Light Propagation in Biological Tissue*

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ABSTRACT

Biological tissue scatters light mainly in the forward direction where the scattering phase function has a narrow peak. This peak makes it difficult to solve the radiative transport equation. However, it is just for forward peaked scattering that the Fokker-Planck equation provides a good approximation, and it is easier to solve than the transport equation. Furthermore, the modification of the Fokker-Planck equation by Leakeas and Larsen provides an even better approximation and is also easier to solve. We demonstrate the accuracy of these two approximations by solving the problem of reflection and transmission of a plane wave normally incident on a slab composed of a uniform scattering medium.

1. INTRODUCTION

Light propagation in a random medium is governed by the radiative transport equation. This equation takes account of scattering from inhomogeneities and of absorption by the medium. Exact solutions to it are only known for relatively simple problems. Because it is an integrodifferential equation, this equation is difficult to solve numerically. Furthermore, biological tissue scatters light strongly in the forward direction.\textsuperscript{1} Hence, the scattering phase function is sharply peaked in the forward direction. This sharp peak makes computing solutions to the radiative transport equation even more difficult.

Kim and Keller proposed replacing the radiative transport equation by the Fokker-Planck equation for light propagation in biological tissue.\textsuperscript{2} The Fokker-Planck equation is applicable to sharply forward peaked scattering.\textsuperscript{3,4} We review the results of the radiance transmitted and backscattered from a slab of a random medium computed using the radiative transport equation and the Fokker-Planck equation. We also show results using a more complicated alternative to the Fokker-Planck equation, due to Leakeas and Larsen\textsuperscript{5} that agrees even better with the radiative transport equation.

2. THE TRANSPORT EQUATION AND THE FOKKER-PLANCK EQUATION

The radiance or specific intensity $\Psi(\omega, r)$ is the radiant power per unit solid angle per unit area perpendicular to the direction of propagation. It depends on the unit direction vector $\omega$ and position vector $r$ and satisfies

$$\omega \cdot \nabla \Psi(\omega, r) + \Sigma_a \Psi(\omega, r) = \int_{\Omega} f(\omega \cdot \omega') \Psi(\omega', r) d\omega' - \Sigma_s \Psi(\omega, r).$$

(1)

Here $\Sigma_a$ is the absorption coefficient, $\Sigma_s$ is the scattering coefficient, and $\Omega$ denotes the unit sphere. The fraction of light incident in the direction $\omega'$ which is scattered into the direction $\omega$ is given by $f(\omega \cdot \omega')/\Sigma_s$. Therefore $\Sigma_s$ is related to the scattering phase function $f$ by

$$\Sigma_s = \int_{\Omega} f(\omega \cdot \omega') d\omega'.$$

(2)
The degree of anisotropy is often measured by $g$, the mean value of the cosine of the scattering angle $\Theta$, where $\cos \Theta = \mathbf{\omega} \cdot \mathbf{\omega}'$, and $g$ is defined by

$$g = \frac{2\pi}{\Sigma_s} \int_0^{\pi} \cos \Theta f(\cos \Theta) \sin \Theta d\Theta.$$  \hfill (3)

When the scattering is isotropic $g = 0$ while when it is all in the forward direction $\Theta = 0$, then $g = 1$.

For $f$ strongly peaked in the forward direction, the integral operator in (1) can be simplified by expanding $\Psi(\mathbf{\omega}, r)$ in a Taylor series with respect to $\mathbf{\omega}'$ about $\mathbf{\omega}' = \mathbf{\omega}$,

$$\Psi(\mathbf{\omega}, r) = \Psi(\mathbf{\omega}, r) + (\mathbf{\omega}' - \mathbf{\omega}) \cdot \nabla_{\mathbf{\omega}} \Psi(\mathbf{\omega}, r) + \frac{1}{2} [(\mathbf{\omega}' - \mathbf{\omega}) \cdot \nabla_{\mathbf{\omega}}]^2 \Psi(\mathbf{\omega}, r) + O[(\mathbf{\omega}' - \mathbf{\omega})^3].$$  \hfill (4)

When (4) is used in the integral in (1), the first term cancels $\Sigma_s \Psi$, the second term vanishes because it is odd, and the third term yields $\frac{1}{2} \Sigma_{tr} L \Psi$. Here $\Sigma_{tr}$ is the transport cross-section defined by $\Sigma_{tr} = \Sigma_s (1 - g)$, and $L$ is the Laplace-Beltrami or angular momentum operator. Pomraning$^3$ and Larsen$^4$ provide complete derivations of this approximation, and Polishchuk et al$^6$ also proposed it.

In terms of the polar coordinates $(\theta, \varphi)$ of $\mathbf{\omega}$, with $\mu = \cos \theta$, $L$ is

$$L = \partial_\mu (1 - \mu^2) \partial_\mu + (1 - \mu^2)^{-1} \partial_\varphi^2.$$  \hfill (5)

The simplification of the integral by using (4), and omission of the term $O[(\mathbf{\omega}' - \mathbf{\omega})^3]$, reduces (1) to the Fokker-Planck equation

$$\mathbf{\omega} \cdot \nabla \Psi(\mu, \varphi, r) + \Sigma_s \Psi(\mu, \varphi, r) = \frac{1}{2} \Sigma_{tr} L \Psi(\mu, \varphi, r).$$  \hfill (6)

The advantage of (6) over (1) is that it is easier to solve a partial differential equation than an integrodifferential equation.

### 3. THE LEAKEAS-LARSEN EQUATION

The eigenfunctions of the operator $S$ on the right side of (1), i.e. the integral operator minus $\Sigma_s$, are the spherical harmonics $Y_{nm}(\mathbf{\omega})$

$$SY_{nm}(\mathbf{\omega}) = \int_\omega f(\mathbf{\omega}' \cdot \mathbf{\omega}) Y_{nm}(\mathbf{\omega}') d\mathbf{\omega}' - \Sigma_s Y_{nm}(\mathbf{\omega}') = (f_n - \Sigma_s) Y_{nm}(\mathbf{\omega}).$$  \hfill (7)

The corresponding eigenvalue is $f_n - \Sigma_s$ where $f_n$ is defined in terms of the Legendre function $P_n$ by

$$f_n = 2\pi \int_{-1}^{1} f(\mu) P_n(\mu) d\mu.$$  \hfill (8)

Since $f_0 = \Sigma_s$, the eigenvalue of $S$ for $n = 0$ is zero, and all the other eigenvalues of $S$ are negative because $f_n < \Sigma_s$. The eigenfunctions of the Laplace-Beltrami operator $L$ are also $Y_{nm}(\mathbf{\omega})$ with eigenvalues $-n(n+1)$

$$LY_{nm}(\mathbf{\omega}) = -n(n+1) Y_{nm}(\mathbf{\omega}).$$  \hfill (9)

In view of (7) and (9), we can write the operator $S$ as a function of the Laplace-Beltrami operator $L$ as

$$S = F(L).$$  \hfill (10)

The function $F$ is just the relation between corresponding eigenvalues of $S$ and $L$,

$$f_n - \Sigma_s = F[-n(n + 1)].$$  \hfill (11)

Instead of the exact relation (10), Leakeas and Larsen$^5$ proposed the rational fraction approximation

$$S \approx \alpha L (I - \beta L)^{-1}.$$  \hfill (12)
For $n = 0$ the eigenvalue of $S$ and that of the operator on the right side of (12) are both zero. Leakeas and Larsen\textsuperscript{5} chose $\alpha$ and $\beta$ to make the eigenvalues agree for $n = 1$ and $n = 2$, which yields

$$\alpha = \frac{1}{2}(s_1 - f_1)(1 + 2\beta), \quad \beta = \frac{2s_1 - 3f_1 + f_2}{6(f_1 - f_2)}.$$  \tag{13}

By using (12) to replace $S$ in (1) we obtain the Leakeas-Larsen equation

$$\omega \cdot \nabla \Psi + \Sigma_0 \Psi = \alpha L(I - \beta L)^{-1}\Psi.$$  \tag{14}

\section*{4. REFLECTION AND TRANSMISSION OF A PLANE WAVE BY A SLAB OR HALF-SPACE}

We now consider the radiance produced by a plane wave normally incident upon a slab composed of a scattering medium bounded by the planes $z = 0$ and $z = z_0$. Then $\Psi = \Psi(\mu, z)$ will depend only upon $z$ and $\mu = \cos \theta$, where $\theta$ is the angle between $\omega$ and the positive $z$-axis. In this case (1) reduces to

$$\mu \partial_z \Psi(\mu, z) + \Sigma_0 \Psi(\mu, z) \Psi(\mu, z) = \int_{-1}^{+1} h(\mu, \mu') \Psi(\mu', z) d\mu' - \Sigma_s \Psi(\mu, z).$$  \tag{15}

Here $h(\mu, \mu')$ is the azimuthally integrated phase function. To define it we write $\mu = \cos \theta$, $\mu' = \cos \theta'$, we write $h$ as

$$h(\mu, \mu') = \int_0^{2\pi} f(\mu \mu' + (1 - \mu'^2)^{1/2}(1 - \mu'^2)^{1/2}\cos(\varphi - \varphi')) d(\varphi - \varphi').$$  \tag{16}

At $z = 0$ we prescribe the incident radiance to be normally incident ($\mu = 1$) with intensity $F$,

$$\Psi(\mu, 0) = F\delta(\mu - 1), \quad 0 < \mu \leq 1.$$  \tag{17}

At $z = z_0$ we assume that there is no incident radiance, so

$$\Psi(\mu, z_0) = 0, \quad -1 \leq \mu < 0.$$  \tag{18}

We assume that the refractive index of the slab is the same as that of the surrounding medium, so that there is no reflection at either the front or rear face of the slab.

When $\Psi = \Psi(\mu, z)$, the Fokker-Planck equation (6) and the Leakeas-Larsen equation (14) become respectively

$$\mu \partial_z \Psi(\mu, z) + \Sigma_0 \Psi(\mu, z) = \frac{1}{2} \Sigma_s L \Psi(\mu, z),$$  \tag{19}

$$\mu \partial_z \Psi(\mu, z) + \Sigma_0 \Psi(\mu, z) = \alpha L(I - \beta L)^{-1}\Psi(\mu, z).$$  \tag{20}

Here $L = \partial_\mu (1 - \mu^2) \partial_\mu$ and the boundary conditions are still (17) and (18).

Kim and Keller\textsuperscript{2} give complete details of the numerical method used to compute the following results. We omit those details in this discussion.

\section*{5. NUMERICAL RESULTS}

\subsection*{5.1. Exponential Phase Function}

First we compute the solution for the exponential phase function

$$f(\cos \Theta) = \frac{\Sigma_s \exp \left[ - \frac{(1 - \cos \Theta)}{\varepsilon} \right]}{2\pi \varepsilon \left( 1 - \exp \left[ -2/\varepsilon \right] \right)}.$$  \tag{21}
Figure 1. Backscattered radiance $\Psi(\mu, 0)$ (upper figure) and transmitted radiance $\Psi(\mu, z_0)$ (lower figure) for a slab of thickness $z_0 = 1$ cm with $\Sigma_s = 100$ cm$^{-1}$ and $\Sigma_a = 1$ cm$^{-1}$. The exponential phase function with $\epsilon = 0.10$ is used.

Here $\epsilon$ is a small positive parameter. The corresponding function $h$ is, with $I_0$ the modified Bessel function,

$$h(\mu, \mu') = \frac{\Sigma_s}{\epsilon} \frac{\exp[-(1-\mu\mu')/\epsilon]}{1-\exp[-2/\epsilon]} I_0 \left( \epsilon^{-1} \sqrt{1-\mu^2} \right).$$  \hspace{1cm} (22)

The computed backscattered radiance $\Psi(\mu, 0), -1 \leq \mu \leq 0$ is shown in upper plot in Fig. 1 and the computed transmitted radiance $\Psi(\mu, z_0), 0 \leq \mu \leq 1$ is shown in the lower plot in Fig. 1. Here $z_0 = 1.0$ cm, $\Sigma_a = 1.0$ cm$^{-1}$, $\Sigma_s = 100.0$ cm$^{-1}$, $N = 128$ and $\epsilon = 0.1$. Then the anisotropy factor is $g = 0.90$ and the transport cross-section is $\Sigma_{tr} = 10$ cm$^{-1}$. Thus the slab thickness is 100 scattering mean free paths or 10 transport mean free paths. The Fokker-Planck approximation is close to the transport equation solution for the transmitted radiance, except near the grazing angle $\theta = \pi/2$. It is only fairly close for the backscattered radiance. The Leakeas-Larsen approximation is very close to the transport equation solution in both cases at all angles. The solution of the modified Leakeas-Larsen equation is close to the transport equation solution for transmission and less close for backscattering.

Fig. 2 shows results for the same problem with $\epsilon = 0.01$. Then $g = 0.99$ and $\Sigma_{tr} = 1$ cm$^{-1}$. Thus the phase function $f$ has a much sharper peak, but the slab thickness is just one transport mean free path. The
consequence is that the Fokker-Planck approximation and modified Leakeas-Larsen approximation are almost indistinguishable from the transport equation solution for transmission, and very close to it for backscattering. The Leakeas-Larsen result is indistinguishable from the transport result in both cases.

Fig. 3 shows results for $\epsilon = 0.1$ but with $z_0 = 0.05$ cm. Then $\Sigma_{tr} = 10$ cm$^{-1}$ so the slab thickness is just one half a transport mean free path and just five scattering mean free paths. For backscattering, the Fokker-Planck and modified Leakeas-Larsen results are still qualitatively correct, but not very close to the transport result. They are closer for transmission. The Leakeas-Larsen result is still very close to the transport result in both cases, except for the transmitted radiance at $\mu = 0.5$.

5.2. Henyey-Greenstein Phase Function

Next we compute the solution for the Henyey-Greenstein phase function

$$f(\cos \Theta) = \frac{\Sigma_s}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \Theta)^{3/2}}.$$  \hspace{1cm} (23)
The constant $g$ is exactly the anisotropy factor, and the corresponding function $h$ is

$$h(\mu, \mu') = \frac{2(1 - g^2)}{\pi(a - b)\sqrt{a + b}} E\left(\frac{2b}{a + b}\right).$$

Here $a = 1 + g^2 - 2g\mu\mu'$, $b = 2g\sqrt{1 - \mu^2}\sqrt{1 - \mu'^2}$ and $E(k)$ is the complete elliptic integral of the second kind. We choose $z_0 = 0.1$ cm, $\Sigma_a = 1$ cm$^{-1}$, $\Sigma_s = 100$ cm$^{-1}$, $g = 0.90$ and $N = 128$.

Fig. 4 shows the results. It is well known that the Fokker-Planck equation is not a good approximation to the transport equation when $f$ is given by (23) because $f$ is not sufficiently peaked, and Fig. 4 confirms that. Even the Leakeas-Larsen result is not close to the transport result at all angles, being the least close for backscattering. For large values of $z_0$, the two approximations are somewhat closer to the transport results, but not significantly.

6. CONCLUSIONS

The Fokker-Planck equation provides a good approximation to the transmitted radiance given by the scalar transport equation, provided that the scattering phase function is sharply peaked in the forward direction.
Figure 4. Backscattered radiance (upper figure) and transmitted radiance (lower figure) for the Henyey-Greenstein phase function with $g = 0.90$ in a slab of thickness $z_0 = 0.1 \text{ cm}$ with $\Sigma_s = 100 \text{ cm}^{-1}$ and $\Sigma_a = 1 \text{ cm}^{-1}$.

It is qualitatively correct, but less accurate, for the backscattered radiance. The Leakeas-Larsen equation is much more accurate for both the transmitted and backscattered radiance, yielding results which are almost indistinguishable from those of the transport equation.

These results suggest that to calculate light propagation in biological tissue, for which the scattering is strongly peaked in the forward direction, the Fokker-Planck and Leakeas-Larsen equations should be very useful.

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