideality of an illuminator). Ideal concentrator designs are known only for some planar problems. In general, for all known 3D problems, some light is reflected backwards into the aperture. Such light is said to be rejected. Analytical design approaches, such as the Hottel string method and the flow line method, have been developed for planar designs and can be applied to 3D design with rotational symmetry (cones) or cylindrical symmetry (troughs). However, many practical designs use faceted reflectors (as in Fig. 1) because they are cheaper to manufacture, lighter, and provide excellent performance. For those designs there is no accurate analytical approach. Moreover, the quantities of interest, such as the light flux incident on the receiver, are not rotationally or cylindrically symmetric. Therefore, computational methods are necessary.

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Miñano and Benítez developed the computational SMS design method,[5] which is based on wavefront calculations. Ilan *et al.* developed an approach based on derivative-free optimization with ray tracing.[6] However, there is a need for adaptable, accurate and fast computational methods for designing concentrators with competing specifications.

3. RADIATIVE TRANSFER THEORY AND THE GREEN'S FUNCTION

Nonimaging optics deals with incoherent light sources, such as sunlight, which can be described by its radiance. Chandrasekhar developed the theory of radiative transfer, which describes the propagation, absorption and scattering of radiance.[7] It is therefore natural to think of nonimaging optics in the context of radiative transfer theory. In this vein, we present the governing partial differential equation (PDE) problem and develop a Green's function formalism.

We denote the radiance by $L(\mathbf{x}, \mathbf{s})$, which is a function of position $\mathbf{x}$ in $\mathbb{R}^d$ and the direction $\mathbf{s}$ on the unit sphere S^{d-1}. In steady-state, the radiance is governed by radiative transfer equation (RTE),

$$\mathbf{s} \cdot \vec{\nabla} L \ + \mu_t L = \mu_s \int_{S^{d-1}} p(\mathbf{s}, \mathbf{s}') L(\mathbf{x}, \mathbf{s}') \, d\mathbf{s}' \ + Q(\mathbf{x}, \mathbf{s}) \ , \tag{1}$$

where μ_t and μ_s are related to absorption and scattering coefficients, p is the scattering phase function, and Q is a source term. The RTE needs to be supplemented with boundary conditions. To this end, let $\mathbf{x}$ be in D, which may be either a bounded domain or an unbounded region. For a well-posed problem, boundary conditions need to be prescribed for the radiance over the entire boundary of D (or asymptotically at infinity) in all directions that enter D, *i.e.*,

$$L(\mathbf{x}, \mathbf{s}) = \mathcal{B}_{\text{in}} \ , \quad \mathbf{x} \in \partial D, \quad \mathbf{s} \cdot \mathbf{n} < 0 \ , \tag{2}$$

where $\mathbf{n}$ is the outward unit normal to ∂D and $\mathcal{B}_{\text{in}}$ describe sources (entrance aperture), sinks (exit aperture), and / or a reflection operator (discussed below) along the boundary.

For many problems in nonimaging optics, light propagates in free space. In that case the RTE (1) reduces to

$$\mathbf{s} \cdot \vec{\nabla} L \ = 0 \ . \tag{3}$$

Equation (3) is a PDE for the radiance. Because it is a local equation, it is much simpler than (1). However, solving Eq. (3) for nonimaging problems is challenging due to the dimensionality of the problem and the geometry of the reflector. We seek to express the radiance using a Green's function. Case[8] showed that the solution of the general RTE problem [Eqns. (1) and (2)] can be expressed as

$$L(\mathbf{x}, \mathbf{s}) = \int_{\partial D} \int_{S^{d-1}} \Gamma(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}') \left(\mathbf{n}(\mathbf{x}') \cdot \mathbf{s}'\right) L(\mathbf{x}', \mathbf{s}') \, d\mathbf{s}' d\mathbf{x}' \ , \tag{4}$$

where the function Γ is the fundamental solution of the RTE in free space (without the boundary conditions), the spatial integral is carried over the boundary of the domain, and $\mathbf{n}(\mathbf{x}')$ is the unit normal to ∂D at $\mathbf{x}'$. The term $\mathbf{n}(\mathbf{x}') \cdot \mathbf{s}'$ is derived from Eq. (1) using the divergence theorem, *i.e.*, it accounts for the conservation of flux.

Equation (4) provides a representation for the general solution of Eq. (3) inside D and on its boundary. It is important to note that the integral in Eq. (4) depends on the radiance on the boundary of D in *all* directions, whereas boundary conditions (2) provide only the incoming radiation. Therefore, in general, Eq. (4) is an integral equation for the radiance.

As a first step for this formulation, we need to find the fundamental solution that appears in Eq. (4). Consider Eq. (3) with a source term Q,

$$\mathbf{s} \cdot \vec{\nabla} L \ = Q(\mathbf{x}, \mathbf{s}) \ . \tag{5}$$

The solution of Eq. (5) can be expressed as

$$L(\mathbf{x},\mathbf{s}) = \int_{\mathbb{R}^d}\int_{S^{d-1}} \Gamma(\mathbf{x},\mathbf{x}',\mathbf{s},\mathbf{s}')Q(\mathbf{x}',\mathbf{s}')\,d\mathbf{s}'d\mathbf{x}' \; , \tag{6}$$

where Γ is the fundamental solution, which satisfies Eq. (5) with a point source at position $\mathbf{x}'$ in direction $\mathbf{s}'$, *i.e.*,

$$\mathbf{s}\cdot\vec{\nabla}\Gamma(\mathbf{x},\mathbf{x}',\mathbf{s},\mathbf{s}') \;=\; \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{s}-\mathbf{s}'), \tag{7}$$

where δ is the Dirac delta function. To solve this equation, we introduce the ray function.

4. THE RAY FUNCTION

Rays arise naturally in the description of light propagation. For our theory we need to define a *ray function*. As a preliminary, we recall that the indicator function* of a set A is defined by

$$\mathbb{1}_A(\mathbf{x}) = \begin{cases} 1 & \mathbf{x}\in A \\ 0 & \mathbf{x}\notin A\,. \end{cases} \tag{8}$$

Lange showed that the normal derivative of the indicator function is a surface delta function,[9] *i.e.*,

$$-\mathbf{n}\cdot\vec{\nabla}\mathbb{1}_A(\mathbf{x}) = \delta_{\partial A}(\mathbf{x}), \tag{9}$$

where $\mathbf{n}$ is the outward normal to the surface of A and $\delta_{\partial A}(\mathbf{x})$ is the delta function of the surface (or boundary) of A.

In one-dimension it is the well-known that

$$H'(x) = \delta(x)\;,$$

where H isHeaviside's step function,

$$H(x) \;=\; \mathbb{1}_{x>0} = \begin{cases} 1 & x>0 \\ 0 & x<0\,. \end{cases}$$

This is special case of relation (9), since $H(x)$ is the indicator function of the half-line $x>0$, $\delta(x)$ is concentrated around $x=0$, *i.e.*, at the boundary of the half-line $x>0$, and $\mathbf{n}=-1$.

In general, a ray is an oriented half-line that emanates from a point $\mathbf{x}_0\in\mathbb{R}^d$ (or at infinity) and has a fixed direction on the unit sphere $\mathbf{s}\in S^{d-1}$. The set of points that describes this ray is

$$R(\mathbf{x}_0,\mathbf{s}) \;=\; \{\mathbf{x}_0 + t\mathbf{s} \mid \forall t\geq 0\}\,. \tag{10}$$

We define a *ray function* as the indicator function of the ray set (10),

$$\mathbb{1}_{R(\mathbf{x}_0,\mathbf{s})}(\mathbf{x}) \;=\; \begin{cases} 1 & \mathbf{x}\in R(\mathbf{x}_0,\mathbf{s}) \\ 0 & \mathbf{x}\notin R(\mathbf{x}_0,\mathbf{s})\,. \end{cases} \tag{11}$$

The ray function has translational symmetry. In particular,

$$\mathbb{1}_{R(\mathbf{x}_0,\mathbf{s})}(\mathbf{x}) = \mathbb{1}_{R(\mathbf{0},\mathbf{s})}(\mathbf{x}-\mathbf{x}_0)\,. \tag{12}$$

The boundary of the ray $R(\mathbf{x}_0,\mathbf{s})$ is the point $\mathbf{x}_0$ from which it emanates. Although $\mathbf{x}_0$ is a singular point of $R(\mathbf{x}_0,\mathbf{s})$, from symmetry considerations it follows that the outward normal to $R(\mathbf{x}_0,\mathbf{s})$ at $\mathbf{x}_0$ is defined and has the direction of $-\mathbf{s}$. In addition, it follows from relation (9) that the normal derivative of the ray function is

$$\mathbf{s}\cdot\vec{\nabla}\mathbb{1}_{R(\mathbf{x}_0,\mathbf{s})}(\mathbf{x}) = \delta_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_0). \tag{13}$$

*Also known as the characteristic function and Dirac measure.

5. THE FUNDAMENTAL SOLUTION

By inspection, a particular solution of Eq. (7) is

$$\Gamma(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}') = \mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})\ \delta(\mathbf{s} - \mathbf{s}')\ , \tag{14}$$

where we have used identity (13) with $\mathbf{x}_0 = \mathbf{x}'$. Equation (14) is the fundamental solution. It follows from (12) and (14) that Γ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$, symmetric in $\mathbf{s}$ and $\mathbf{s}'$, and has the translational symmetry

$$\Gamma(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}') = \Gamma(\mathbf{x} - \mathbf{x}', \mathbf{s} - \mathbf{s}')\ . \tag{15}$$

Using (14) and (6), a solution of (5) can be expressed as

$$L(\mathbf{x}, \mathbf{s}) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})\ \delta(\mathbf{s} - \mathbf{s}') Q(\mathbf{x}', \mathbf{s}')\, d\mathbf{s}' d\mathbf{x}'\ . \tag{16}$$

Loosely speaking, this describes the radiance in free space as propagating along rays, which are the characteristic curves of the transport equation (5).

The general solution of Eq. (5) may also include any solution in the nullspace of this equation (with $Q = 0$). This nullspace contains all the rays that emanate from infinity in any direction, along which the radiance is constant. For the problems considered here there is no such background radiation. Hence, we impose the *no background radiation condition*,

$$L(\mathbf{x}, \mathbf{s}) = 0\ , \qquad \text{as} \quad |\mathbf{x}| \to \infty, \quad \mathbf{s} \cdot \mathbf{n}(\mathbf{x}) < 0\ , \tag{17}$$

where $\mathbf{n}(\mathbf{x})$ is any outward normal direction, *i.e.*, $\mathbf{x} \sim \mathbf{n}(\mathbf{x})|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$. The particular solution (16) naturally satisfies (17) due to the ray function in its integrand. From the uniqueness theory of the RTE it follows that the fundamental solution (16) is the unique solution of (5) subject to condition (17).

5.1 1D case

Specializing this theory to one-dimension, let x be in $\mathbb{R}$ and denote the directions along the $x-$axis by $s \in S^0 = \{\pm 1\}$. The fundamental solution (14) reduces to

$$\Gamma(x, x', s, s')\ =\ \mathbb{1}_{R(x',s)}(x)\ \delta(s - s') \tag{18}$$

and the solution (16) becomes

$$L(x, s) = \int_{x \in \mathbb{R}} \sum_{s' \in S^0} \mathbb{1}_{R(x',s)}(x)\ \delta(s - s') Q(x', s')\, dx'\ , \tag{19}$$

where the integral over s has been replaced with summation over S^0.

As a first example, we choose a point source at the origin in the positive x direction, *i.e.*,

$$Q(x, s) = \delta(x)\delta(s - 1)\ .$$

Then

$$L(x, s) \ = \ \int_{-\infty}^{\infty} \sum_{s'=\pm 1} \mathbb{1}_{R(x',s)}(x)\delta(s - s')\delta(x')\delta(s' - 1)dx' \tag{20}$$

$$= \ \int_{-\infty}^{\infty} \sum_{s'=\pm 1} H(x - x')\delta(s - 1)\delta(x')dx' = H(x)\delta(s - 1)\ , \tag{21}$$

where in the first step we have used

$$\mathbb{1}_{R(x',+1)}(x) = H(x - x')\ .$$

This shows that the radiance is equal to 1 on the half-line $x > 0$ in the positive x direction,, and is zero otherwise.

Similarly, choosing a point source at the origin in the negative x direction, *i.e.*,

$$Q(x, s) = \delta(x)\delta(s + 1)$$

we get

$$L(x, s) = \int_{x' \in \mathbb{R}} \sum_{s' \in S^0} H(x' - x)\delta(s - s')\delta(x')\delta(s' + 1)dx' = H(-x)\delta(s + 1) \, . \tag{22}$$

where we have used

$$\mathbb{1}_{R(x',-1)}(x) = H(x' - x) \, ,$$

which is the step function on the negative half-line $x < -x'$. In this case the radiance is equal to 1 on the half-line $x < 0$ in the negative x direction and is zero otherwise.

5.2 3D case

Next we compute the fundamental solution in three dimensions. Intuitively, we expect that the fundamental solution is proportional to the product of a Heaviside step function along $\mathbf{s}$ and delta functions in the two coordinates orthogonal to $\mathbf{s}$. To derive this result, we will derive the fundamental solution (14) using the Fourier transform method.

Let the Fourier transform of $f(\mathbf{x})$ be

$$F[f(\mathbf{x})](\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \, d\mathbf{x} \, .$$

Formally applying the Fourier transform to Eq. (7) gives

$$\mathbf{s} \cdot i\mathbf{k} \, F[\Gamma(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}')](\mathbf{k}) \; = \; e^{-i\mathbf{k}\cdot\mathbf{x}'}\delta(\mathbf{s} - \mathbf{s}') \, . \tag{23}$$

On the other hand, applying the Fourier transform to relation (13) with $\mathbf{x}_0 = \mathbf{x}'$ gives

$$\mathbf{s} \cdot i\mathbf{k}F[\mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})](\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}'} \, . \tag{24}$$

Comparing (23) and (24) we arrive at

$$F[\Gamma(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}')] = F[\mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})]\delta(\mathbf{s} - \mathbf{s}') \, . \tag{25}$$

Taking the inverse Fourier transform of (25) recovers the fundamental solution (14).

In what follows, we show that

$$F[\mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})](\mathbf{k}) = \frac{e^{-i\mathbf{k}\cdot\mathbf{x}'}}{i(\mathbf{s}\cdot\mathbf{k})} \; + \; \pi\delta(\mathbf{s}\cdot\mathbf{k}) \, . \tag{26}$$

At first inspection, Eq. (26) might seem inconsistent with Eq. (24) because of the additional delta function on the right-hand side of (26). However, this delta function ensures that the ray function satisfies the no-background radiation condition (17). In other words, it removes the degeneracy of the inverse Fourier transform when applied to generalized functions.[10]

First, we use translational symmetry relation (12) and find

$$F[\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x} - \mathbf{x}')] = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x} - \mathbf{x}')d\mathbf{x}. \tag{27}$$

By the shifting principle of Fourier transforms, it follows that

$$F[\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x}-\mathbf{x}')] = e^{i\mathbf{k}\cdot\mathbf{x}'}F[\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x})]. \tag{28}$$

Thus, it is sufficient to find $F[\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x})]$.

We first consider the special case in which $\mathbf{s} = \hat{\mathbf{z}}$. For that case, the ray function is given by

$$\mathbb{1}_{R(0,\hat{\mathbf{z}})}(\mathbf{x}) = \delta(x)\delta(y)H(z)\ . \tag{29}$$

This case is convenient since it separates the dependence on the coordinates. Using

$$\int e^{-ik_x x}\delta(x)\mathrm{d}x = 1\ ,$$
$$\int e^{-ik_y y}\delta(y)\mathrm{d}y = 1\ ,$$

and the result of Burrows and Colwell,[11]

$$\int e^{-ik_z z}H(z)\mathrm{d}z = \pi\delta(k_z) + \frac{1}{ik_z}, \tag{30}$$

we arrive at the result

$$F[\mathbb{1}_{R(0,\hat{\mathbf{z}})}(\mathbf{x})] = \pi\delta(k_z) + \frac{1}{ik_z}. \tag{31}$$

which is Eq. (26) for this special case.

We now discuss how to extend this result to a general direction s. We introduce the polar angle θ and azimuthal angle φ and write

$$\mathbf{s} = \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}. \tag{32}$$

We introduce the unit vectors

$$\hat{\mathbf{v}}_1 = \cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}, \tag{33}$$
$$\hat{\mathbf{v}}_2 = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}, \tag{34}$$
$$\hat{\mathbf{v}}_3 = \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}, \tag{35}$$

that form a rotated orthonormal coordinate system in which $\hat{\mathbf{v}}_3 = \mathbf{s}$. Suppose $\mathbf{x} = \xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 + \xi_3\mathbf{v}_3$. It follows that

$$\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x}) = \delta(\xi_1)\delta(\xi_2)H(\xi_3). \tag{36}$$

Now suppose that $\mathbf{k} = \kappa_1\hat{\mathbf{v}}_1 + \kappa_2\hat{\mathbf{v}}_2 + \kappa_3\hat{\mathbf{v}}_3$. Computing the Fourier transform of $\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x})$ in this rotated coordinate system and using Eq. (31), we find that

$$F[\mathbb{1}_{R(0,\mathbf{s})}(\mathbf{x})] = \pi\delta(\kappa_3) + \frac{1}{i\kappa_3}. \tag{37}$$

Substituting

$$\kappa_3 = \mathbf{k}\cdot\mathbf{s}, \tag{38}$$

we obtain Eq. (26).

5.2.1 Remark

Relation (26) can also be obtained as follows. We decompose the ray function as

$$\mathbb{1}_{R(\mathbf{0},\mathbf{s})}(\mathbf{x}) \;=\; \frac{1}{2}\mathbb{S}_{R(\mathbf{0},\mathbf{s})}(\mathbf{x}) + \frac{1}{2}\mathbb{1}_{(\mathbf{s}\cdot\mathbf{x})}(\mathbf{x}) \;, \tag{39}$$

where we have defined the generalized sgn function along the direction of s as

$$\mathbb{S}_{R(\mathbf{0},\mathbf{s})}(\mathbf{x}) \;=\; \begin{cases} -1 & \mathbf{x}\in R(\mathbf{0},-\mathbf{s}) \\ 1 & \mathbf{x}\in R(\mathbf{0},\mathbf{s}) \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

and $\mathbb{1}_{(\mathbf{s}\cdot\mathbf{x})}(\mathbf{x})$ is the indicator function along the direction of s going through the origin. Following[11] it can be shown in the sense of the Cauchy principle value that

$$F\left[\mathbb{S}_{R(\mathbf{0},\mathbf{s})}(\mathbf{x})\right] \;=\; \frac{2}{i(\mathbf{s}\cdot\mathbf{k})} \;. \tag{41}$$

In addition, a direct calculation yields

$$F[\mathbb{1}_{(\mathbf{s}\cdot\mathbf{x})}] = 2\pi\delta(\mathbf{s}\cdot\mathbf{k}) \;. \tag{42}$$

Relation (26) follows from Eqns. (39)–(42).

6. HALF-SPACE PROBLEM

Using the fundamental solution (14) we conclude from Eq. (4) that the solution of the boundary-value problem can be expressed as

$$L(\mathbf{x},\mathbf{s}) = \int_{\partial D}\int_{S^{d-1}} \mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})\;(\mathbf{n}(\mathbf{x}')\cdot\mathbf{s}')L(\mathbf{x}',\mathbf{s}')\,d\mathbf{s}'d\mathbf{x}' \;. \tag{43}$$

As a simple example of this theory, consider Eq. (3) in the half-space $x>0$ in $\mathbb{R}^3$. The incoming radiation is provided on the $y-z$ plane for all directions within the right hemisphere $S^+ = S^2\cap\{s_1>0\}$, *i.e.*,

$$L(\mathbf{x},\mathbf{s}) = L_{\text{in}}(y,z,\mathbf{s}) \;, \quad \mathbf{x}=\{0,y,z\}, \quad \mathbf{s}\in S^+ \;.$$

In this setup, no radiation is reflected backwards. Therefore, throughout the half-space including the $y-z$ plane, the solution is $L(\mathbf{x},\mathbf{s})=0$ for all directions within the left hemisphere, $s_1\le 0$. Hence, the angular integral in (43) can be restricted to S^+. Using (43) and (14), the radiance in the half-space $x>0$ for any $\mathbf{s}\in S^+$ is

$$\begin{aligned} L(\mathbf{x},\mathbf{s}) &= \iint_{\mathbf{x}'=\{0,y',z'\}}\int_{\mathbf{s}'\in S^+} \mathbb{1}_{R(\mathbf{x}',\mathbf{s}')}(\mathbf{x})\;\delta(\mathbf{s}-\mathbf{s}')\mathbf{n}\cdot\mathbf{s}'L_{\text{in}}(y',z',\mathbf{s}')\,d\mathbf{s}'d\mathbf{x}' \\ &= \iint_{\mathbf{x}'=\{0,y',z'\}} \mathbb{1}_{R(\mathbf{x}',\mathbf{s})}(\mathbf{x})\;\mathbf{n}\cdot\mathbf{s}\;L_{\text{in}}(y',z',\mathbf{s})\,d\mathbf{x}' \;, \end{aligned} \tag{44}$$

where $\mathbf{n}$ is the unit normal to the $y-z$ plane in the positive x direction. Let $\mathbf{s}=(\cos\theta,\sin\theta\cos\phi,\sin\theta\sin\phi)$, where θ and ϕ denote the polar and azimuthal angles, respectively. Thus, $\mathbf{n}\cdot\mathbf{s}=\cos\theta$. For any point $\mathbf{x}$ in the half-space and direction $\mathbf{s}\in S^+$, the indicator function in (44) is nonzero only at one point on the $y-z$ plane, $\mathbf{x}_0'=(0,y_0',z_0')$, which can be obtained by tracing backwards the ray from $\mathbf{x}$ in the direction of $-\mathbf{s}$. A straightforward calculation shows that

$$(y_0',z_0') = (y,z) - x\tan\theta(\cos\phi,\sin\phi) \;.$$

It follows from (44) that

$$L(\mathbf{x},\mathbf{s}) \;=\; L_{\text{in}}(y_0',z_0',\mathbf{s})\cos\theta \;. \tag{45}$$

This shows that the radiance is transported along rays from the source plane into the half-space. The $\cos\theta$ factor accounts for the conservation of radiative flux. This result can be thought of as manifestation of Lambert's cosine law.

7. APPLICATION TO NONIMAGING OPTICS

We consider a general nonimaging concentrator problem, depicted in Fig. 1. We denote by D the interior of the concentrator. Its boundary is comprised of the surfaces of the aperture, receiver, and reflector,

$$\partial D = S_{\text{aperture}} \cup S_{\text{receiver}} \cup S_{\text{reflector}} \,.$$

The boundary conditions are prescribed as follows.

- At the aperture the incoming radiance is prescribed for all directions entering the domain:

$$L(\mathbf{x}, \mathbf{s}) = L_{\text{in}}(\mathbf{x}, \mathbf{s}) \,, \quad \mathbf{x} \in S_{\text{aperture}}, \quad \mathbf{s} \cdot \mathbf{n} < 0 \,. \tag{46a}$$

- No radiation enters from the receiver:

$$L(\mathbf{x}, \mathbf{s}) = 0 \,, \quad \mathbf{x} \in S_{\text{receiver}}, \quad \mathbf{s} \cdot \mathbf{n} < 0 \,. \tag{46b}$$

- The reflector is a specular reflecting surface (Snell's Law):

$$L(\mathbf{x}, \mathbf{s}) = L(\mathbf{x}, \mathcal{R}\mathbf{s}) \quad \mathbf{x} \in S_{\text{reflector}}, \quad \mathbf{s} \cdot \mathbf{n} < 0 \,. \tag{46c}$$

  where $\mathcal{R}\mathbf{s} = \mathbf{s} - 2(\mathbf{s} \cdot \mathbf{n})\mathbf{n}$ is the reflection operator.

Thus, the problem consists of solving Eq. (3) with the boundary conditions (46). We note that for some design, the reflector may be non-contiguous and the receiver may be nonplanar. Moreover, the receiver may be enclosed inside D, such as a cylindrical tube commonly used in solar-thermal applications.[1] We recap that since the light exiting from the aperture and / or the receiver is unknown to begin with, Eq. (43) is an integral equation for the radiance.

Many computational methods have been developed to solve the RTE (1). These fall under two main categories:

1. Particle methods, including deterministic ray tracing or Monte Carlo simulations.
2. Mesh-based methods, where the spatial domain and directions are discretized on a mesh or finite elements.

Of these, ray-tracing is the most commonly used computational method in nonimaging optics. In standard ray-tracing, rays emanate from chosen points and directions along the source's surface and their trajectories are traced until each ray eventually exists the receiver or aperture. Consequently, a limitation of ray tracing is that it does not allow us to specify points on the receiver. This, in turn, makes it difficult to estimate the irradiance (radiance integrated on the outgoing hemisphere) on the receiver. Another limitation of ray tracing and other methods is that any change in the light source requires another ray tracing computation. This is important for estimating concentrators' performance as the sun moves across the sky.

The transport equation (3) and Green's function formalisms open the door for developing new ways for calculating irradiance and flux in nonimaging optics designs. To outline such an approach, the angular integral in (43) can be broken up into the incoming and outgoing contributions on the boundary of D, *i.e.*,

$$L(\mathbf{x}, \mathbf{s}) = \int_{\partial D} \int_{S^+} \mathbb{1}_{R(\mathbf{x}', \mathbf{s})}(\mathbf{x}) \, (\mathbf{n}(\mathbf{x}') \cdot \mathbf{s}') L(\mathbf{x}', \mathbf{s}') \, d\mathbf{s}' d\mathbf{x}' \; + \int_{\partial D} \int_{S^-} \mathbb{1}_{R(\mathbf{x}', \mathbf{s})}(\mathbf{x}) \, (\mathbf{n}(\mathbf{x}') \cdot \mathbf{s}') L(\mathbf{x}', \mathbf{s}') \, d\mathbf{s}' d\mathbf{x}' \,. \tag{47}$$

Each of these integrals can be broken up to the contributions over the source, reflector, and absorber, and the boundary conditions (46) can then be applied appropriately. In analogy to Green's function methods applied to second-order PDEs, one can seek the *surface Green's function* for the problem on the bounded domain, so that the solution can be expressed as

$$L(\mathbf{x}, \mathbf{s}) = \int_{S_{\mathrm{aperture}}} \int_{S^+} G(\mathbf{x}, \mathbf{x}', \mathbf{s}, \mathbf{s}') \, (\mathbf{n}(\mathbf{x}') \cdot \mathbf{s}') L_{\mathrm{in}}(\mathbf{x}', \mathbf{s}') \, d\mathbf{s}' d\mathbf{x}' \, , \tag{48}$$

where the integration is carried over the source in the incoming directions. Thus, the problem is transformed to finding the surface Green's function G. Once G has been found, the radiance, irradiance and fluxes can be readily calculated using (48). We note that G depends only on the geometry of the concentrator and the kind of boundary conditions (emissive or reflective), but not on the specific distribution of the radiance that is incident on the aperture.

This approach offers several potential advantages over known methods:

1. The integration is reduced to surfaces, compared with volumetric mesh-based methods for solving the RTE.
2. The outgoing irradiance can be computed accurately at specified points on the receiver, which enables accurate computation of the flux as well.
3. If the distribution of the incoming light on the aperture varies in time, as in the case of the sun moving across the sky, the outgoing fluxes can be computed for all the different light distributions "in one shot".

8. SUMMARY AND CONCLUSIONS

Nonimaging optics problems are described using radiative transfer theory. It is shown that the radiance can be expressed as a solution of an integral equation, which depend on *ray functions*. The properties of ray function are studied and their Fourier theory is developed. This formalism may be useful for nonimaging optics designs, because it opens the door for developing new computational approaches.

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