Optical and Millimeter Wave Pulse Propagation Through Fog and Rain Layers

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INTRODUCTION

Pulse propagation in random media is studied by solving the two-frequency radiative transfer equation and transforming the solution into the time domain via Fourier Transforms. We briefly review several known techniques of solving the radiative transfer equation and describe a simple and effective finite element method which allows for the investigation of pulse propagation through highly anisotropic media. Specifically, pulse delay and broadening are calculated by solving the radiative transfer equation with empirically determined coefficients for fog and rain layers. In addition, first-order multiple scattering and diffusion approximations are compared to the solutions of the radiative transfer equation, and are found to be useful depending on the values of the single-scattering albedo, optical depth and asymmetry parameter.

THE RADIATIVE TRANSFER EQUATION

The two-frequency radiative transfer equation for the diffuse component of the specific intensity in plane-parallel geometry using the narrow-band approximation,

\[
\mu \frac{dI(\tau, \mu)}{d\tau} = -\eta I(\tau, \mu) + \frac{1}{2} \int_{-1}^{1} p_o(\mu, \mu') I(\tau, \mu') d\mu' + \varepsilon_{r1}(\tau, \mu),
\]

where \( \eta = 1 - \omega_N/\tau_o \), and

\[
\varepsilon_{r1}(\tau, \mu) = p_o(\mu, 1) \exp(-\eta \tau)/(4\pi),
\]

has been derived to model pulse propagation through random media[1]. Here, the independent variables are the cosine of the polar angle, \( \mu = \cos \theta \), the normalized frequency difference, \( \omega_N = L(\omega_1 - \omega_2)/c \), and the optical depth, \( \tau = \rho \sigma_t z \) (0 \( \leq z \leq L \)), where \( \rho \) is the number density and \( \sigma_t \) is the total scattering cross-section. The maximum optical depth is defined as \( \tau_o = \rho \sigma_t L \). The source term, \( \varepsilon_{r1} \), corresponds to a plane-wave directed normally to the plane with unit flux density[2].

The kernel of the integral operator is defined as,

\[
p_o(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} p(\mu, \psi, \mu', \phi') d\psi' d\phi \quad (3)
\]

where

\[
p(\mu, \psi, \mu', \phi') = \frac{W_0 (1 - \tilde{\mu}^2)}{(1 + \tilde{\mu}^2 - 2\mu \cos \xi)^{3/2}},
\]

is the Henyey-Greenstein phase function, \( W_0 \) is the single-scattering albedo, \( \tilde{\mu} \) is the mean cosine of the phase function, and

\[
\cos \xi = \mu \mu' + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} \cos(\phi - \phi')
\]

is the cosine of the angle between incident and scattered wave vectors.

In order to consider a mathematically well-posed problem, we need to prescribe boundary conditions. For our calculations, we assume that the refractive index of the background medium of the plane-parallel slab is the same as that of the surrounding medium, and that the only source comes from the incident wave. This yields the boundary conditions,

\[
I(\tau = 0, \mu) = 0 \quad \text{for} \quad 0 < \mu \leq 1 \quad \text{and} \quad I(\tau = \tau_o, \mu) = 0 \quad \text{for} \quad -1 \leq \mu < 0. \quad (6)
\]

If we consider a tenuous medium where the optical depth is small, or the single scattering albedo is small, then the first-order multiple scattering approximation for the transmitted diffuse intensity may be valid[2]. Furthermore, if the optical depth is large, then a diffusion approximation can be used which assumes that the diffuse intensity develops a nearly isotropic angular distribution due to profound multiple scattering effects. This assumption leads to a diffusion equation for the angularly averaged diffuse intensity[1-4]. In this case, we must also consider approximate boundary conditions.

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which state that there is zero total flux directed in towards the medium[1,2]. The diffusion approximation works best in nearly lossless and isotropic media at large optical depths[3].

**NUMERICAL METHODS**

Exact analytical solutions of (1) are generally not available, and thus, one must rely on accurate numerical methods. There exist many numerical treatments for radiative transfer equations[5] including the method of invariant imbedding, the spherical harmonics method, and the method of discrete ordinates. These methods, however, require a large number of points or modes to accurately resolve the angular distribution of the specific intensity for highly anisotropic media. For our calculations, we chose a finite element method[6] which allows for highly anisotropic scattering with minimal computational cost. This numerical method considers the specific intensity as a Galerkin expansion in \( \mu \) of piecewise continuous linear functions. In addition, this method conserves the flux independently of the phase function[6].

Because the frequency dependence in (1) is decoupled, the solution of (1) with (6) can be computed for each frequency[3]. The spectrum is then inverted into the time domain by an inverse Fast Fourier Transform. Special care must be taken to resolve the spectrum width as well as the number of modes. Since the characteristic pulse shape typically has a sharp rise at early times, this necessitates a very wide spectrum which may not be practical to compute. Therefore, the presence of noise in our pulse plots comes from an under-resolved spectrum.

In order to calculate the received intensity for a given Field of View (FOV) angle, \( \theta_o \), we evaluate the integral,

\[
I_R(\tau_o) = 2\pi \int_{\cos \theta_o}^{1} \exp \left( - (\cos^{-1}(\mu)/\theta_o)^2 \right) I(\tau_o, \mu) d\mu, \tag{7}
\]

by quadrature. After calculating the received intensity in the time domain, \( I_R(\tau_o, t) \), we calculate the delay,

\[
\langle t \rangle = \left( \int t I_R(\tau_o, t) dt \right) / \left( \int I_R(\tau_o, t) dt \right), \tag{8}
\]

and the spread,

\[
(\Delta t)^2 = \left( \int (t - \langle t \rangle)^2 I_R(\tau_o, t) dt \right) / \left( \int I_R(\tau_o, t) dt \right), \tag{9}
\]

by FFT techniques.

![Graph showing the received intensity (dBs) vs. \((t - t_0)/t_0\) for different FOVs.](image)

**RESULTS**

Radiative transfer computations were done with 50 ns optical pulses through 1 km fog layers for wavelengths of 0.5, 1.0, and 10 \( \mu \)m. The parameters for these computations are summarized in Table 1.

<table>
<thead>
<tr>
<th>Wavelength</th>
<th>Optical Depth</th>
<th>Albedo</th>
<th>Mean Cosine</th>
</tr>
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<tbody>
<tr>
<td>0.5 ( \mu )m</td>
<td>14.830</td>
<td>~1.0</td>
<td>0.8411</td>
</tr>
<tr>
<td>1.0 ( \mu )m</td>
<td>16.760</td>
<td>~1.0</td>
<td>0.8299</td>
</tr>
<tr>
<td>10 ( \mu )m</td>
<td>8.837</td>
<td>0.6537</td>
<td>0.8792</td>
</tr>
</tbody>
</table>

**Table 2. Parameters for rain layers[8].**

<table>
<thead>
<tr>
<th>Rain Rate</th>
<th>Optical Depth</th>
<th>Albedo</th>
<th>Mean Cosine</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5 mm/hr</td>
<td>2.759</td>
<td>0.5080</td>
<td>0.5790</td>
</tr>
<tr>
<td>25 mm/hr</td>
<td>14.125</td>
<td>0.5460</td>
<td>0.9457</td>
</tr>
</tbody>
</table>
In addition, computations were done with 10 ns wide 100 GHz pulses through 5 km rain layers. The parameters for these computations are summarized in Table 2. These computations were compared to the first-order scattering and diffusion approximation solutions for a variety of fields of view in order to ascertain their ability to approximate the radiative transfer solution accurately.

The field of view of the receiver greatly affects the spread of the received pulse. A narrow field of view excludes intensities arriving at later times resulting in a narrower pulse. This is accompanied by a lower received intensity as well as a shorter delay time because of the exclusion of intensities from wider angles. This effect can clearly be seen in Fig. 1 and Fig. 2.

The first-order multiple scattering approximation seems most applicable to the "light" rain calculations where the albedo is small, $W < 1$, and higher order scattering effects are negligible, $r < 1$. However, as the rain rate increases, the role of multiple scattering becomes more profound, and the first-order scattering solution becomes inaccurate. The diffusion approximation seems most applicable to the fog computations where multiple scattering dominates. However, for most of these cases, the mean cosine of the phase function is large, $\bar{\mu} \sim 1$, which impedes the diffusion limit process [3]. Thus, the diffusion approximation's performance for optical pulses in fog may be insufficiently accurate. An example comparing the first-order scattering solution and the diffusion approximation to the radiative transfer solution can be seen in Fig. 3.

Fig. 2. Pulse statistics for 10 ns pulses at 100 GHz through 5 km rain layers for rain rates of 2.5 mm/hr, and 25 mm/hr. Here $t_r = 16.67 \mu s$.

Fig. 3. Radiative transfer, first-order multiple scattering, and diffusion approximation calculations of a 50 ns optical pulse through 1 km of fog in the vicinity of Point-Loma for $\lambda = 10.0 \mu m$. Here the field of view is $20^\circ$, and $t_r = 3.33 \mu s$.

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References